Pre-conditioners for low Mach number flows

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Low Mach flows

In principle, the compressible equations can model flow at all Mach numbers. Mathematically we can show that as $M \to 0$ the solution of the compressible NS equations converge to the solution of the incompressible NS equations (Majda).

But compressible schemes have problems when used in situations where there is a large region of low Mach number. In particular if $M < 0.1$ throughout the flow domain, then two problems are encountered.

- slow convergence to steady solution
- inaccurate computation of pressure

Slow convergence

At small Mach numbers, convective and acoustic eigenvalues are vastly different. For $M \to 0$

$$\text{Condition no.} \qquad \chi = \frac{\text{maximum speed}}{\text{minimum speed}} = \frac{|u| + a}{|u|} = 1 + \frac{1}{M}$$

During computations, acoustic waves travel much faster than convective waves. These acoustic waves are of no interest and they need to go out of the domain in order achieve steady state. This can require lot of time iterations since the time step is restricted by the large acoustic speeds.

The presence of vastly different time scales is also referred to as leading to a **stiffness problem**. Such problems occur in other contexts, e.g., chemical reactions.

Early attempts aimed at equalizing the time scales (eigenvalues) so that steady state is reached quickly. For unsteady problems, it requires using pseudo-timestepping.

Accuracy problem

Asymptotic analysis of NS equations at low Mach numbers

$$\rho(x,t) = \rho_0(t) + M^2 \rho_1(x,t) + \mathcal{O}\left(M^4\right)$$
$$p(x,t) = p_0(t) + M^2 p_1(x,t) + \mathcal{O}\left(M^4\right)$$
$$u(x,t) = u_0(x,t) + M u_1(x,t) + \mathcal{O}\left(M^2\right), \qquad \nabla \cdot u_0 = 0$$

If the conditions in the far-field/inflow are independent of time, then ρ_0, p_0 are constant and equal to far-field values. Then the leading order pressure perturbations are of the order of M^2.

One can do asymptotic analysis of Finite Volume scheme. Then it can be shown that upwind schemes like Roe scheme lead to pressure solution of the form (Guillard & Viozat, 1999)

$$p(x,t) = p_0(t) + M p_1(x,t) + \mathcal{O}\left(M^2\right)$$

The leading order pressure perturbations scale wrongly with the Mach number. This is a spurious solution and leads to large error in pressure. Due to this wrong scaling, upwind schemes add too much dissipation at low Mach numbers.

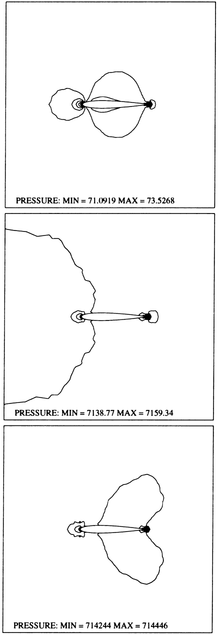


Fig. 2. Isovalues of the pressure, on a 3114 node mesh for $M_\infty = 0.1$ (top), $M_\infty = 0.01$ (middle), $M_\infty = 0.001$ (bottom).

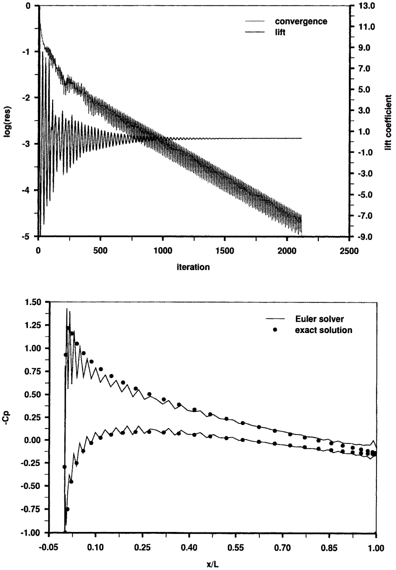
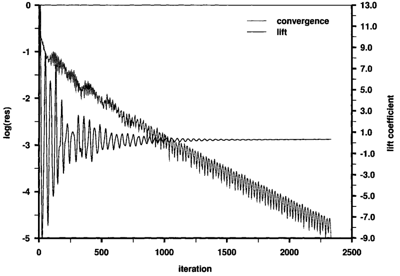


