Riemann Solvers

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in/~praveen

March 18, 2013

Godunov scheme

- At any time t^n, FV solution is constant in each cell

$$U(x, t^n) = U_i^n, \quad x_{i-1/2} < x < x_{i+1/2}$$

- Riemann problem at every cell interface
- Godunov's idea
  1. Solve Riemann problem for U^n at every cell interface exactly
  2. Evolve the Riemann solution upto next time level $t^{n+1} = t^n + \Delta t$
  3. Average the solution at t^{n+1} to get cell average values U^{n+1}
- Solution of Riemann problem at $x_{i+1/2}$

$$U_{i+1/2}\left(\frac{x - x_{i+1/2}}{t - t^n}\right)$$

waves moving to left and right

Godunov scheme

- Waves from successive Riemann problems must not intersect

$$\Delta t < \frac{h}{2S^n}$$

- $S^n =$ maximum wave speed of all Riemann problems
- Average solution at new time level

$$U_i^{n+1} = \frac{1}{h}\left[\int_0^{\frac{1}{2}h} U_{i-1/2}\left(\frac{\xi}{\Delta t}\right)\mathrm{d}\xi + \int_{-\frac{1}{2}h}^{0} U_{i+1/2}\left(\frac{\xi}{\Delta t}\right)\mathrm{d}\xi\right]$$

- Difficult to implement numerically when expansion waves are present
- CFL condition is more restrictive

Godunov scheme

- Exact solution of Riemann problem

$$\tilde{U}(x,t) = U_{i+1/2}\left(\frac{x - x_{i+1/2}}{t - t^n}\right), \quad x_i \le x \le x_{i+1}, \quad t^n \le t \le t^{n+1}$$

- Satisfies integral conservation law

$$\begin{aligned}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{U}(x,t^{n+1})\mathrm{d}x &= \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{U}(x,t^n)\mathrm{d}x + \int_0^{\Delta t} F[\tilde{U}(x_{i-1/2},t)]\mathrm{d}t \\ &\quad - \int_0^{\Delta t} F[\tilde{U}(x_{i+1/2},t)]\mathrm{d}t\end{aligned}$$

But

$$\int_0^{\Delta t} F[\tilde{U}(x_{i+1/2},t)]\mathrm{d}t = \int_0^{\Delta t} F[U_{i+1/2}(0)]\mathrm{d}t = F[U_{i+1/2}(0)]\Delta t$$

etc., so that we finally have

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{h}\left[F(U_{i+1/2}(0)) - F(U_{i-1/2}(0))\right]$$

Godunov scheme

- Godunov flux

$$F_{i+1/2} = F(U_i, U_{i+1}) = F(U_{i+1/2}(0))$$

- right moving waves from $U_{i-1/2}$ should not reach $x_{i+1/2}$ and vice versa: CFL condition

$$\Delta t < \frac{h}{S^n}$$

- Accurate but expensive - not used for practical computations
- Recall: Linear system of equations $F = AU$, A constant matrix

$$F(U_{i+1/2}(0)) = A^+ U_i + A^- U_{i+1}$$

Here, Godunov scheme is identical to upwind scheme

Roe scheme

- Roe's idea: Solve Riemann problem approximately
- Linearize the non-linear problem - then solve Riemann problem

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\partial U}{\partial t} + \bar{\mathrm{A}}\frac{\partial U}{\partial x} = 0$$

- Matrix $\bar{\mathrm{A}} = \bar{\mathrm{A}}(U_l, U_r)$ must satisfy certain consistency conditions
  1. Consistency: $\bar{\mathrm{A}}(U, U) = A(U)$
  2. Hyperbolicty: $\bar{\mathrm{A}}$ has all real eigenvalues and linearly independent eigenvectors.
  3. Conservation: $F(U_r) - F(U_l) = \bar{\mathrm{A}}(U_l, U_r)(U_r - U_l)$

  This is known as a **Roe-type linearization** of the non-linear hyperbolic PDE.

