# High order methods

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# High resolution schemes

- We have seen two notions of stability: Fourier stability and maximum stability (monotone scheme, positive scheme, LED scheme)
- Fourier stability is always necessary; it determines stability to small perturbations.
- A scheme which is not Fourier stable is useless since it will cause blow-up of solution.
- Maximum stability is a stronger notion of stability. It prevents solution from oscillating. It is required when solution has discontinuities or steep gradients or is not smooth.
- Maximum stable first order schemes: upwind, Lax-Friedrichs
- Second order scheme: Lax-Wendroff, Fourier stable under CFL condition, but not stable in maximum norm.
- Basic idea of high order scheme: blend first order and second order scheme with a switching function, called **limiter**. If solution is locally in danger of developing oscillations, switch from second order to first order scheme. It is second order in smooth regions.

# Godunov's order barrier theorem

$$v_i^{n+1} = \sum_j b_j v_{i+j}^n, \qquad \{b_j\} \text{ are some constants}$$

Taylor series in space for exact solution $u$

$$u_{i+j}^n = u_i^n + \sum_{m=1}^{\infty} \frac{(jh)^m}{m!} \frac{\partial^m u}{\partial x^m}$$

Taylor series in time for exact solution $u$

$$u_i^{n+1} = u_i^n + \sum_{m=1}^{\infty} \frac{(\Delta t)^m}{m!} \frac{\partial^m u}{\partial t^m}$$

Local truncation error: scheme is $p$'th order accurate if

$$\tau_i^n = \frac{1}{\Delta t}\left( u_i^{n+1} - \sum_j b_j u_{i+j}^n \right) = \mathcal{O}\left(h^p\right)$$

# Godunov's order barrier theorem

Use PDE: $u_t = -a u_x$, $u_{tt} = a^2 u_{xx}$, etc. Then

$$\begin{aligned} \tau_i^n \Delta t &= (1 - \sum_j b_j) u_i^n - (\sigma + \sum_j j b_j) h u_x \\ &\quad + (\sigma^2 - \sum_j j^2 b_j) \frac{1}{2} h^2 u_{xx} + \mathcal{O}\left(h^3\right) \end{aligned}$$

For first order accuracy, we need

$$\sum_j b_j = 1, \qquad \sum_j j b_j = -\sigma \qquad \Longrightarrow \qquad \tau_i^n = \mathcal{O}\left(h\right)$$

For second order accuracy, we need

$$\sum_j j^2 b_j = \sigma^2 \qquad \Longrightarrow \qquad \tau_i^n = \mathcal{O}\left(h^2\right)$$

# Godunov's order barrier theorem

Assume that the scheme is positive, $b_j \geq 0$ and second order accurate. Then, by Cauchy-Schwartz inequality

$$\begin{aligned} \sigma^2 &= \left( \sum_j j b_j \right)^2 = \left( \sum_j j \sqrt{b_j} \sqrt{b_j} \right)^2 \\ &\leq (\sum_j j^2 b_j)(\sum_j b_j) = \sigma^2 \end{aligned}$$

This is possible only if equality holds in Cauchy-Schwartz inequality, i.e., if

$$j\sqrt{b_j} = c\sqrt{b_j}, \qquad \text{for some constant } c$$

This implies that $j = -\sigma$ and requires $\sigma$ to be an integer in which case we obtain exact solution. But in general, this condition cannot be satisfied.

Any linear, positive scheme for $u_t + a u_x = 0$ is at most first order accurate.

# Second order upwind scheme

Assume $a > 0$. The semi-discrete SOU scheme is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}(3v_i - 4v_{i-1} + v_{i-2}) = \frac{2a}{h}(v_{i-1} - v_i) - \frac{a}{2h}(v_{i-2} - v_i)$$

Write as first order upwind scheme + correction

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}(v_i - v_{i-1}) - \frac{a}{h}\left[\frac{1}{2}(v_i - v_{i-1}) - \frac{1}{2}(v_{i-1} - v_{i-2})\right]$$

Define difference ratios to measure local solution smoothness

$$r_{i-1} = \frac{v_i - v_{i-1}}{v_{i-1} - v_{i-2}}, \qquad r_i = \frac{v_{i+1} - v_i}{v_i - v_{i-1}}$$

