Finite volume method for boundary value problem

Discontinuous coefficients and high Peclet number

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in/~praveen

January 21, 2013

Discontinuous coefficients

$$-\frac{\mathrm{d}}{\mathrm{d}x}a(x)\frac{\mathrm{d}u}{\mathrm{d}x} = q, \quad x \in \Omega = (0,1)$$

Finite difference scheme

$$-\frac{1}{2}[D_x^+(aD_x^-) + D_x^-(aD_x^+)]u_j = q_j, \quad j = 1,2,\ldots,n-1$$

or, for $j = 1,2,\ldots,n-1$

$$\frac{1}{2h^2}[-(a_{j-1}+a_j)u_{j-1} + (a_{j-1}+2a_j+a_{j+1})u_j - (a_j+a_{j+1})u_{j+1}] = q_j$$

Boundary condition

$$u(0) = 0, \qquad u(1) = 1$$

Discontinuous coefficient

$$a(x) = \begin{cases} \epsilon & 0 \le x \le x^* \\ 1 & x^* < x \le 1 \end{cases}$$

Discontinuous coefficients

Jump condition: au' is continuous across the discontinuity

$$\epsilon \lim_{x \uparrow x^*} u'(x) = \lim_{x \downarrow x^*} u'(x)$$

Exact solution ($q = 0$)

$$u(x) = \begin{cases} \alpha x & 0 \leq x < x^* \\ \epsilon \alpha x + 1 - \epsilon \alpha & x^* \leq x \leq 1 \end{cases}$$

where

$$\alpha = \frac{1}{(1 - \epsilon)x^* + \epsilon}$$

Discontinuous coefficients

Finite difference solution: Consider the uniform grid $x_j = jh$, $j = 0, 1, \ldots, n$ with $h = \frac{1}{n}$. Assume $x_k < x^* \leq x_{k+1}$. Postulate FD solution as ($u_0 = 0$ and $u_n = 1$ is satisfied)

$$u_j = \begin{cases} \alpha_h j & 0 \leq j \leq k \\ \beta_h j - \beta_h n + 1 & k+1 \leq j \leq n \end{cases}$$

where (apply FD equation at $j = k$ and $j = k+1$)

$$\alpha_h = \left[\epsilon \frac{1-\epsilon}{1+\epsilon} + \epsilon(n-k) + k \right]^{-1}, \quad \beta_h = \epsilon \alpha_h$$

At $x = x_k = hk$, FD solution is

$$u_k = \frac{x_k}{\epsilon h \frac{1-\epsilon}{1+\epsilon} + (1-\epsilon) x_k + \epsilon}$$

Discontinuous coefficients

Case $x^* = x_{k+1}$: Exact solution is

$$u(x_k) = \frac{x_k}{(1-\epsilon)x_{k+1} + \epsilon}$$

and error in FD solution

$$u_k - u(x_k) = \mathcal{O}\left(\epsilon \frac{1-\epsilon}{1+\epsilon} h\right)$$

Case $x^* = x_k + h/2$: Exact solution is

$$u(x_k) = \frac{x_k}{(1-\epsilon)x_k + \epsilon + \frac{1}{2}h(1-\epsilon)}$$

and error in FD solution

$$u_k - u(x_k) = \mathcal{O}\left(\frac{(1-\epsilon)^2}{\epsilon(1+\epsilon)} h\right)$$

Discontinuous coefficients

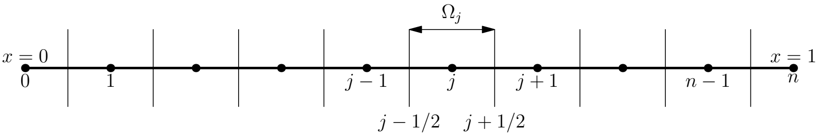
$a(x)$	error
constant	0
continuous	$\mathcal{O}\left(h^2\right)$
discontinuous	$\mathcal{O}\left(h\right)$

FD scheme loses its optimal accuracy when coefficients are discontinuous.

