# Linear hyperbolic systems

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in/~praveen

March 14, 2013

# Linear hyperbolic system

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \qquad U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}, \qquad F(U) = AU, \qquad A \in \mathbb{R}^{m \times m}$$

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0, \qquad U(x,0) = U^0(x)$$

System $U_t + AU_x = 0$ is said to be *hyperbolic* provided

- $A$ has $m$ real eigenvalues

$$\lambda_1 < \lambda_2 < \ldots < \lambda_m$$

- The eigenvectors form a basis for $\mathbb{R}^m$.

# General solution

Eigenvalues and eigenvectors

$$A r_k = \lambda_k r_k, \qquad r_k \in \mathbb{R}^m, \qquad R = [r_1, r_2, \ldots, r_m] \in \mathbb{R}^{m \times m}$$

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m), \qquad AR = R\Lambda, \qquad A = R\Lambda R^{-1}$$

$$\frac{\partial U}{\partial t} + R\Lambda R^{-1} \frac{\partial U}{\partial x} = 0 \qquad \Longrightarrow \qquad R^{-1}\frac{\partial U}{\partial t} + \Lambda R^{-1} \frac{\partial U}{\partial x} = 0$$

Define the characteristic variables by

$$W = R^{-1} U \qquad \Longrightarrow \qquad \frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = 0$$

These equations become decoupled

$$\frac{\partial W_i}{\partial t} + \lambda_i \frac{\partial W_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, m$$

# General solution

Initial condition for $W$

$$W(x,0) = W^0(x) = R^{-1}U^0(x)$$

Solution is

$$W_i(x,t) = W_i^0(x - \lambda_i t) = [R^{-1}U^0(x - \lambda_i t)]_i$$

Solution $U$ is obtained

$$U(x,t) = RW(x,t) = \sum_{i=1}^{m} W_i(x,t) r_i = \sum_{i=1}^{m} W_i^0(x - \lambda_i t) r_i$$

# Riemann problem

Consider initial condition

$$U^0(x) = \begin{cases} U_l & x < 0 \\ U_r & x > 0 \end{cases}$$

Initial condition for $W$

$$W^0(x) = R^{-1}U^0(x) = \begin{cases} R^{-1}U_l & x < 0 \\ R^{-1}U_r & x > 0 \end{cases} = \begin{cases} W_l & x < 0 \\ W_r & x > 0 \end{cases}$$

Solution for $W_i$, $i = 1, 2, \ldots, m$

$$W_i(x,t) = W_i^0(x - \lambda_i t) = \begin{cases} W_{l,i} & x/t < \lambda_i \\ W_{r,i} & x/t > \lambda_i \end{cases}$$

# Riemann problem

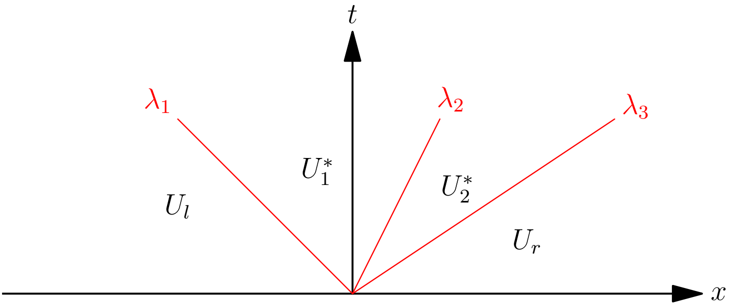
Solution $U$

$$
\begin{aligned}
U(x,t) &= \sum_i W_i^0(x-\lambda_i t) r_i \\
&= \sum_{i: x/t<\lambda_i} W_{l,i} r_i + \sum_{i: x/t>\lambda_i} W_{r,i} r_i
\end{aligned}
$$

**Example**: System of 3 equations $(m = 3)$

$$
U(x,t) = \begin{cases} U_l & x/t < \lambda_1 \\ U_1^* & \lambda_1 < x/t < \lambda_2 \\ U_2^* & \lambda_2 < x/t < \lambda_3 \\ U_r & x/t > \lambda_3 \end{cases}
$$

# Riemann problem




The intermediate states are given by

$$\begin{aligned} U_1^* &= W_{r,1} r_1 + W_{l,2} r_2 + W_{l,3} r_3 \\ U_2^* &= W_{r,1} r_1 + W_{r,2} r_2 + W_{l,3} r_3 \end{aligned}$$

It is easy to check that the jump in the intermediate states is

$$U_m^* - U_{m-1}^* = (W_{r,m} - W_{l,m}) r_m$$

# Riemann problem

with the convention that $U_0^* = U_l$ and $U_n^* = U_r$. Hence the *initial discontinuity breaks into $n$ discontinuity waves which propagate at speeds* $\lambda_i$, $i = 1, \ldots, n$.