Introduce switching functions

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}\left[(v_i - v_{i-1}) + \frac{1}{2}\Psi(r_i)(v_i - v_{i-1}) - \frac{1}{2}\Psi(r_{i-1})(v_{i-1} - v_{i-2})\right]$$

# Second order upwind scheme

which can be written as

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}\left[1 + \frac{1}{2}\Psi(r_i) - \frac{1}{2}\frac{\Psi(r_{i-1})}{r_{i-1}}\right](v_i - v_{i-1})$$

This scheme is positive provided

$$\frac{\Psi(r_{i-1})}{r_{i-1}} - \Psi(r_i) \le 2$$

We have lot of freedom in choosing the function $\Psi$. Let us restrict $\Psi$ to be a positive function

$$\Psi(r) \ge 0, \qquad r \ge 0$$

When $\Psi = 0$, the scheme becomes first order accurate. If there is shock around some grid point, i.e., $r_i < 0$, we want the scheme to become first order accurate. Hence

$$\Psi(r) = 0, \qquad r \le 0$$

# Second order upwind scheme

These conditions imply that

$$0 \leq \Psi(r) \leq 2r$$

Symmetry property: Backward and forward differences are treated in the same manner

$$\frac{\Psi(r)}{r} = \Psi\left(\frac{1}{r}\right)$$

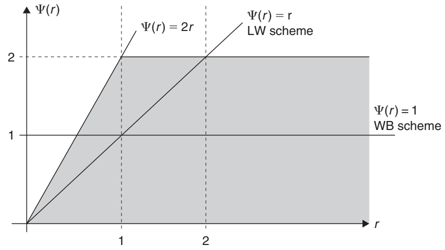
This leads to the condition

$$\Psi(r) \leq 2$$

Combining all the above conditions, we get

$$0 \leq \Psi(r) \leq \min(2, 2r)$$

# Second order upwind scheme

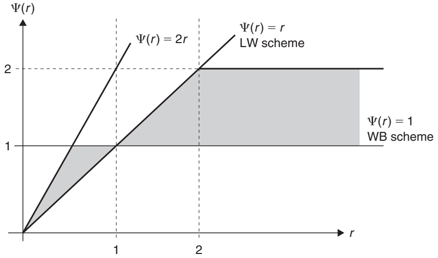



If $v$ is a linear function of $x$, then the scheme should give exact solution. In this case $r_i = 1$ and we should have

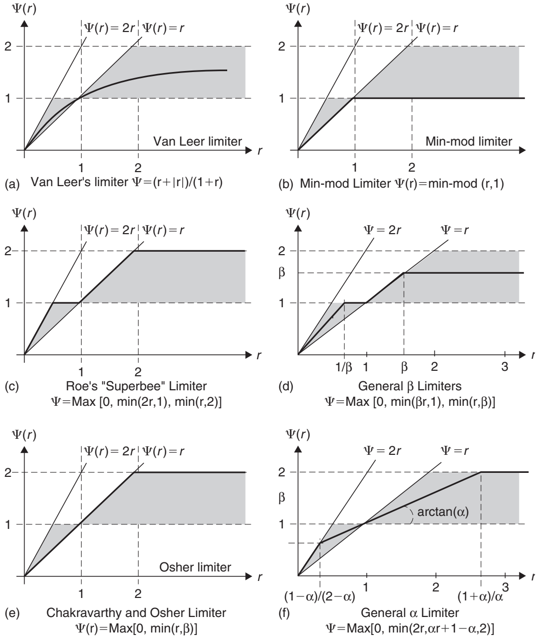
$$\Psi(1) = 1$$

This condition is required to achieve second order accuracy in smooth regions.

# Second order upwind scheme



# Limiter functions



# Lax-Wendroff scheme

$$v_i^{n+1} = \frac{1}{2}\sigma(1+\sigma)v_{i-1}^n + (1-\sigma^2)v_i^n - \frac{1}{2}\sigma(1-\sigma)v_{i+1}^n$$

If $a > 0$ so that $\sigma > 0$, then coefficient of $v_{i+1}^n$ is negative and the scheme is not monotone.