Figure 9.10: Inviscid 2-D flow around symmetric Joukowsky airfoil (10% thickness). Structured grid, $M_\infty = 10^{-2}$, $\alpha = 3°$, central spatial discretisation, explicit multistage time-stepping scheme, no preconditioning. Shown are the convergence history (top), and comparison of the pressure coefficient with the exact potential solution (bottom).



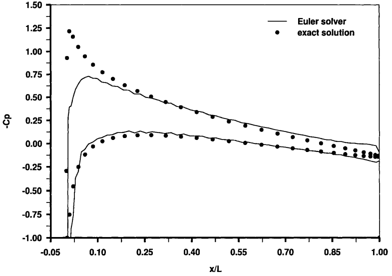


Figure 9.11: Inviscid 2-D flow around symmetric Joukowsky airfoil (10% thickness). Structured grid, $M_\infty = 10^{-2}$, $\alpha = 3°$, 2nd-order Roe's upwind discretisation, explicit multistage time-stepping scheme, no preconditioning. Shown are the convergence history (top), and comparison of the pressure coefficient with the exact potential solution (bottom).

Artificial compressibility method

$$u_t + uu_x + vu_y + p_x = 0, \qquad v_t + uv_x + vv_y + p_y = 0$$

$$u_x + v_y = 0 \qquad \text{There is no time evolution equation for } p \text{ !!!}$$

Chorin (1967) proposed to use an artificial equation of state

$$p = \beta^2 \rho$$

and modify the continuity equation

$$\frac{1}{\beta^2} p_t + u_x + v_y = 0$$

In matrix form this is

$$\begin{bmatrix} \beta^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p \\ u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} p \\ u \\ v \end{bmatrix}_x + \begin{bmatrix} 0 & 0 & 1 \\ 0 & v & 0 \\ 1 & 0 & v \end{bmatrix} \begin{bmatrix} p \\ u \\ v \end{bmatrix}_y = 0$$

These equations can be discretized and integrated in time until steady solution is reached.

Compressible Euler equations in terms of $Q = [p \;\; u \;\; v \;\; s]^\top$

$$Q_t + A(Q)Q_x + B(Q)Q_y = 0$$

$$A(Q) = \begin{bmatrix} u & \rho a^2 & 0 & 0 \\ 1/\rho & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix}, \qquad B(Q) = \begin{bmatrix} v & 0 & \rho a^2 & 0 \\ 0 & v & 0 & 0 \\ 1/\rho & 0 & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}$$

Preconditioned Euler equations

$$P^{-1}Q_t + A(Q)Q_x + B(Q)Q_y = 0$$

P is chosen such that the above system of equations has eigenvalues which are close to one another. The choice of P is motivated by Chorin's artificial compressibility method.

One can work with other set of primitive variables, e.g., (p, u, v, T) or (ρ, u, v, p) which leads to different pre-conditioners.

Choi and Merkle (1984)

They used (ρ, u, v, p) variables but in terms of (p, u, v, s) their preconditioner is

$$P = \begin{bmatrix} \beta^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \beta = \mathcal{O}(M)$$

The eigenvalues of $P(A(Q)n_x + B(Q)n_y)$ where (n_x, n_y) is a unit vector are

$$\lambda_{2,3} = un_x + vn_y$$

$$\lambda_{1,4} = \frac{1}{2}\left[\left(1+\beta^2\right)\lambda_{2,3} \pm \sqrt{\left(1-\beta^2\right)^2 \lambda_{2,3}^2 + 4\beta^2 a^2}\right]$$

If $\beta = \mathcal{O}(M)$ then as $M \to 0$

$$\lambda_{1,4} \approx \frac{1 \pm \sqrt{5}}{2}\lambda_{2,3}$$

The maximum condition number is about 2.6.

