A posteriori error estimation for elliptic problems

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Partial differential equation

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \forall\ v \in H_0^1(\Omega) \tag{1}$$

where

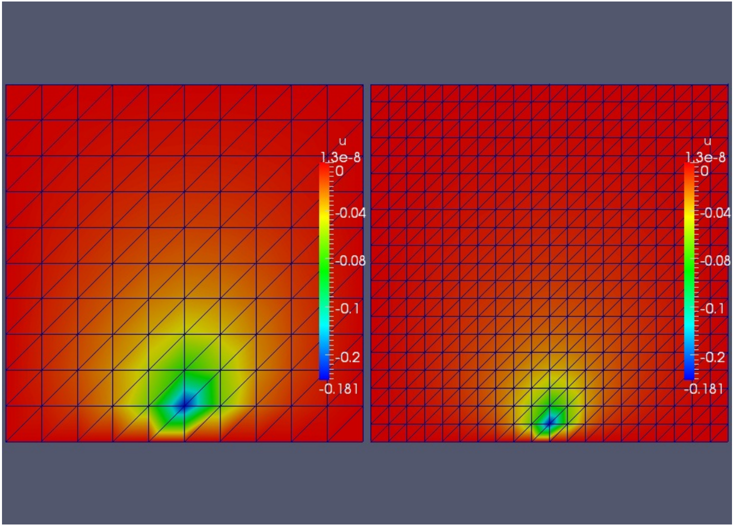
$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad \ell(v) = \int_\Omega f v$$

Galerkin method

Find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\ v_h \in V_h \tag{2}$$

Example: $u = \frac{1}{\pi}\tan^{-1}(y/x)$



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A priori error estimate

If we take

$$V_h = X_h^1 = \{v \in \mathcal{C}^0(\bar{\Omega}) : v|_K \in \mathbb{P}_1\}$$

and if $u \in H^2(\Omega)$ then we have the error estimate

$$|u - u_h|_{1,\Omega} \le C \left(\sum_{K \in \mathcal{T}_h} h_K^2 |u|_{2,K}^2 \right)^{\frac{1}{2}} \le Ch|u|_{2,\Omega}$$

This is not precisely computable since it depends on the exact solution u.

- We can estimate the element seminorms $|u|_{2,K}$ by using the numerical solution u_h and some finite difference approximation, etc.
- Then $h_K|u_h|_{2,K}$ indicates the error from element K.
- Identify elements with largest error indicator and refine them.
- Or equi-distribute the error: given some error tolerance ϵ, let $N_K =$ number of elements. Then refine all elements for which

$$h_K|u_h|_{2,K} > \frac{\epsilon}{CN_K}$$

L^2 a posteriori error estimate

Let Ω be a convex polygonal domain and let u and u_h be the solutions to (1) and (2), respectively. Then

$$\|u - u_h\|_{0,\Omega} \leq C \left(\sum_{K \in \mathcal{T}_h} h_K^2 \eta_K^2 \right)^{\frac{1}{2}}$$

where

$$\eta_K = h_K \|f + \Delta u_h\|_{0,K} + \frac{1}{2} h_K^{\frac{1}{2}} \|[\![n \cdot \nabla u_h]\!]\|_{0,\partial K \setminus \Gamma} \tag{3}$$

where $[\![n \cdot \nabla u_h]\!]$ denotes the jump across ∂K in the normal derivative $n \cdot \nabla u_h$.

Proof: We use the duality argument which was also used to prove the L^2 a-priori error estimate. Let $e_h = u_h - u$ be the error and let φ be the solution of the adjoint problem, $-\Delta\varphi = e_h$ in Ω and $\varphi = 0$ on Γ:

$$\text{find} \quad \varphi \in H_0^1(\Omega) \quad \text{such that} \quad a(v, \varphi) = \int_\Omega e_h v \qquad \forall\ v \in H_0^1(\Omega)$$

Note that $\varphi \in H^2(\Omega)$ and $\|\varphi\|_{2,\Omega} \leq C \|e_h\|_{0,\Omega}$.