Define ratios

$$R_i = \frac{v_i - v_{i-1}}{v_{i+1} - v_i} = \frac{\text{backward difference}}{\text{forward difference}}$$

If the solution $v_{i-1}, v_i, v_{i+1}$ is smooth then $R_i = \mathcal{O}(1)$. In fact if solution is linear in $x$, then $R_i = 1$.

# Modification of Lax-Wendroff scheme

Step 1: Write high order scheme as low order positive scheme + correction term

$$
\begin{aligned}
u_i^{n+1} &= u_i^n - \sigma(u_i^n - u_{i-1}^n) \\
&\quad -\frac{\sigma}{2}(1-\sigma)(u_{i+1}^n - u_i^n) + \frac{\sigma}{2}(1-\sigma)(u_i^n - u_{i-1}^n)
\end{aligned}
$$

Step 2: Introduce switching function in correction terms

$$
\begin{aligned}
u_i^{n+1} &= u_i^n - \sigma(u_i^n - u_{i-1}^n) \\
&\quad -\frac{\sigma}{2}(1-\sigma)\Psi(R_i)(u_{i+1}^n - u_i^n) + \frac{\sigma}{2}(1-\sigma)\Psi(R_{i-1})(u_i^n - u_{i-1}^n)
\end{aligned}
$$

or, re-arranging

$$
u_i^{n+1} = u_i^n - \sigma\left\{1 + \frac{1}{2}(1-\sigma)\left[\frac{\Psi(R_i)}{R_i} - \Psi(R_{i-1})\right]\right\}(u_i^n - u_{i-1}^n)
$$

# Modification of Lax-Wendroff scheme

Scheme is positive if

$$\Psi(R_{i-1}) - \frac{\Psi(R_i)}{R_i} \leq \frac{2}{1-\sigma}$$

Assuming symmetry property for limiter, this condition is satisfied by choosing

$$0 \leq \Psi(R) \leq \min\left(\frac{2R}{\sigma}, \frac{2}{1-\sigma}\right)$$

Since by CFL condition $0 \leq \sigma \leq 1$, we can take the more restrictive condition

$$0 \leq \Psi(R) \leq \min(2, 2R)$$

This is the same condition as we obtained for the SOU scheme and the allowed region for $\Psi$ is as before.
For better accuracy, one should use the condition involving the CFL number $\sigma$, see Hirsch, page 387.

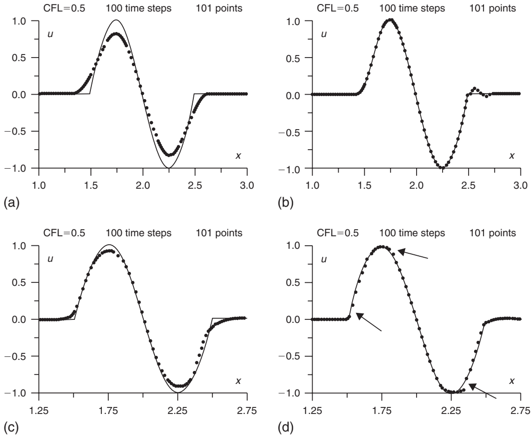



**Figure 8.3.5** *Effects of limiters on the linear convection of a sinusoidal wave (a) first order upwind scheme (b) second order upwind scheme (c) second order upwind scheme with min-mod limiter (d) second order upwind scheme with superbee limiter.*

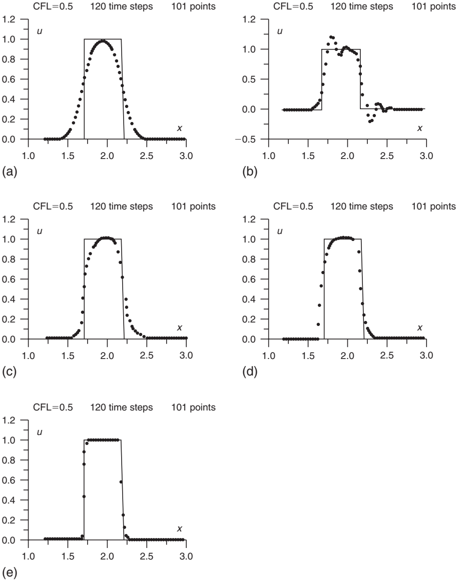



**Figure 8.3.6** *Effects of limiters on the linear convection of a square wave after 120 time steps: (a) first order upwind scheme, (b) second order upwind scheme, (c) second order upwind scheme with min-mod limiter, (d) second order upwind scheme with Van Leer limiter and (e) second order upwind scheme with superbee limiter.*