Turkel (1987)

Turkel used $Q = [p, u, v, s]$ variables and proposed the preconditioner matrix with two parameters α and β as

$$P^{-1} = \begin{bmatrix} \frac{1}{\rho\beta^2} & 0 & 0 & 0 \\ \frac{\alpha u}{\rho\beta^2} & 1 & 0 & 0 \\ \frac{\alpha v}{\rho\beta^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \beta^2 = \mathcal{O}\left(u^2 + v^2\right)$$

If $\alpha = 0$, we obtain preconditioner of Choi-Merkle. The eigenvalues are

$$\lambda_{2,3} = un_x + vn_y$$

$$\lambda_{1,4} = \frac{1}{2}\left[\left(1 - \alpha + \frac{\beta^2}{a^2}\right)\lambda_{2,3} \pm \sqrt{\left(1 - \alpha + \frac{\beta^2}{a^2}\right)^4 \lambda_{2,3}^2 + 4\left(1 - \frac{\lambda_{2,3}^2}{a^2}\right)\beta^2}\right]$$

If $\alpha = 1$ then the condition number is one and all waves propagate at same speed.

Pre-conditioner for conserved variables

Precondition equations in some primitive variables Q

$$P^{-1}(Q)Q_t + A(Q)Q_x + B(Q)Q_y = 0$$

We want preconditioner for conserved variables U

$$P^{-1}(U)U_t + A(U)U_x + B(U)U_y = 0$$

To find $P(U)$ transform this to Q variables. Let $J = \frac{\partial U}{\partial Q}$

$$P^{-1}(U)\frac{\partial U}{\partial Q}Q_t + A(U)\frac{\partial U}{\partial Q}Q_x + B(U)\frac{\partial U}{\partial Q}Q_y = 0$$

$$J^{-1}P^{-1}(U)JQ_t + \underbrace{J^{-1}A(U)J}_{A(Q)}Q_x + \underbrace{J^{-1}B(U)J}_{B(Q)}Q_y = 0$$

Hence

$$P^{-1}(Q) = J^{-1}P^{-1}(U)J, \qquad P(U) = JP(Q)J^{-1}$$

Scheme for pre-conditioned equations

We have to construct a scheme for pre-conditioned system

$$U_t + PAU_x + PBU_y = 0$$

Now P, A, B are for the conserved variables.

We recall that the upwind scheme is equivalent to applying central scheme to the modified equation

$$U_t + AU_x + BU_y - \frac{\Delta x}{2}(|A|U_x)_x - \frac{\Delta y}{2}(|B|U_y)_y = 0$$

Similarly for the preconditioned system, we can construct a scheme for the modified equation

$$U_t + PAU_x + PBU_y - \frac{\Delta x}{2}(|PA|U_x)_x - \frac{\Delta y}{2}(|PB|U_y)_y = 0$$

To obtain conservation form, multiply by P^{-1}

$$P^{-1}U_t + F_x + G_y - \frac{\Delta x}{2}(P^{-1}|PA|U_x)_x - \frac{\Delta y}{2}(P^{-1}|PB|U_y)_y = 0$$

Pre-conditioned JST scheme

The dissipation in JST scheme must be based on the form of the modified equation for the pre-conditioned system. For the wave speed we must use the spectral radius of $|PA|$, $|PB|$.

$$\lambda_{i+\frac{1}{2},j} = \text{spectral radius of } |PA|_{i+\frac{1}{2},j}$$

The scalar dissipation is

$$d^{(2)}_{i+\frac{1}{2},j} = \varepsilon^{(2)}_{i+\frac{1}{2},j} \lambda_{i+\frac{1}{2},j} P^{-1}_{i+\frac{1}{2},j} (U_{i+1,j} - U_{i,j})$$

$$d^{(4)}_{i+\frac{1}{2},j} = -\varepsilon^{(4)}_{i+\frac{1}{2},j} \lambda_{i+\frac{1}{2},j} P^{-1}_{i+\frac{1}{2},j} (U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j})$$