**Solution on $x/t = 0$:**
For future use, we compute the solution along $x/t = 0$. It is given by

$$U_R(0) = \sum_{i:\lambda_i > 0} W_{l,i} r_i + \sum_{i:\lambda_i < 0} W_{r,i} r_i$$

and the corresponding flux is

$$F(U_R(0)) = AU_R(0) = \sum_{i:\lambda_i > 0} \lambda_i W_{l,i} r_i + \sum_{i:\lambda_i < 0} \lambda_i W_{r,i} r_i$$

This can be re-written as

$$F(U_R(0)) = \sum_i \lambda_i^+ W_{l,i} r_i + \sum_i \lambda_i^- W_{r,i} r_i$$

# Riemann problem

where we have defined

$$\begin{aligned}
\lambda^+ &= \max(0,\lambda) = \frac{1}{2}(\lambda + |\lambda|) \ge 0 \\
\lambda^- &= \min(0,\lambda) = \frac{1}{2}(\lambda - |\lambda|) \le 0
\end{aligned}$$

The above flux can also be written as

$$F(U_R(0)) = A^+ U_l + A^- U_r, \quad A^\pm = R\Lambda^\pm R^{-1}$$

with $\Lambda^\pm = \mathrm{diag}(\lambda_1^\pm, \ldots, \lambda_n^\pm)$. Another formula is obtained using the second definition of $\lambda^\pm$;

$$\begin{aligned}
F(U_R(0)) &= \frac{1}{2}(F_l + F_r) - \frac{1}{2}\sum_i |\lambda_i|(W_{r,i} - W_{l,i})r_i \\
&= \frac{1}{2}(F_l + F_r) - \frac{1}{2}|A|(U_r - U_l), \qquad |A| = R|\Lambda|R^{-1}
\end{aligned}$$

# Upwind scheme

The system of conservation laws can be transformed to a set of decoupled linear advection equations

$$\frac{\partial W_i}{\partial t} + \lambda_i \frac{\partial W_i}{\partial x} = 0, \quad 1 \leq i \leq n$$

which represent waves moving with velocity $\lambda_i$. We can try to build a scheme for the system of conservation laws by applying the upwind scheme to the above advection equations. For the grid point $j$ we have

$$\frac{W_{i,j}^{n+1} - W_{i,j}^{n}}{\Delta t} + \lambda_i^{+} \frac{W_{i,j}^{n} - W_{i,j-1}^{n}}{\Delta x} + \lambda_i^{-} \frac{W_{i,j+1}^{n} - W_{i,j}^{n}}{\Delta x} = 0$$

or using matrices

$$\frac{W_{j}^{n+1} - W_{j}^{n}}{\Delta t} + \Lambda^{+} \frac{W_{j}^{n} - W_{j-1}^{n}}{\Delta x} + \Lambda^{-} \frac{W_{j+1}^{n} - W_{j}^{n}}{\Delta x} = 0$$

# Upwind scheme

Transforming back to the conserved variables $U$, the above scheme becomes

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + A^+ \frac{U_j^n - U_{j-1}^n}{\Delta x} + A^- \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0$$

We could have obtained this scheme using the CIR splitting technique; separating the Jacobian $A$ into positive and negative parts

$$A = A^+ + A^-, \qquad \frac{\partial U}{\partial t} + A^+ \frac{\partial U}{\partial x} + A^- \frac{\partial U}{\partial x} = 0$$

and using backward and forward differencing for the $A^+$ and $A^-$ terms respectively,

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + A^+ \frac{U_j^n - U_{j-1}^n}{h} + A^- \frac{U_{j+1}^n - U_j^n}{h} = 0$$

we obtain exactly the same scheme.

# Upwind scheme

**Finite difference flux splitting**

Another way to arrive at this scheme is to start with flux splitting. The eigenvalue splitting leads to the flux splitting

$$F = A^+U + A^-U = F^+ + F^-$$

so that conservation law can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial F^+}{\partial x} + \frac{\partial F^-}{\partial x} = 0$$

and we use backward and forward differencing for the $F^+$ and $F^-$ terms respectively.

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_j^+ - F_{j-1}^+}{h} + \frac{F_{j+1}^- - F_j^-}{h} = 0$$

# Upwind scheme

We can write this as a finite volume scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{(F_j^+ + F_{j+1}^-) - (F_{j-1}^+ + F_j^-)}{h} = 0$$

with the numerical flux

$$F_{j+\frac{1}{2}} = F_j^+ + F_{j+1}^-$$

This has **upwind property** in the following sense: If all eigenvalues are positive, i.e., all the waves are moving to the right, then

$$F_j^+ = F_j, \qquad F_{j+1}^- = 0, \qquad F_{j+\frac{1}{2}} = F_j$$

The flux is entirely determined from the left state $U_j$ which is physically meaningful. Conversely if all eigenvalues are negative, then

$$F_j^+ = 0, \qquad F_{j+1}^- = F_{j+1}, \qquad F_{j+\frac{1}{2}} = F_{j+1}$$

the flux is now entirely determined from the right state $U_{j+1}$.