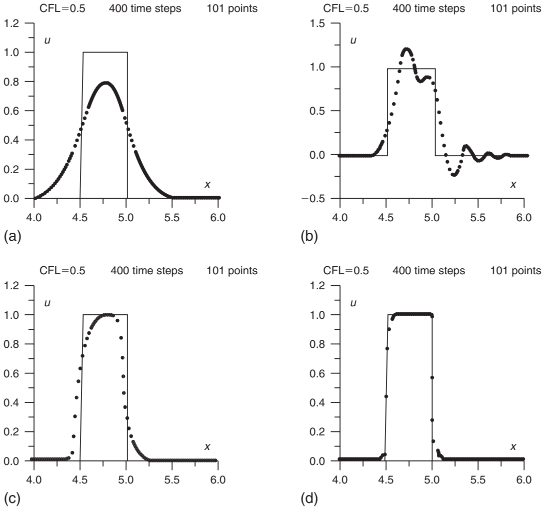



**Figure 8.3.7** *Effects of limiters on the linear convection of a square wave after 400 time steps: (a) first order upwind scheme, (b) second order upwind scheme, (c) second order upwind scheme with Van Leer limiter and (d) second order upwind scheme with superbee limiter.*

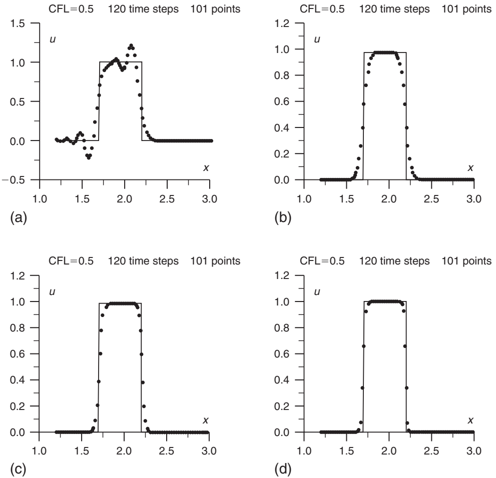



**Figure 8.3.8** *Effects of limiters on the linear convection of a square wave after 120 time steps: (a) standard LW scheme, (b) second order high-resolution LW scheme with min-mod limiter, (c) second order high-resolution LW scheme with Van Leer limiter and (d) second order high-resolution LW scheme with superbee limiter.*

# FVM for Non-linear conservation law

$$v_i^{n+1} = v_i^n - \lambda(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n) = H(\ldots, v_{i-1}^n, v_i^n, v_{i+1}^n, \ldots)$$

**Definition: Monotone scheme**

The scheme is monotone if $H$ is an increasing function in all its arguments. If $H$ is differentiable then it is monotone if

$$\frac{\partial}{\partial v_k} H(\ldots, v_{i-1}, v_i, v_{i+1}, \ldots) \geq 0, \qquad k = \ldots, i-1, i, i+1, \ldots$$

**Remark**: A linear scheme, e.g.,

$$H(v_{i-1}, v_i, v_{i+1}) = av_{i-1} + bv_i + cv_{i+1}$$

is monotone if all coefficients are positive, i.e., $a \geq 0$, $b \geq 0$, $c \geq 0$.

### Theorem

Consider a 3 point scheme with numerical flux $g(\cdot,\cdot)$ i.e.,

$$g_{i+\frac{1}{2}} = g(v_i, v_{i+1})$$

The scheme is monotone if $g$ is an increasing function in the first argument and a decreasing function in the second argument.

### Theorem

A monotone scheme converges to the unique entropy solution.