Numerical flux

$$F_{i+\frac{1}{2},j} = \frac{1}{2}(F_{i,j} + F_{i+1,j}) - d^{(2)}_{i+\frac{1}{2},j} - d^{(4)}_{i+\frac{1}{2},j}$$

Semi-discrete scheme

$$P^{-1}_{i,j} \frac{\mathrm{d}}{\mathrm{d}t} U_{i,j} + \frac{F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j}}{\Delta x} + \frac{G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

Preconditioned Roe scheme

Standard Roe flux

$$F_{j+\frac{1}{2}} = \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}|A(\hat{U})|(U_{j+1} - U_j)$$

$|A| = R|\Lambda|R^{-1}$ is evaluated at the Roe averaged state $\hat{U}$ of U_j, U_{j+1}.

Roe-Turkel scheme (Viozat, 1997)

$$F_{j+\frac{1}{2}} = \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}P(\hat{U})^{-1}|P(\hat{U})A(\hat{U})|(U_{j+1} - U_j)$$

This is used in a finite volume scheme

$$\frac{\mathrm{d}U_j}{\mathrm{d}t} + \frac{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}}{\Delta x} = 0$$

We ignore the pre-conditioner matrix on the time derivative.

Preconditioned Roe scheme

With this preconditioned flux, it can be shown that the pressure solution of the FV scheme is of the form (Guillard & Viozat, 1999)

$$p(x,t) = p_0(t) + M^2 p_1(x,t) + \mathcal{O}\left(M^2\right)$$

The pressure perturbations now scale correctly with the Mach number.

We have a standard finite volume scheme which is time accurate. It can be time marched using any RK scheme, etc.

However, with above scheme, it is found that for stability of the explicit numerical scheme, the time step must be (Birken & Meister, 2005)

$$\Delta t = \mathcal{O}\left(M^2\right)$$

which is too restrictive. Hence an implicit scheme is necessary for computational efficiency.

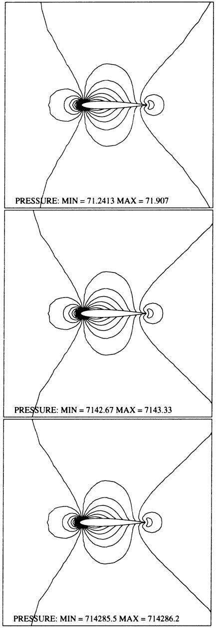


Fig. 5. Isovalues of the pressure, on a 3114 node mesh for $M_\infty = 0.1$ (top), $M_\infty = 0.01$ (middle), $M_\infty = 0.001$ (bottom). Preconditioned dissipation.

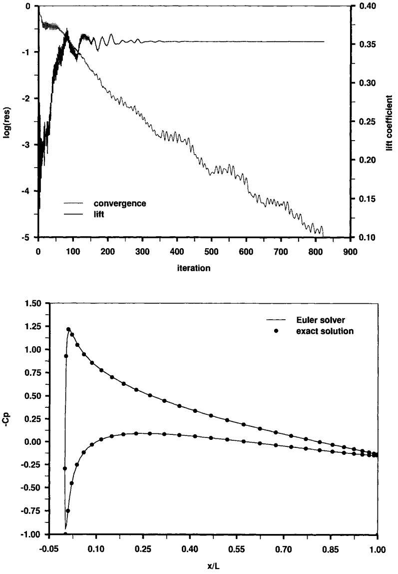


Figure 9.12: Inviscid 2-D flow around symmetric Joukowsky airfoil (10% thickness). Structured grid, $M_\infty = 10^{-2}$, $\alpha = 3°$, central spatial discretisation, explicit multistage time-stepping scheme, Weiss-Smith preconditioning. Shown are the convergence history (top), and comparison of the pressure coefficient with the exact potential solution (bottom).