Finite volume scheme

Non-overlapping *finite volumes* (vertex-based or vertex-centered)

$$\Omega_j = (x_j - \frac{h}{2}, x_j + \frac{h}{2}), \quad j = 1, 2, \ldots, n-1$$



Integrate over each cell Ω_j

$$\begin{aligned} -\int_{\Omega_j} (au')' \mathrm{d}x &= \int_{\Omega_j} q \mathrm{d}x \\ -[(au')_{j+\frac{1}{2}} - (au')_{j-\frac{1}{2}}] &\approx hq_j \end{aligned}$$

Finite volume scheme

$$\begin{aligned} u_{j+1} - u_j &= \int_{x_j}^{x_{j+1}} u' \mathrm{d}x \\ &= \int_{x_j}^{x_{j+1}} \frac{1}{a}(au')\mathrm{d}x \\ &\approx (au')_{j+\frac{1}{2}} \int_{x_j}^{x_{j+1}} \frac{1}{a}\mathrm{d}x \\ &\approx (au')_{j+\frac{1}{2}} \frac{h}{a_{j+\frac{1}{2}}} \end{aligned}$$

where

$$\frac{1}{a_{j+\frac{1}{2}}} = \frac{1}{2}\left(\frac{1}{a_j} + \frac{1}{a_{j+1}}\right) \approx \frac{1}{h}\int_{x_j}^{x_{j+1}} \frac{1}{a}\mathrm{d}x$$

Flux at $x = x_{j+\frac{1}{2}}$

$$(au')_{j+\frac{1}{2}} = a_{j+\frac{1}{2}} \frac{u_{j+1} - u_j}{h}$$

Finite volume scheme

Finite volume scheme

$$-\left[a_{j+\frac{1}{2}}\frac{u_{j+1}-u_j}{h} - a_{j-\frac{1}{2}}\frac{u_j-u_{j-1}}{h}\right] = hq_j, \quad j = 1, 2, \ldots, n-1$$

with

$$u_0 = 0, \qquad u_n = 1$$

Discontinuous coefficient, $x^* = x_k + h/2$: Discontinuity coinciding with cell face $x_{k+\frac{1}{2}}$

$$a_{j+\frac{1}{2}} = \begin{cases} \epsilon, & 1 \le j < k \\ \frac{2\epsilon}{1+\epsilon}, & j = k \\ 1, & k < j \le n-1 \end{cases}$$

Postulating FVM solution as before, we get

$$\alpha_h = \frac{h}{(1-\epsilon)x^* + \epsilon}, \qquad \beta_h = \epsilon\alpha_h$$

Finite volume scheme

The FVM solution is exact !!! In general, the error of FV solution is $\mathcal{O}\left(h^2\right)$.

<u>Discontinuous coefficient, $x^* = x_k$</u>: Discontinuity inside a cell $a_{j+\frac{1}{2}}$ is again given as in previous case and the error in u_k is $\mathcal{O}\left(h^2\right)$ (show this).

When dealing with problems with discontinuous coefficients and/or solutions, finite volume methods are more accurate than finite difference methods. FVM has better mathematical basis since it is based on weak formulation.

If mesh Peclet number is large

$\mu > 0$ and $U > 0$ are constants

$$-\mu u'' + U u' = 0, \quad x \in (0, L)$$

Non-dimensionalize: $x \to x/L$

$$-\frac{1}{\mathrm{Pe}} u'' + u' = 0, \quad x \in (0, 1)$$

Peclet number:

$$\mathrm{Pe} = \frac{UL}{\mu} = \frac{\text{convection}}{\text{diffusion}}$$

General solution

$$u(x) = A + B\mathrm{e}^{x\mathrm{Pe}}$$

Boundary conditions

$$u(0) = a, \qquad u(1) = b$$

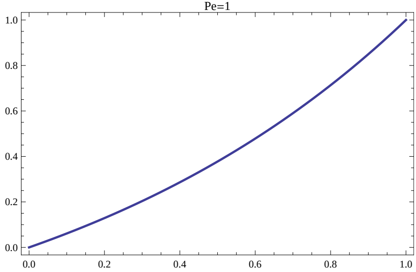
If mesh Peclet number is large

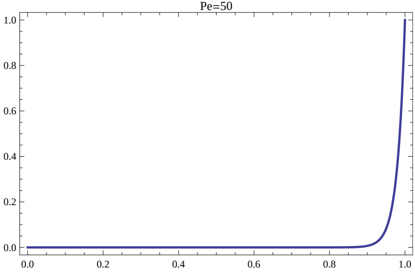
Solution

$$u(x) = a + (b-a)\frac{\mathrm{e}^{(x-1)\mathrm{Pe}} - \mathrm{e}^{-\mathrm{Pe}}}{1 - \mathrm{e}^{-\mathrm{Pe}}}$$

Example

$$a = 0, \qquad b = 1, \qquad u(x) = \frac{\mathrm{e}^{(x-1)\mathrm{Pe}} - \mathrm{e}^{-\mathrm{Pe}}}{1 - \mathrm{e}^{-\mathrm{Pe}}}$$



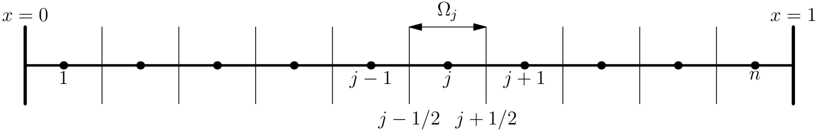


Boundary layer is present near $x = 1$ for large value of Pe

Boundary layer thickness $= \mathcal{O}\left(\frac{1}{\mathrm{Pe}}\right)$ for $\mathrm{Pe} \gg 1$

If mesh Peclet number is large

Finite volume scheme (cell-centered)



Integrating over finite volume Ω_j

$$-\frac{1}{\text{Pe}}[u'_{j+\frac{1}{2}} - u'_{j-\frac{1}{2}}] + u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} = 0$$

Fluxes: for $j = 1, 2, \ldots, n-1$

$$u_{j+\frac{1}{2}} = \frac{1}{2}(u_j + u_{j+1}), \qquad u'_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}$$

Boundary fluxes

$$u_{\frac{1}{2}} = a, \quad u_{n+\frac{1}{2}} = b, \quad u'_{\frac{1}{2}} = \frac{u_1 - a}{h/2}, \quad u'_{n+\frac{1}{2}} = \frac{b - u_n}{h/2}$$

If mesh Peclet number is large

System of finite volume equations

$$\begin{aligned}
(1+\frac{6}{P})u_1 + (1-\frac{2}{P})u_2 &= (2+\frac{4}{P})a \\
-(1+\frac{2}{P})u_{j-1} + \frac{4}{P}u_j + (1-\frac{2}{P})u_{j+1} &= 0, \quad j = 2, 3, \ldots, n-1 \\
-(1+\frac{2}{P})u_{n-1} + (-1+\frac{6}{P})u_n &= (-2+\frac{4}{P})b
\end{aligned}$$

with the mesh Peclet number $\boxed{P = h\mathrm{Pe}}$.

<u>Case $P = 0$</u>: No convection, only diffusion

$$u_j = A + Bj$$

Determine A, B from first and last equations.

If mesh Peclet number is large

Case $P = 2$:

$$u_j = a, \quad j = 1, 2, \ldots, n$$

Case $P \neq 0, 2$: Postulate solution of the form

$$u_j = A + Bz^j \quad \Longrightarrow \quad z = \frac{2+P}{2-P}$$

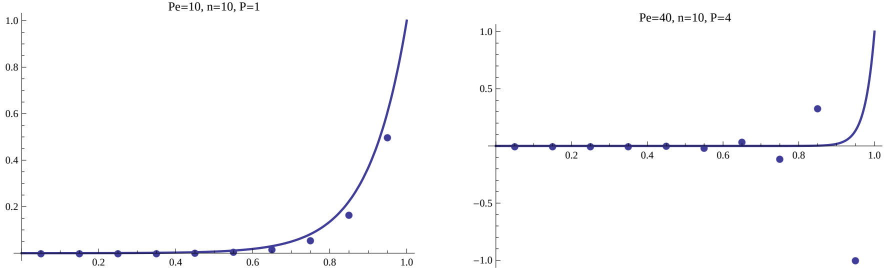
Determine A, B from first and last equations.