Roe scheme

- Parameter vector

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} 1 \\ u \\ H \end{bmatrix}$$

- Define

$$Z(\alpha) = Z_l + \alpha(Z_r - Z_l), \qquad Z(0) = Z_l, \qquad Z(1) = Z_r$$

- U and F are homogeneous of degree two in the parameter vector Z

$$\begin{aligned} \Delta U &= U_r - U_l = U(Z(1)) - U(Z(0)) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha} U(Z(\alpha)) \mathrm{d}\alpha \\ &= \int_0^1 \underbrace{U'(Z(\alpha))}_{\text{linear in } \alpha} (Z_r - Z_l) \mathrm{d}\alpha \\ &= D(\bar{Z}) \Delta Z, \qquad \bar{Z} = \frac{1}{2}(Z_l + Z_r) \end{aligned}$$

Roe scheme

- Similarly, we can show for the flux difference

$$\Delta F = C(\bar{Z})\Delta Z$$

- Matrices C, D are given by

$$D(Z) = \begin{bmatrix} 2Z_1 & 0 & 0 \\ Z_2 & Z_1 & 0 \\ \frac{1}{\gamma}Z_3 & \frac{\gamma-1}{\gamma}Z_2 & \frac{1}{\gamma}Z_1 \end{bmatrix}, \quad C(Z) = \begin{bmatrix} Z_2 & Z_1 & 0 \\ \frac{\gamma-1}{\gamma}Z_3 & \frac{\gamma+1}{\gamma}Z_2 & \frac{\gamma-1}{\gamma}Z_1 \\ 0 & Z_3 & Z_2 \end{bmatrix}$$

- Hence we have

$$\Delta F = \hat{A}\Delta U, \quad \hat{A} = C(\bar{Z})D^{-1}(\bar{Z})$$

Roe scheme

- Explicit computation gives

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{\gamma}\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)^2 & (3-\gamma)\frac{\bar{Z}_2}{\bar{Z}_1} & \gamma - 1 \\ \frac{\gamma-1}{2}\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)^2 - \frac{\bar{Z}_2\bar{Z}_3}{\bar{Z}_1^2} & \frac{\bar{Z}_3}{\bar{Z}_1} - (\gamma-1)\left(\frac{\bar{Z}_2}{\bar{Z}_1}\right)^2 & \gamma\frac{\bar{Z}_2}{\bar{Z}_1} \end{bmatrix}$$

- Define *Roe averages*

$$\bar{u} = \frac{\bar{Z}_2}{\bar{Z}_1} = \frac{u_l\sqrt{\rho}_l + u_r\sqrt{\rho}_r}{\sqrt{\rho}_l + \sqrt{\rho}_r}, \quad \bar{H} = \frac{\bar{Z}_3}{\bar{Z}_1} = \frac{H_l\sqrt{\rho}_l + H_r\sqrt{\rho}_r}{\sqrt{\rho}_l + \sqrt{\rho}_r}$$

In terms of these average quantities, the matrix $\hat{A}$ is

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)\bar{u}^2 & (3-\gamma)\bar{u} & \gamma-1 \\ \bar{u}[\frac{1}{2}(\gamma-1)\bar{u}^2 - \bar{H}] & \bar{H} - (\gamma-1)\bar{u}^2 & \gamma\bar{u} \end{bmatrix}$$

Roe scheme

- $\bar{U}$ = Conserved vector corresponding to $\bar{u}$, $\bar{H}$
- Then
$$\hat{A} = A(\bar{U})$$
- Hence if we take $\bar{\mathrm{A}} = \hat{A}$, all three conditions satisfied
- $U_R(x/t)$ be the solution of the Riemann problem for the Roe-linearized equation
- The flux is given by $F(U_R(0))$ and we know from linear systems
$$F(U_R(0)) = \frac{1}{2}(F_l + F_r) - \frac{1}{2}|\bar{\mathrm{A}}(U_l, U_r)|(U_r - U_l)$$
- Has upwind property

Roe scheme: entropy violation

- Roe scheme – derived from linearized problem – has only contact discontinuities – no expansion wave
- sonic expansion wave – Roe scheme can give rise to entropy violating shocks
- Eigenvalue of $\bar{\mathrm{A}}$ becomes zero – loss of numerical dissipation

$$\bar{\lambda}_1 = \bar{u} - \bar{a}, \quad \bar{\lambda}_2 = \bar{u}, \qquad \bar{\lambda}_3 = \bar{u} + \bar{a}$$

- Entropy fix – do not allow eigenvalue to become zero

$$|\hat{\lambda}_i| = \begin{cases} \frac{\bar{\lambda}_i^2}{4\epsilon\bar{a}} + \epsilon\bar{a} & \text{if } |\bar{\lambda}_i| < 2\epsilon\bar{a} \\ |\bar{\lambda}_i| & \text{otherwise} \end{cases}$$

This fix is applied to $\lambda_1 = u - a$ and $\lambda_3 = u + a$.