Taking $v = e_h$ in adjoint problem, the error can be written as

$$\begin{aligned}
\|e_h\|_{0,\Omega}^2 &= a(e_h, \varphi) = a(u_h - u, \varphi) = a(u_h, \varphi) - (f, \varphi) \\
&= \sum_K [(\nabla u_h, \nabla \varphi)_K - (f, \varphi)_K] \\
&= \sum_K [(-\Delta u_h - f, \varphi)_K + (n \cdot \nabla u_h, \varphi)_{\partial K}] \\
&= \sum_K \left[(-\Delta u_h - f, \varphi)_K + \frac{1}{2}(\llbracket n \cdot \nabla u_h \rrbracket, \varphi)_{\partial K \setminus \Gamma}\right]
\end{aligned}$$

where the factor of $\frac{1}{2}$ appears in the second term since each interior edge belongs to two elements. Since

$$a(e_h, v_h) = a(u_h - u, v_h) = 0 \qquad \forall\ v_h \in V_h$$

we can replace φ with $\varphi - I_h \varphi$ to obtain

$$\begin{aligned}
&\|e_h\|_{0,\Omega}^2 = |a(e_h, \varphi - I_h \varphi)| \\
\leq &\sum_K \left[\|\Delta u_h + f\|_K \|\varphi - I_h \varphi\|_K + \frac{1}{2} \|\llbracket n \cdot \nabla u_h \rrbracket\|_{\partial K \setminus \Gamma} \|\varphi - I_h \varphi\|_{\partial K \setminus \Gamma}\right]
\end{aligned}$$

We have the interpolation error estimates

$$\|\varphi - I_h\varphi\|_K \leq Ch_K^2|\varphi|_{2,K} \qquad \|\varphi - I_h\varphi\|_{\partial K} \leq Ch_K^{\frac{3}{2}}|\varphi|_{2,K} \tag{4}$$

which leads to the desired result since

$$\begin{aligned}
\|e_h\|_{0,\Omega}^2 &\leq C\sum_K \left[h_K^2 \|\Delta u_h + f\|_K |\varphi|_{2,K} + \frac{1}{2}h_K^{\frac{3}{2}} \|[\![n\cdot\nabla u_h]\!]\|_{\partial K\setminus\Gamma} |\varphi|_{2,K}\right] \\
&= C\sum_K h_K\eta_K|\varphi|_{2,K} \\
&\leq C\left(\sum_K h_K^2\eta_K^2\right)^{\frac{1}{2}}\left(\sum_K |\varphi|_{2,K}^2\right)^{\frac{1}{2}} \\
&= C\left(\sum_K h_K^2\eta_K^2\right)^{\frac{1}{2}}|\varphi|_{2,\Omega} \leq C\left(\sum_K h_K^2\eta_K^2\right)^{\frac{1}{2}}\|\varphi\|_{2,\Omega} \\
&\leq C\left(\sum_K h_K^2\eta_K^2\right)^{\frac{1}{2}}\|e_h\|_{0,\Omega}
\end{aligned}$$

To show the second inequality in (4) we will show that for any $w \in H^1(K)$

$$\|w\|_{\partial K} \le C\left(h_K^{-\frac{1}{2}}\|w\|_K + h_K^{\frac{1}{2}}|w|_{1,K}\right) \tag{5}$$

which then leads to

$$\begin{aligned}\|\varphi - I_h\varphi\|_{\partial K} &\le C\left(h_K^{-\frac{1}{2}}\|\varphi - I_h\varphi\|_K + h_K^{\frac{1}{2}}|\varphi - I_h\varphi|_{1,K}\right)\\ &\le C\left(h_K^{-\frac{1}{2}}h_K^2|\varphi|_{2,K} + h_K^{\frac{1}{2}}h_K|\varphi|_{2,K}\right)\\ &\le Ch_K^{\frac{3}{2}}|\varphi|_{2,K}\end{aligned}$$

To prove (5) we map any element K to the reference element $\hat{K}$ by the affine transformaton $x = F_K(\hat{x}) = B_K\hat{x} + b_K$; then

$$\|w\|_{\partial K} \le h_K^{\frac{d-1}{2}}\|\hat{w}\|_{\partial\hat{K}}, \qquad d = 2, 3 \tag{6}$$

From the trace theorem, we have continuity of trace operator which means

$$\|\hat{w}\|_{\partial\hat{K}} \le C\|\hat{w}\|_{1,\hat{K}} \le C(\|\hat{w}\|_{\hat{K}} + |\hat{w}|_{1,\hat{K}}) \tag{7}$$

Now we convert the norms to the original element K using a previous result

$$|\hat{w}|_{m,\hat{K}} \leq C \left\|B_K\right\|^m |\det B_K|^{-\frac{1}{2}} |w|_{m,K} \qquad \forall\, w \in H^m(K)$$