$$u_j = a + \frac{b-a}{z^n - 1}\left(-1 + \frac{2-P}{2} z^j\right)$$

P	z	Solution
$0 < P < 2$	$z > 0$	monotonic
$P > 2$	$z < 0$	oscillatory

If mesh Peclet number is large

Non-oscillatory solution is guaranteed provided $h < \frac{2}{\mathrm{Pe}}$. If $\mathrm{Pe} \gg 1$, then we need a very fine mesh, $h \ll 1$.



Solution is oscillatory at higher Pe numbers; does not satisfy maximum principle

Matlab code: `convdiff_central_ccfvm.m`

If mesh Peclet number is large

Adaptive grid: Boundary layer thickness

$$\Delta = \frac{4}{\text{Pe}}$$

Put 1/3 of total cells in the boundary layer.

Grid refined in the boundary layer

Fig. 4.4. Local grid refinement in boundary layer.

If mesh Peclet number is large

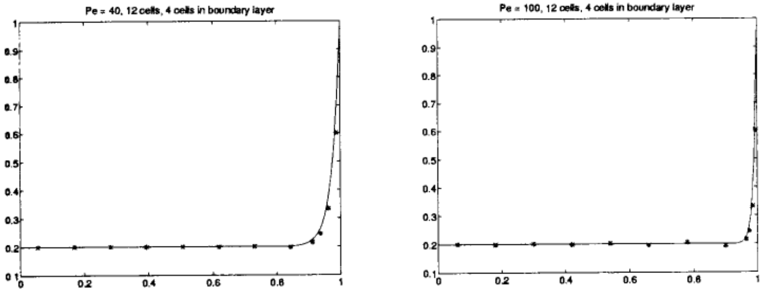


Fig. 4.5. Numerical solution with local grid refinement in boundary layer (*: numerical solution: –: exact solution.)

Oscillation $\neq$ instability

- Stability refers to influence of perturbations in data (initial and boundary conditions) on the solution
- Consider a perturbation Δa of the boundary value a. Then

$$\Delta u_j = \frac{z^n - z^j + z^j \frac{P}{2}}{z^n - 1} \Delta a$$

- Solution is stable in norm $\|\cdot\|$ if there is a constant C independent of h such that

$$\|\Delta u\| \leq C|\Delta a|$$

- Case $z < -1$:

$$\|\Delta u\|_\infty \leq C|\Delta a|, \qquad C = 2\left(2 + \frac{|P|}{2}\right)$$

Oscillation $\neq$ instability

- Case $-1 \le z < 0$ and perturbations in b are left as an exercise.
- As $h \to 0$, the condition $h < 2/\mathrm{Pe}$ will be satisfied; the numerical solution converges to the exact solution.
- In practical computations, we use a finite grid h. It is desirable to have a scheme free of oscillations for all values of h.

Upwind scheme

$$\frac{\mathrm{d}}{\mathrm{d}x}(Uu) - \frac{\mathrm{d}}{\mathrm{d}x} D \frac{\mathrm{d}u}{\mathrm{d}x} = q$$

Finite volume scheme

$$(Uu)_{j+\frac{1}{2}} - (Uu)_{j-\frac{1}{2}} - [(Du')_{j+\frac{1}{2}} - (Du')_{j-\frac{1}{2}}] = q_j h$$

Central scheme for convection term

$$(Uu)_{j+\frac{1}{2}} = \frac{1}{2} U_{j+\frac{1}{2}} (u_j + u_{j+1})$$

Upwind scheme for convection term

$$(Uu)_{j+\frac{1}{2}} = \begin{cases} U_{j+\frac{1}{2}} u_j, & U_{j+\frac{1}{2}} \geq 0 \\ U_{j+\frac{1}{2}} u_{j+1}, & U_{j+\frac{1}{2}} < 0 \end{cases}$$