**Remark**: The Godunov scheme is a monotone scheme. The Murman-Roe scheme is not a monotone scheme.

### Theorem

Any differentiable monotone scheme is at most first order accurate.

# Total variation

$$\mathrm{TV}(v) = \sum_i |v_i - v_{i-1}|$$

$\mathrm{TV}$ measures the amount of oscillation in the solution $v$. If $v$ develops new wiggles, then its $\mathrm{TV}$ increases.

**Definition: Total Variation Diminishing (TVD) scheme**

The scheme $H$ is said to be TVD if

$$\mathrm{TV}(v^{n+1}) \leq \mathrm{TV}(v^n)$$

**Theorem**

$$\text{Monotone scheme} \Rightarrow \text{TVD scheme} \Leftrightarrow \text{Monotonicity preserving scheme}$$

## Harten's incremental form

$$v_i^{n+1} = v_i^n + C^n_{i+\frac{1}{2}}(v^n_{i+1} - v^n_i) - D^n_{i-\frac{1}{2}}(v^n_i - v^n_{i-1})$$

$$C_{i+\frac{1}{2}} = C(\ldots, v_i, v_{i+1}, \ldots), \qquad D_{i-\frac{1}{2}} = D(\ldots, v_{i-1}, v_i, \ldots)$$

$$\begin{aligned}
v_i^{n+1} &= v_i^n - \lambda(g^n_{i+\frac{1}{2}} - g^n_{i-\frac{1}{2}}) \\
&= v_i^n - \lambda(g^n_{i+\frac{1}{2}} - f^n_i + f^n_i - g^n_{i-\frac{1}{2}}) \\
&= v_i^n - \lambda\frac{g^n_{i-\frac{1}{2}} - f^n_i}{v^n_i - v^n_{i-1}}(v^n_i - v^n_{i-1}) - \lambda\frac{g^n_{i+\frac{1}{2}} - f^n_i}{v^n_{i+1} - v^n_i}(v^n_{i+1} - v^n_i)
\end{aligned}$$

### Theorem

The scheme $H$ is TVD if

$$C_{i+\frac{1}{2}} \geq 0, \qquad D_{i+\frac{1}{2}} \geq 0, \qquad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1$$

Proof:

# Reconstruction approach

- First order FV: piecewise constant solution

$$v_i^{n+1} = v_i^n - \lambda[g(v_i^n, v_{i+1}^n) - g(v_{i-1}^n, v_i^n)]$$

  or semi-discrete scheme

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{g(v_i, v_{i+1}) - g(v_{i-1}, v_i)}{h} = 0$$

- Higher order scheme: Reconstruct solution inside each cell by a polynomial $p_i(x)$

$$v_i = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p_i(x)\mathrm{d}x$$

  Evaluate $p_i(x)$ at the cell faces $x_{i-\frac{1}{2}}$, $x_{i+\frac{1}{2}}$

$$v_{i-\frac{1}{2}}^R = p_i(x_{i-\frac{1}{2}}), \qquad v_{i+\frac{1}{2}}^L = p_i(x_{i+\frac{1}{2}})$$

# Reconstruction approach

- At any cell face $x_{i+\frac{1}{2}}$ we have reconstructed states

$$v^L_{i+\frac{1}{2}} = p_i(x_{i+\frac{1}{2}}), \qquad v^R_{i+\frac{1}{2}} = p_{i+1}(x_{i+\frac{1}{2}}), \qquad v^L_{i+\frac{1}{2}} \neq v^R_{i+\frac{1}{2}}$$

- Semi-discrete scheme

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{g(v^L_{i+\frac{1}{2}}, v^R_{i+\frac{1}{2}}) - g(v^L_{i-\frac{1}{2}}, v^R_{i-\frac{1}{2}})}{h} = 0$$

- Discretise in time using high order time integration scheme, e.g., RK scheme.