Roe scheme formulae

$$F(U_l, U_r) = \frac{1}{2}(F_l + F_r) - \frac{1}{2}|\bar{\mathrm{A}}(U_l, U_r)|(U_r - U_l)$$

Eigenvectors of $\bar{\mathrm{A}}$

$$r_1 = \begin{bmatrix} 1 \\ \bar{u} - \bar{a} \\ \bar{H} - \bar{u}\bar{a} \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \bar{u} \\ \frac{1}{2}\bar{u}^2 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ \bar{u} + \bar{a} \\ \bar{H} + \bar{u}\bar{a} \end{bmatrix}$$

where

$$\bar{a}^2 = (\gamma - 1)[\bar{H} - \frac{1}{2}\bar{u}^2]$$

Write jump in terms of eigenvectors

$$\Delta U = U_r - U_l = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3$$

Roe scheme formulae

or

$$\begin{aligned}\alpha_1 + \alpha_2 + \alpha_3 &= \Delta U_1 \\ \alpha_1(\bar{u} - \bar{a}) + \alpha_2 \bar{u} + \alpha_3(\bar{u} + \bar{a}) &= \Delta U_2 \\ \alpha_1(\bar{H} - \bar{u}\bar{a}) + \alpha_2 \frac{\bar{u}^2}{2} + \alpha_3(\bar{H} + \bar{u}\bar{a}) &= \Delta U_3\end{aligned}$$

Solution is

$$\begin{aligned}\alpha_2 &= \frac{\gamma - 1}{\bar{a}^2}[(\bar{H} - \bar{u}^2)\Delta U_1 + \bar{u}\Delta U_2 - \Delta U_3] \\ \alpha_1 &= \frac{1}{2\bar{a}}[(\bar{u} + \bar{a})\Delta U_1 - \Delta U_2 - \bar{a}\alpha_2] \\ \alpha_3 &= \Delta U_1 - \alpha_1 - \alpha_2\end{aligned}$$

Then the flux is

$$F(U_l, U_r) = \frac{1}{2}(F_l + F_r) - \frac{1}{2}\sum_{j=1}^{3} \alpha_j |\hat{\lambda}_j| r_j$$

Roe scheme for general system

- General system of conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad U, F \in \mathbb{R}^n$$

- Entropy-entropy flux pair $(\eta, \theta) : U \to \mathbb{R}$, η convex

$$\theta'(U) = \eta'(U) F'(U)$$

Theorem of Harten and Lax

If the hyperbolic system has an entropy-entropy flux pair, then it admits a Roe-type linearization.

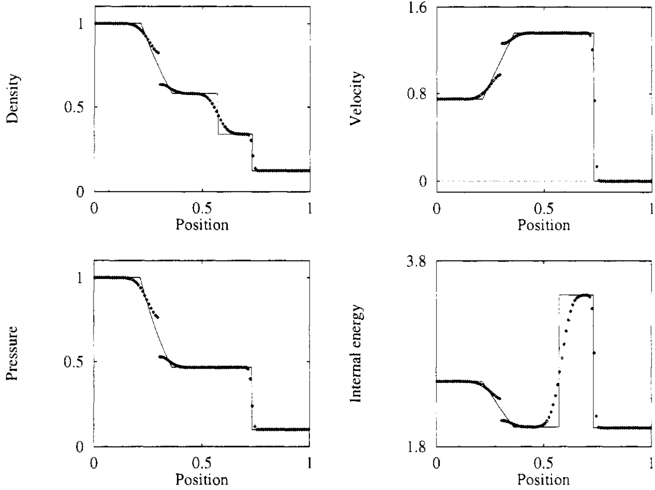

Fig. 11.4. Godunov's method with Roe's Riemann solver (no entropy fix) for Test 1, $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions compared at time 0.2

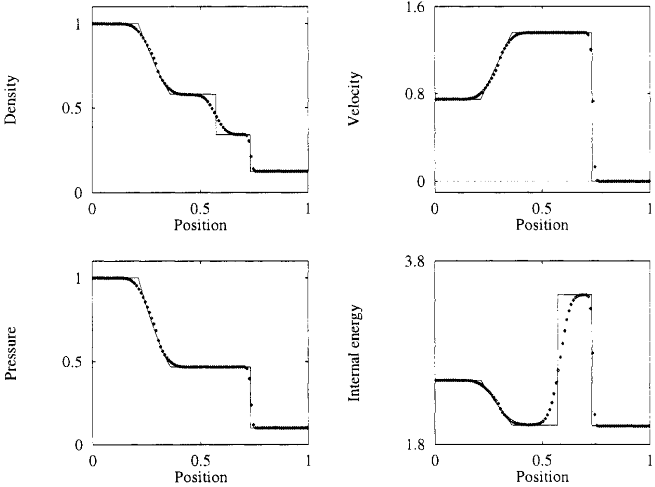

Fig. 11.5. Godunov's method with Roe's Riemann solver applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2