Note that

$$|\det B_K| = 2|K| > C\rho_K^d \geq C\kappa^d h_K^d$$

where we use the fact that we have a regular triangulation. Moreover recall that

$$\|B_K\| \leq \frac{h_K}{\hat{\rho}}$$

Hence for $m = 0, 1$ we get

$$\|\hat{w}\|_{\hat{K}} \leq Ch_K^{-\frac{d}{2}} \|w\|_K \quad \text{and} \qquad |\hat{w}|_{1,\hat{K}} \leq Ch_K^{1-\frac{d}{2}} |w|_{1,K} \tag{8}$$

Combining (6), (7), (8) we obtain

$$\|w\|_{\partial K} \leq Ch_K^{\frac{d-1}{2}} (\|\hat{w}\|_{\hat{K}} + |\hat{w}|_{1,\hat{K}}) \leq C \left(h_K^{-\frac{1}{2}} \|w\|_K + h_K^{\frac{1}{2}} |w|_{1,K} \right)$$

Clement interpolation

The standard interpolation requires function to be in H^2. For functions which are only in H^1, we have to use other types of interpolation operations. Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω. Given a node x_j, let

$$\omega_j = \bigcup \{K \in \mathcal{T}_h : x_j \in K\}, \qquad \tilde{\omega}_K = \bigcup \{\omega_j : x_j \in K\}$$

The number of triangles that belong to ω_j and $\tilde{\omega}_K$ is bounded.

Since $\mathcal{T}_h$ is shape-regular, the area of $\tilde{\omega}_K$ can be bounded as

$$|\tilde{\omega}_K| \le c(\kappa) h_K^2$$

For $v \in H^1$ we cannot evaluate the functions pointwise since they may not be continuous. We will use a local averaging procedure to associate a function value to each node of the mesh.

Let X_h^1 denote the space of piecewise $\mathbb{P}_1$ functions, and let $\{\varphi_j\}_j$ be the standard hat basis functions.

Clement interpolation

Clement interpolation

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω. Then there exists a linear mapping $C_h : H^1(\Omega) \to X_h^1$ such that for all $K \in \mathcal{T}_h$

$$\begin{aligned} \|v - C_h v\|_{m,K} &\leq Ch_K^{1-m} |v|_{1,\tilde{\omega}_K} \qquad m = 0, 1 \\ \|v - C_h v\|_{0,\partial K} &\leq Ch_K^{\frac{1}{2}} |v|_{1,\tilde{\omega}_K} \end{aligned}$$

Proof: Given a nodal point x_j, let

$$Q_j : L^2(\omega_j) \to \mathbb{P}_0$$

be the L^2 projection onto the constant functions, i.e.,

$$\int_{\omega_j} Q_j v = \int_{\omega_j} v \quad \Longrightarrow \quad Q_j v = \frac{1}{|\omega_j|} \int_{\omega_j} v$$

It follows by applying Bramble-Hilbert lemma that

$$\|v - Q_j v\|_{0,\omega_j} \leq Ch_j |v|_{1,\omega_j} \tag{9}$$

Clement interpolation

where h_j is the diameter of ω_j. In order to cope with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$ we can modify the operator and set

$$\tilde{Q}_j v = \begin{cases} 0 & \text{if } x_j \in \Gamma_D \\ Q_j v & \text{otherwise} \end{cases}$$

Using a proof similar to Poincare-Friedrichs inequality we obtain

$$\left\| v - \tilde{Q}_j v \right\|_{0,\omega_j} = \|v\|_{0,\omega_j} \le C h_j |v|_{1,\omega_j} \qquad \text{if } x_j \in \Gamma_D$$

Next we define the Clement interpolation as

$$C_h v = \sum_j (\tilde{Q}_j v)\varphi_j \in X_h^1$$

The shape functions $\{\varphi_j\}_j$ form a partition of unity. Hence

$$v - C_h v = \sum_j v\varphi_j - \sum_j (\tilde{Q}_j v)\varphi_j = \sum_j (v - (\tilde{Q}_j v))\varphi_j$$

Clement interpolation

and hence

$$\begin{aligned}\|v - C_h v\|_{0,K} &\leq \sum_j \left\| v - (\tilde{Q}_j v))\varphi_j \right\|_{0,K} \leq \sum_j \left\| v - \tilde{Q}_j v \right\|_{0,\omega_j} \\ &\leq C \sum_j h_j |v|_{1,\omega_j} \leq C h_K |v|_{1,\tilde{\omega}_K}\end{aligned}$$

The case of $m = 1$ is left for further studies.