# Solution Reconstruction

Given the cell average values $\{v_i\}$, we want to reconstruct the solution in $x$
Simplest approach: piecewise linear reconstruction

$$p_i(x) = v_i + s_i \frac{(x - x_i)}{h}, \qquad x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}$$

Note that this already satisfies

$$\frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p_i(x) = v_i$$

How to estimate the slope $s_i$ ? Several possible choices

$$s_i^b = v_i - v_{i-1}, \qquad s_i^f = v_{i+1} - v_i, \qquad s_i^c = \frac{1}{2}(v_{i+1} - v_{i-1})$$

If $v$ is smooth, then central difference $s_i^c$ is the most accurate. But if there is a discontinuity in $v$ around $i$ then we should take the smoothest possible

# Solution Reconstruction

data/stencil. One possibility is to choose the estimate with smallest absolute value of slope.

$$\text{minmod}(a, b, c) = \begin{cases} \text{sign}(a)\min(|a|, |b|, |c|) & \text{if } \text{sign}(a) = \text{sign}(b) = \text{sign}(c) \\ 0 & \text{otherwise} \end{cases}$$

Linear reconstruction with minmod limiter

$$s_i = \text{minmod}(s_i^b, s_i^c, s_i^f)$$

The reconstructed solution has TVD property

$$\text{TV}(p) = \text{TV}(v)$$

This scheme can still lead to diffusion of shocks and clipping of local extrema. A slightly relaxed version of slope which gives more accurate shocks, but may create some small oscillations is

$$s_i = \text{minmod}(\theta s_i^b, s_i^c, \theta s_i^f), \qquad 1 \le \theta \le 2$$

# MUSCL scheme of Van Leer

From Taylor formula

$$v(x) = v(x_j) + (x - x_j)v_x(x_j) + \frac{1}{2}(x - x_j)^2 v_{xx}(x_j) + \mathcal{O}\left(\Delta x^3\right)$$

But $v(x_j) \neq v_j$ while we want conservation, so ignoring terms $\mathcal{O}\left(\Delta x\right)^3$ and above

$$\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(x)\mathrm{d}x = v_j \quad \Longrightarrow \quad v(x_j) = v_j - \frac{\Delta x^2}{24} v_{xx}(x_j)$$

Hence

$$v(x) = v_j + (x - x_j)v_x(x_j) + \frac{1}{2}\left[(x - x_j)^2 - \frac{\Delta x^2}{12}\right] v_{xx}(x_j) + \mathcal{O}\left(\Delta x^3\right)$$

# MUSCL scheme of Van Leer

Degree two polynomial in cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$p_j(x) = v_j + (x - x_j)\frac{v_{j+1} - v_{j-1}}{2\Delta x} + \frac{3\kappa}{2}\left[(x - x_j)^2 - \frac{\Delta x^2}{12}\right]\frac{v_{j-1} - 2v_j + v_{j+1}}{\Delta x^2}$$

where we have introduced a parameter $\kappa$. If $\kappa = \frac{1}{3}$ then we obtain third order accuracy in the reconstruction. Using this approximation we can get the states at the cell faces

$$v^+_{j-\frac{1}{2}} = p_j(x_{j-\frac{1}{2}}) = v_j - \frac{1}{4}[(1+\kappa)\Delta v_{j-\frac{1}{2}} + (1-\kappa)\Delta v_{j+\frac{1}{2}}]$$
$$v^-_{j+\frac{1}{2}} = p_j(x_{j+\frac{1}{2}}) = v_j + \frac{1}{4}[(1-\kappa)\Delta v_{j-\frac{1}{2}} + (1+\kappa)\Delta v_{j+\frac{1}{2}}]$$

which can be used to compute the flux

$$g_{j+\frac{1}{2}} = g(v^-_{j+\frac{1}{2}}, v^+_{j+\frac{1}{2}})$$

# MUSCL scheme of Van Leer

In order to make the scheme TVD we limit the reconstructed states $v^{\pm}_{j+\frac{1}{2}}$. We first write

$$v^{+}_{j-\frac{1}{2}} = v_j - \frac{1}{4}[(1+\kappa)\frac{\Delta v_{j-\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\Delta v_{j+\frac{1}{2}} + (1-\kappa)\frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}\Delta v_{j-\frac{1}{2}}]$$

$$v^{-}_{j+\frac{1}{2}} = v_j + \frac{1}{4}[(1-\kappa)\frac{\Delta v_{j-\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\Delta v_{j+\frac{1}{2}} + (1+\kappa)\frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}\Delta v_{j-\frac{1}{2}}]$$

