Linear hyperbolic systems

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Linear hyperbolic system

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}, \quad F(U) = AU, \quad A \in \mathbb{R}^{m \times m}
\]

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \quad U(x, 0) = U^0(x)
\]

System \( U_t + AU_x = 0 \) is said to be **hyperbolic** provided

- \( A \) has \( m \) real eigenvalues
  \[ \lambda_1 < \lambda_2 < \ldots < \lambda_m \]
- The eigenvectors form a basis for \( \mathbb{R}^m \).
General solution

Eigenvalues and eigenvectors

\[ Ar_k = \lambda_k r_k, \quad r_k \in \mathbb{R}^m, \quad R = [r_1, r_2, \ldots, r_m] \in \mathbb{R}^{m \times m} \]

\[ \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m), \quad AR = R\Lambda, \quad A = R\Lambda R^{-1} \]

\[ \frac{\partial U}{\partial t} + R\Lambda R^{-1} \frac{\partial U}{\partial x} = 0 \implies R^{-1} \frac{\partial U}{\partial t} + \Lambda R^{-1} \frac{\partial U}{\partial x} = 0 \]

Define the characteristic variables by

\[ W = R^{-1}U \implies \frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = 0 \]

These equations become decoupled

\[ \frac{\partial W_i}{\partial t} + \lambda_i \frac{\partial W_i}{\partial x} = 0, \quad i = 1, 2, \ldots, m \]
General solution

Initial condition for $W$

$$W(x, 0) = W^0(x) = R^{-1}U^0(x)$$

Solution is

$$W_i(x, t) = W_i^0(x - \lambda_i t) = [R^{-1}U^0(x - \lambda_i t)]_i$$

Solution $U$ is obtained

$$U(x, t) = RW(x, t) = \sum_{i=1}^{m} W_i(x, t)r_i = \sum_{i=1}^{m} W_i^0(x - \lambda_i t)r_i$$
Riemann problem

Consider initial condition

\[ U^0(x) = \begin{cases} 
U_l & x < 0 \\
U_r & x > 0 
\end{cases} \]

Initial condition for \( W \)

\[ W^0(x) = R^{-1}U^0(x) = \begin{cases} 
R^{-1}U_l & x < 0 \\
R^{-1}U_r & x > 0 
\end{cases} = \begin{cases} 
W_l & x < 0 \\
W_r & x > 0 
\end{cases} \]

Solution for \( W_i, i = 1, 2, \ldots, m \)

\[ W_i(x, t) = W^0_i(x - \lambda_i t) = \begin{cases} 
W_{l,i} & x/t < \lambda_i \\
W_{r,i} & x/t > \lambda_i 
\end{cases} \]
Riemann problem

Solution $U$

$$U(x, t) = \sum_i W^0_i (x - \lambda_i t) r_i$$

$$= \sum_{i: x/t < \lambda_i} W_{l,i} r_i + \sum_{i: x/t > \lambda_i} W_{r,i} r_i$$

Example: System of 3 equations ($m = 3$)

$$U(x, t) = \begin{cases} 
U_l & x/t < \lambda_1 \\
U^*_1 & \lambda_1 < x/t < \lambda_2 \\
U^*_2 & \lambda_2 < x/t < \lambda_3 \\
U_r & x/t > \lambda_3 
\end{cases}$$
The intermediate states are given by

\[
U_1^* = W_{r,1}r_1 + W_{l,2}r_2 + W_{l,3}r_3
\]
\[
U_2^* = W_{r,1}r_1 + W_{r,2}r_2 + W_{l,3}r_3
\]

It is easy to check that the jump in the intermediate states is

\[
U_m^* - U_{m-1}^* = (W_{r,m} - W_{l,m})r_m
\]
Riemann problem
with the convention that $U^*_0 = U_l$ and $U^*_n = U_r$. Hence the initial discontinuity breaks into $n$ discontinuity waves which propagate at speeds $\lambda_i$, $i = 1, \ldots, n$.

Solution on $x/t = 0$:
For future use, we compute the solution along $x/t = 0$. It is given by

$$U_R(0) = \sum_{i: \lambda_i > 0} W_{l,i} r_i + \sum_{i: \lambda_i < 0} W_{r,i} r_i$$

and the corresponding flux is

$$F(U_R(0)) = AU_R(0) = \sum_{i: \lambda_i > 0} \lambda_i W_{l,i} r_i + \sum_{i: \lambda_i < 0} \lambda_i W_{r,i} r_i$$

