Compressible Navier-Stokes equations in two dimensions

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March 26, 2013

Compressible NS equations

$$U_t + F_x + G_y = P_x + Q_y$$

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \qquad F = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ (E+p)u \end{bmatrix}, \qquad G = \begin{bmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ (E+p)v \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{xx}u + \tau_{xy}v - q_x \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{yx}u + \tau_{yy}v - q_y \end{bmatrix}$$

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right]$$

Newton's constitutive law

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Compressible NS equations

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Fourier law for heat conduction

$$q_x = -\kappa \frac{\partial T}{\partial x}, \qquad q_y = -\kappa \frac{\partial T}{\partial y}$$

We also need a model for viscosity coefficient. For gases under normal conditions, Sutherland law is commonly used.

$$\mu(T) = \mu_r \left(\frac{T}{T_r} \right)^{\frac{3}{2}} \frac{T_r + T_0}{T + T_0}$$

Euler equations

Conservation form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

Flux jacobians

$$A(U) = \frac{\partial F}{\partial U}, \qquad B(U) = \frac{\partial G}{\partial U}$$

Quasi-linear form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + B(U)\frac{\partial U}{\partial y} = 0$$

Hyperbolic property is the existence of wave-like solutions. So try a wave-like solution

$$U(x, y, t) = \hat{U}\mathrm{e}^{\mathrm{i}(xk_x + yk_y - \lambda t)}$$

Here k_x, k_y, λ are real numbers. Then

$$(-\lambda I + Ak_x + Bk_y)\hat{U} = 0$$

Euler equations

For non-trivial solutions to exist

$$\det(-\lambda I + Ak_x + Bk_y) = 0$$

This means that λ is an eigenvalue of the matrix

$$Ak_x + Bk_y$$

It is clear that $\hat{U}$ is the corresponding eigenvector. We want all solutions to be wave-like. If the set of eigenvectors span the space of solutions, then any solution can be written as a linear combination of eigenvectors. This motivates the following definition.

Hyperbolicity

The quasi-linear system is hyperbolic if for every admissible state U and for every vector $(k_x, k_y) \in \mathbb{R}^2$, the matrix $A(U)k_x + B(U)k_y$

- has real eigenvalues
- a complete set of eigenvectors

Euler equations

For the Euler equations, one can show that the eigenvalues are

$$uk_x + vk_y - a|k|, \qquad uk_x + vk_y, \qquad uk_x + vk_y, \qquad uk_x + vk_y + a|k|$$

where

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}, \qquad |k| = (k_x^2 + k_y^2)^{\frac{1}{2}}$$

The flux Jacobians can be diagonalized by the matrix of eigenvectors

$$A = R_A \Lambda_A R_A^{-1}, \qquad B = R_B \Lambda_B R_B^{-1}$$

$$\Lambda_A = \mathrm{diag}(u-a, u, u, u+a), \qquad \Lambda_B = \mathrm{diag}(v-a, v, v, v+a)$$

Assume that $R_A = R_B = R$ which is not true for Euler equations. In this case we say that the flux Jacobians A, B can be simultaneously diagonalized by the matrix R.

$$\frac{\partial U}{\partial t} + R\Lambda_A R^{-1}\frac{\partial U}{\partial x} + R\Lambda_B R^{-1}\frac{\partial U}{\partial y} = 0$$

Euler equations

Defining a new variable

$$\mathrm{d}W = R^{-1}\mathrm{d}U$$

we get

$$\frac{\partial W}{\partial t} + \Lambda_A \frac{\partial W}{\partial x} + \Lambda_B \frac{\partial W}{\partial y} = 0$$

This is a set of decoupled convections equations. It contains four waves travelling in four directions. Then it would be straightforward to construct upwind schemes.

In the case of Euler equations, we cannot simultaneously diagonalize the Jacobians. This means that the solutions contain waves moving in all directions (atleast in subsonic regions).