Let us introduce the parameter

$$R_j = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

# MUSCL scheme of Van Leer

to measure the local smoothness of the function. Then we introduce a limiter function $\psi$ into the reconstruction scheme

$$\begin{aligned} v^+_{j-\frac{1}{2}} &= v_j - \frac{1}{4}[(1+\kappa)\psi(1/R_j)\Delta v_{j+\frac{1}{2}} + (1-\kappa)\psi(R_j)\Delta v_{j-\frac{1}{2}}] \\ v^-_{j+\frac{1}{2}} &= v_j + \frac{1}{4}[(1-\kappa)\psi(1/R_j)\Delta v_{j+\frac{1}{2}} + (1+\kappa)\psi(R_j)\Delta v_{j-\frac{1}{2}}] \end{aligned} \tag{1}$$

In smooth regions we expect $R_j \approx 1$ and we should have $\psi(R_j) \approx R_j$. In particular, we need $\psi(1) = 1$ in order to obtain second order accuracy in smooth regions.

## Theorem

The finite volume scheme with monotone Lipschitz continuous numerical flux

$$v_j^{n+1} = v_j^n - \lambda(g^n_{j+\frac{1}{2}} - g^n_{j-\frac{1}{2}}), \qquad g_{j+\frac{1}{2}} = g(v^-_{j+\frac{1}{2}}, v^+_{j+\frac{1}{2}})$$

where the states $v^{\pm}_{j+\frac{1}{2}}$ are obtained by the $\kappa$ parameter MUSCL scheme (1) is TVD if $\psi$ satisfies

$$0 \le \psi(R) \le \frac{3-\kappa}{1-\kappa} - (1+\alpha)\frac{1+\kappa}{1-\kappa}, \qquad 0 \le \frac{\psi(R)}{R} \le 2+\alpha$$

where $\alpha \in [-2, 2(1-\kappa)/(1+\kappa)]$ under the CFL time step restriction

$$\lambda\frac{(2-(2+\alpha)\kappa)}{1-\kappa}C_j \le 1, \qquad C_j = \max_{u,v}\left|\frac{\partial g}{\partial u}(u, v^+_{j+\frac{1}{2}}) - \frac{\partial g}{\partial v}(v^-_{j-\frac{1}{2}}, v)\right|$$

where the maximum is taken over all $u$ between $v^-_{j-\frac{1}{2}}$, $v^-_{j+\frac{1}{2}}$ and all $v$ between $v^+_{j-\frac{1}{2}}$, $v^+_{j+\frac{1}{2}}$.

## Some limiters

The limiter function

$$\psi_{MM}(R) = \max(0, \min(R, \beta)), \qquad \beta \in [1, (3-\kappa)/(1-\kappa)]$$

satisfies the conditions of the theorem. Let us choose $\kappa = -1$ and $\beta = 1$. Then we obtain the minmod limiter

$$\psi_{MM}(R) = \max(0, \min(R, 1))$$

Van Leer limiter

$$\psi(R) = \frac{R + |R|}{1 + |R|}$$

Van Albada limiter

$$\psi(R) = \frac{R^2 + R}{1 + R^2}$$

Unfortunately TVD schemes lose their accuracy near smooth local extrema. This leads to *clipping of local extrema*.

**Theorem (Osher)**

The TVD discretizations all reduce to at most first order accuracy at non-sonic critical points, i.e., points $u^*$ at which $f'(u^*) \neq 0$ and $u^*_x = 0$

We have to relax the strict TVD condition to develop uniformly high order accurate schemes.

# Method of lines

Integrate over space only

$$\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{g_{j+\frac{1}{2}}(t) - g_{j-\frac{1}{2}}(t)}{\Delta x} = 0, \qquad g_{j+\frac{1}{2}}(t) = g(v^-_{j+\frac{1}{2}}(t), v^+_{j+\frac{1}{2}}(t))$$

The two states are obtained by some piecewise polynomial reconstruction $\mathrm{TV}$,

$$v^-_{j+\frac{1}{2}}(t) = p_j(x_{j+\frac{1}{2}}, t), \qquad v^+_{j+\frac{1}{2}}(t) = p_{j+1}(x_{j+\frac{1}{2}}, t)$$

The space discretization is second order accurate. We need atleast a second order discretization in time so that the overall scheme is second order accurate.