This can be re-written as

$$F(U_R(0)) = \sum_i \lambda_i^+ W_{l,i} r_i + \sum_i \lambda_i^- W_{r,i} r_i$$
Riemann problem
where we have defined

$$\lambda^+ = \max(0, \lambda) = \frac{1}{2}(\lambda + |\lambda|) \geq 0$$

$$\lambda^- = \min(0, \lambda) = \frac{1}{2}(\lambda - |\lambda|) \leq 0$$

The above flux can also be written as

$$F(U_R(0)) = A^+U_l + A^-U_r, \quad A^\pm = R\Lambda^\pm R^{-1}$$

with $\Lambda^\pm = \text{diag}(\lambda_1^\pm, \ldots, \lambda_n^\pm)$. Another formula is obtained using the second definition of $\lambda^\pm$;

$$F(U_R(0)) = \frac{1}{2}(F_l + F_r) - \frac{1}{2} \sum_i |\lambda_i|(W_{r,i} - W_{l,i})r_i$$

$$= \frac{1}{2}(F_l + F_r) - \frac{1}{2}|A|(U_r - U_l), \quad |A| = R|\Lambda|R^{-1}$$
Upwind scheme

The system of conservation laws can be transformed to a set of decoupled linear advection equations

$$\frac{\partial W_i}{\partial t} + \lambda_i \frac{\partial W_i}{\partial x} = 0, \quad 1 \leq i \leq n$$

which represent waves moving with velocity $\lambda_i$. We can try to build a scheme for the system of conservation laws by applying the upwind scheme to the above advection equations. For the grid point $j$ we have

$$\frac{W_{i,j}^{n+1} - W_{i,j}^n}{\Delta t} + \lambda_i^+ \frac{W_{i,j}^n - W_{i,j-1}^n}{\Delta x} + \lambda_i^- \frac{W_{i,j+1}^n - W_{i,j}^n}{\Delta x} = 0$$

or using matrices

$$\frac{W_{j}^{n+1} - W_{j}^n}{\Delta t} + \Lambda^+ \frac{W_{j}^n - W_{j-1}^n}{\Delta x} + \Lambda^- \frac{W_{j+1}^n - W_{j}^n}{\Delta x} = 0$$
Upwind scheme

Transforming back to the conserved variables $U$, the above scheme becomes

$$
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + A^{+} \frac{U_{j}^{n} - U_{j-1}^{n}}{\Delta x} + A^{-} \frac{U_{j+1}^{n} - U_{j}^{n}}{\Delta x} = 0
$$

We could have obtained this scheme using the CIR splitting technique; separating the Jacobian $A$ into positive and negative parts

$$
A = A^{+} + A^{-}, \quad \frac{\partial U}{\partial t} + A^{+} \frac{\partial U}{\partial x} + A^{-} \frac{\partial U}{\partial x} = 0
$$

and using backward and forward differencing for the $A^{+}$ and $A^{-}$ terms respectively,

$$
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + A^{+} \frac{U_{j}^{n} - U_{j-1}^{n}}{h} + A^{-} \frac{U_{j+1}^{n} - U_{j}^{n}}{h} = 0
$$

we obtain exactly the same scheme.
Upwind scheme

Finite difference flux splitting

Another way to arrive at this scheme is to start with flux splitting. The eigenvalue splitting leads to the flux splitting

\[ F = A^+ U + A^- U = F^+ + F^- \]

so that conservation law can be written as

\[ \frac{\partial U}{\partial t} + \frac{\partial F^+}{\partial x} + \frac{\partial F^-}{\partial x} = 0 \]

and we use backward and forward differencing for the \( F^+ \) and \( F^- \) terms respectively.

\[ \frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + \frac{F_{j}^{+} - F_{j-1}^{+}}{h} + \frac{F_{j+1}^{-} - F_{j}^{-}}{h} = 0 \]
Upwind scheme
We can write this as a finite volume scheme

\[
\frac{U_{j}^{n+1} - U_{j}^{n}}{\Delta t} + \frac{(F_{j}^{+} + F_{j+1}^{-}) - (F_{j-1}^{+} + F_{j}^{-})}{h} = 0
\]

with the numerical flux

\[
F_{j+\frac{1}{2}} = F_{j}^{+} + F_{j+1}^{-}
\]

This has upwind property in the following sense: If all eigenvalues are positive, i.e., all the waves are moving to the right, then

\[
F_{j}^{+} = F_{j}, \quad F_{j+1}^{-} = 0, \quad F_{j+\frac{1}{2}} = F_{j}
\]

The flux is entirely determined from the left state \( U_{j} \) which is physically meaningful. Conversely if all eigenvalues are negative, then

\[
F_{j}^{+} = 0, \quad F_{j+1}^{-} = F_{j+1}, \quad F_{j+\frac{1}{2}} = F_{j+1}
\]

the flux is now entirely determined from the right state \( U_{j+1} \).