Error norm on ∂K: We make use of inequality (5).

$$\begin{aligned}\|v - C_h v\|_{0,\partial K} &\leq C \left(h_K^{-\frac{1}{2}} \|v - C_h v\|_{0,K} + h_K^{\frac{1}{2}} |v - C_h v|_{1,K} \right) \\ &\leq C \left(h_K^{-\frac{1}{2}} h_K |v|_{1,\tilde{\omega}_K} + h_K^{\frac{1}{2}} |v|_{1,\tilde{\omega}_K} \right) \\ &= C h_K^{\frac{1}{2}} |v|_{1,\tilde{\omega}_K}\end{aligned}$$

□

We next derive an error estimate for the Galerkin solution assuming that the true solution $u \in H_0^1(\Omega)$ only.

H^1 semi-norm error estimate

Let $\mathcal{T}_h$ be a shape-regular triangulation. Then the Galerkin solution satisfies the a-posteriori error estimate

$$|u - u_h|_{1,\Omega} \leq C \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{\frac{1}{2}}$$

where η_K is given by equation (3).

Proof: We start by using a duality argument to compute the semi-norm of the error

$$|u - u_h|_{1,\Omega} = \sup_{w \in H_0^1(\Omega)} \frac{(\nabla(u - u_h), \nabla w)_\Omega}{|w|_{1,\Omega}} = \sup_{w \in H_0^1(\Omega)} \frac{L(w)}{|w|_{1,\Omega}}$$

Then

$$
\begin{aligned}
L(w) &= (\nabla(u-u_h), \nabla w)_\Omega = (f, w)_\Omega - (\nabla u_h, \nabla w)_\Omega \\
&= \sum_K [(f, w)_K - (\nabla u_h, \nabla w)_K] \\
&= \sum_K \big[(f, w)_K - (-\Delta u_h, w)_K - (n \cdot \nabla u_h, w)_{\partial K \setminus \Gamma}\big] \\
&= \sum_K \left[(\Delta u_h + f, w)_K - \frac{1}{2}(\llbracket n \cdot \nabla u_h \rrbracket, w)_{\partial K \setminus \Gamma}\right]
\end{aligned}
$$

The Galerkin solution satisfies

$$
a(u-u_h, w_h) = (\nabla(u-u_h), \nabla w_h)_\Omega = 0 \qquad \forall\ w_h \in V_h
$$

Let us take $w_h = C_h w$ so that

$$\begin{aligned}
L(w) &= L(w - C_h w) \\
&= \sum_K \left[(\Delta u_h + f, w - C_h w)_K - \frac{1}{2}(\llbracket n \cdot \nabla u_h \rrbracket, w - C_h w)_{\partial K \setminus \Gamma} \right] \\
&\leq \sum_K \left[\|\Delta u_h + f\|_K \|w - C_h w\|_K + \frac{1}{2} \|\llbracket n \cdot \nabla u_h \rrbracket\|_{\partial K \setminus \Gamma} \|w - C_h w\|_{\partial K} \right] \\
&\leq C \sum_K \left[h_K \|\Delta u_h + f\|_K |w|_{1,\tilde{\omega}_K} + \frac{1}{2} h_K^{\frac{1}{2}} \|\llbracket n \cdot \nabla u_h \rrbracket\|_{\partial K \setminus \Gamma} |w|_{1,\tilde{\omega}_K} \right] \\
&= C \sum_K \left[h_K \|\Delta u_h + f\|_K + \frac{1}{2} h_K^{\frac{1}{2}} \|\llbracket n \cdot \nabla u_h \rrbracket\|_{\partial K \setminus \Gamma} \right] |w|_{1,\tilde{\omega}_K} \\
&= C \sum_K \eta_K |w|_{1,\tilde{\omega}_K} \\
&\leq C \left(\sum_K \eta_K^2 \right)^{\frac{1}{2}} \left(\sum_K |w|_{1,\tilde{\omega}_K}^2 \right)^{\frac{1}{2}} \leq C \left(\sum_K \eta_K^2 \right)^{\frac{1}{2}} |w|_{1,\Omega}
\end{aligned}$$