Let us write the system of ODE as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = L(v)$$

Let us also assume that the first order in time discretization is *stable*, i.e.,

$$\boxed{\Delta t \le \Delta t_1 \quad \Longrightarrow \quad \|v + \Delta t L(v)\| \le \|v\|}$$

# First order time integration scheme

First order scheme: Forward Euler

$$v^{n+1} = v^n + \Delta t L(v^n)$$

Let us apply this scheme to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$$

which yields

$$v^{n+1} = [1 + \lambda \Delta t] v^n$$

while the exact solution is

$$u^{n+1} = \mathrm{e}^{\lambda \Delta t} v^n = [1 + \lambda \Delta t + \mathcal{O}\left(\Delta t\right)^2] v^n$$

The numerical scheme agrees with the exact solution upto $\mathcal{O}\left(\Delta t\right)$.

# Second order time integration scheme

Second order scheme (2-stage)

$$
\begin{aligned}
v^{(0)} &= v^n \\
v^{(1)} &= v^{(0)} + \Delta t L(v^{(0)}) \\
v^{(2)} &= \frac{1}{2} v^{(0)} + \frac{1}{2}[v^{(1)} + \Delta t L(v^{(1)})] \\
v^{n+1} &= v^{(2)}
\end{aligned}
$$

Applying this scheme to the ODE $\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$ yields

$$
v^{n+1} = [1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2] v^n
$$

which agrees with the exact solution upto $\mathcal{O}\left(\Delta t\right)^2$. Since the scheme is a convex combination, we obtain stability for $\Delta t \le \Delta t_1$

$$
\left\|v^{n+1}\right\| \le \frac{1}{2}\left\|v^{(0)}\right\| + \frac{1}{2}\left\|v^{(1)} + \Delta t L(v^{(1)})\right\| \le \frac{1}{2}\|v^n\| + \frac{1}{2}\left\|v^{(1)}\right\| \le \|v^n\|
$$

Such time integration schemes are known as *Strong Stability Preserving RK* schemes.

# SSP RK schemes

Third order scheme (3-stage)

$$
\begin{aligned}
v^{(0)} &= v^n \\
v^{(1)} &= v^{(0)} + \Delta t L(v^{(0)}) \\
v^{(2)} &= \frac{3}{4} v^{(0)} + \frac{1}{4}[v^{(1)} + \Delta t L(v^{(1)})] \\
v^{(3)} &= \frac{1}{3} v^{(0)} + \frac{2}{3}[v^{(2)} + \Delta t L(v^{(2)})] \\
v^{n+1} &= v^{(3)}
\end{aligned}
$$

A general $m$-stage RK scheme is of the form

$$
\begin{aligned}
v^{(0)} &= v^n \\
v^{(i)} &= \sum_{k=0}^{i-1} \left[ \alpha_{ik} v^{(k)} + \Delta t \beta_{ik} L(v^{(k)}) \right], \quad i = 1, \ldots, m \\
v^{n+1} &= v^{(m)}
\end{aligned}
$$

By consistency (take $L(v) \equiv 0$), we must have

$$\sum_{k=0}^{i-1} \alpha_{ik} = 1, \qquad i = 1, \ldots, m$$

**Lemma (Stability of SSP RK scheme)**

$$\text{If} \qquad \Delta t \le \Delta t_1 \quad \Longrightarrow \quad \|v + \Delta t L(v)\| \le \|v\|$$

then the $m$-stage RK scheme is stable under CFL condition

$$\Delta t \le c \Delta t_1, \qquad c = \min_{i,k} \frac{\alpha_{ik}}{\beta_{ik}}$$

provided that $\alpha_{ik} \ge 0$, $\beta_{ik} \ge 0$.

**Remark**: If the first order time scheme is TVD, then the high order SSPRK scheme is also TVD under a suitable time step condition.