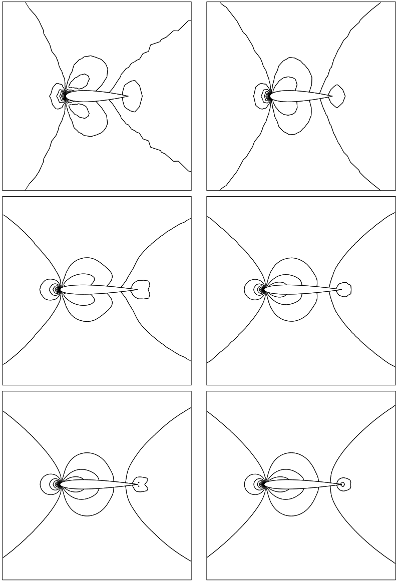


Figure 8: Isovalues of the velocity at second-order accuracy, without limiters, for $M_\infty = 0.1$ with the Roe scheme (left) and the Roe-Turkel scheme (right) on a mesh of 800 nodes (top), 3114 nodes (middle) and 12284 nodes (bottom). Interval between isovalues: 0.05. Min/Max: 0, 2.

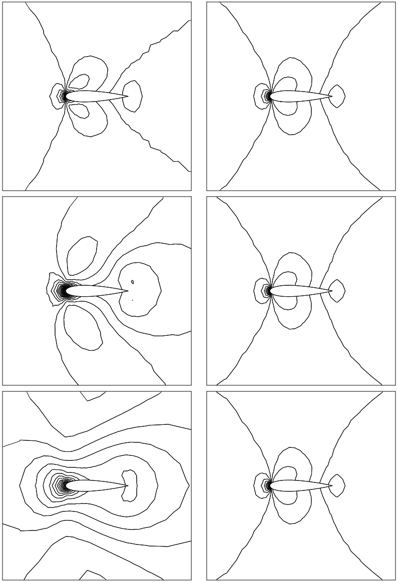


Figure 9: Isovalues of the velocity at second-order accuracy, without limiters, on a coarse mesh (800 nodes) for $M_\infty = 0.1$ (top), $M_\infty = 0.01$ (middle), $M_\infty = 0.001$ (bottom) with the Roe scheme (left) and the Roe-Turkel scheme (right). For $10^{-6} < M_\infty < 0.1$ it has been observed with the Roe-Turkel scheme that the velocity contours change only a little. Interval between isovalues: 0.05. Min/Max: 0, 2.

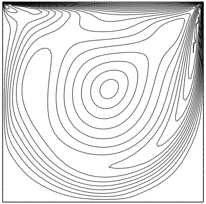


Figure 19: Isovalues of Mach number. Interval between isovalues: 5.10^{-5}. Min/Max: 10^{-5}, 10^{-3}. Second-order accurate Roe-Turkel scheme, no limiters, $Re = 1000$. Mesh 81×81 refined near the upper wall (7586 nodes).

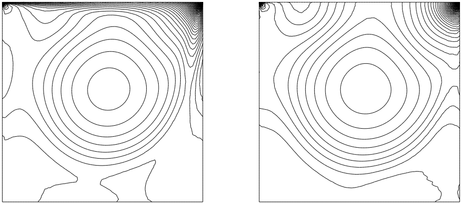


Figure 20: Isovalues of the fluctuations ($x_{\text{represented}} = \frac{x - x_{min}}{x_{max} - x_{min}}$) of density (left) and of pressure (right). $\rho_{min} = 0.99930016$, $\rho_{max} = 0.99930925$. $p_{min} = 714283$, $p_{max} = 714290$. Second-order accurate Roe-Turkel scheme, no limiters, $Re = 1000$, $M{=}10^{-3}$. Interval between isovalues: 0.002. Min/Max: 0.11, 0.14. Mesh 81×81 refined near the upper wall

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