which yields the desired result. □

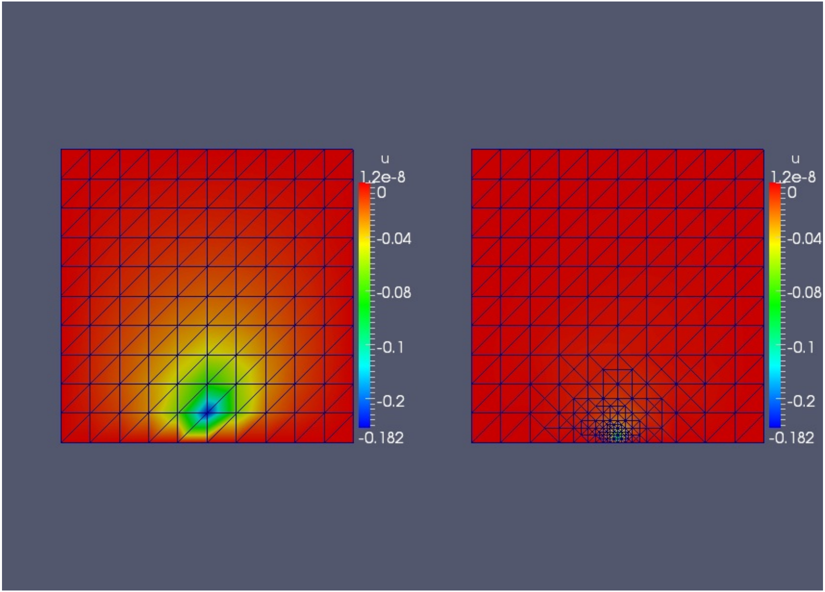
Galerkin method with $\mathbb{P}_1$ functions

In this case $\Delta u_h \equiv 0$ so that

$$\eta_K = h_K \left\| f \right\|_K + \frac{1}{2} h_K^{\frac{1}{2}} \left\| [\![n \cdot \nabla u_h]\!] \right\|_{\partial K}$$

- Compute η_K for each $K \in \mathcal{T}_h$
- Sort the values $\{\eta_K\}$ in decreasing order.
- Select some fraction of elements with highest value of η_K and flag them for division
- Divide each element (into two or four); divide neighbouring elements to avoid hanging nodes.
- If dividing K into two elements, select largest edge of K and divide it.

Example: $u = \frac{1}{\pi} \tan^{-1}(y/x)$



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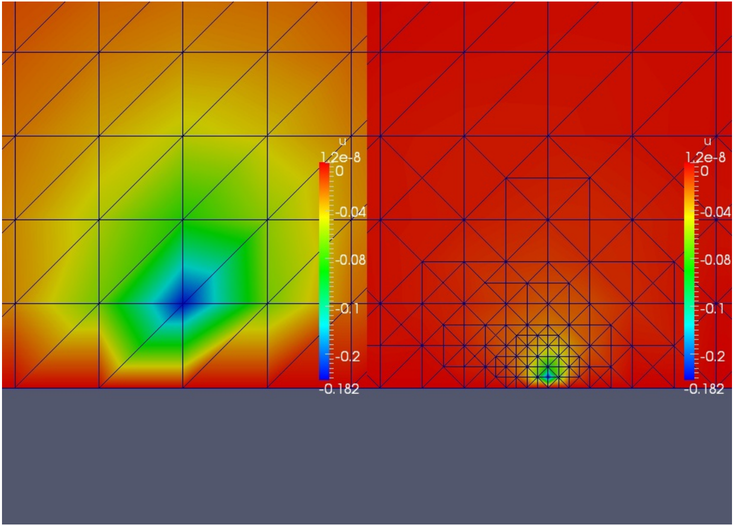

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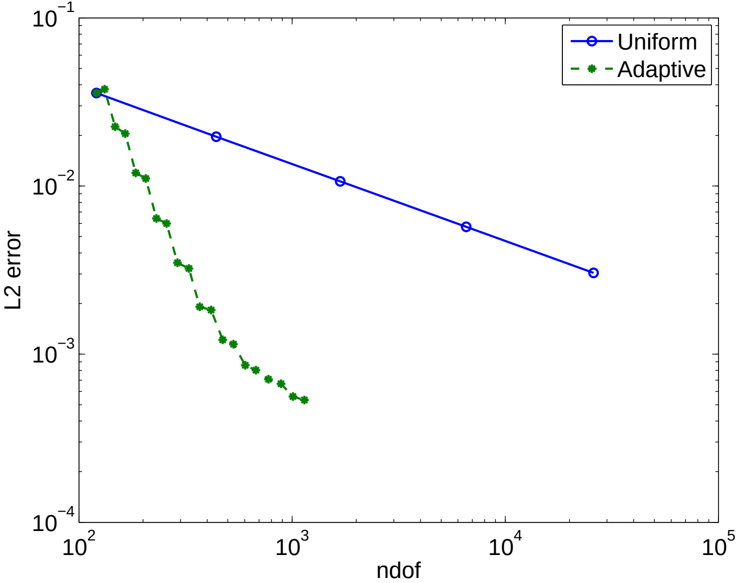
Example: $u = \frac{1}{\pi}\tan^{-1}(y/x)$



Uniform refinement shows convergence rate ≈ 1.0