Upwind scheme

or

$$(Uu)_{j+\frac{1}{2}} = \frac{1}{2}(U_{j+\frac{1}{2}} + |U_{j+\frac{1}{2}}|)u_j + \frac{1}{2}(U_{j+\frac{1}{2}} - |U_{j+\frac{1}{2}}|)u_{j+1}$$

The flux Uu is *biased in the upstream direction*. Define

$$U^+ = \frac{1}{2}(U + |U|) \geq 0, \qquad U^- = \frac{1}{2}(U - |U|) \leq 0$$

so that

$$(Uu)_{j+\frac{1}{2}} = U^+_{j+\frac{1}{2}} u_j + U^-_{j-\frac{1}{2}} u_{j+1}$$

The viscous flux can be discretized as before.

$$(Du')_{j+\frac{1}{2}} = D_{j+\frac{1}{2}} \frac{u_{j+1} - u_j}{h}, \qquad D_{j+\frac{1}{2}} = \frac{2D_j D_{j+1}}{D_j + D_{j+1}}$$

Upwind scheme

Scheme can be written as

$$\underbrace{\left(-U^+_{j-\frac{1}{2}} - \frac{D_{j-\frac{1}{2}}}{h}\right)}_{<0} u_{j-1} + \underbrace{\left(U^+_{j+\frac{1}{2}} - U^-_{j-\frac{1}{2}} + \frac{D_{j-\frac{1}{2}} + D_{j+\frac{1}{2}}}{h}\right)}_{>0} u_j + \underbrace{\left(U^-_{j+\frac{1}{2}} - \frac{D_{j+\frac{1}{2}}}{h}\right)}_{<0} u_{j+1} = q_j h$$

This scheme satisfies maximum principle and hence solutions are non-oscillatory. There is no condition on the mesh Peclet number. However the scheme is only first order accurate because the convective flux is first order accurate

$$U^+_{j+\frac{1}{2}} u_j + U^-_{j-\frac{1}{2}} u_{j+\frac{1}{2}} = (Uu)(x_{j+\frac{1}{2}}) + \mathcal{O}\left(h\right)$$

Upwind scheme

while centered flux is second order accurate

$$\frac{1}{2}U_{j+\frac{1}{2}}(u_j + u_{j+1}) = (Uu)(x_{j+\frac{1}{2}}) + \mathcal{O}\left(h^2\right)$$

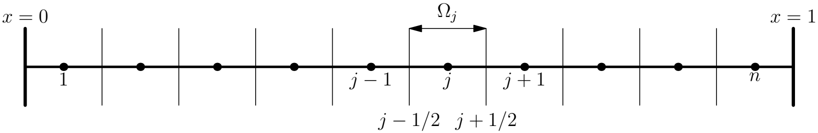
Upwind flux can also be written as: central flux + dissipation

$$(Uu)_{j+\frac{1}{2}} = \frac{1}{2}U_{j+\frac{1}{2}}(u_j + u_{j+1}) - \frac{1}{2}|U_{j+\frac{1}{2}}|(u_{j+1} - u_j)$$

Formal accuracy is not everything. Each component may be built accurately, but the final scheme may still be inaccurate as seen in the case of central difference scheme at large mesh Peclet numbers.

- Obtain exact FD solution for the problem discussed before.
- Try solving it numerically with an upwind scheme.

Example of upwind scheme: $-\frac{1}{\mathrm{Pe}}u'' + u' = 0$



Integrating over finite volume Ω_j

$$-\frac{1}{\mathrm{Pe}}[u'_{j+\frac{1}{2}} - u'_{j-\frac{1}{2}}] + u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} = 0$$

Fluxes: for $j = 1, 2, \ldots, n-1$

$$u_{j+\frac{1}{2}} = u_j, \qquad u'_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h}$$

Boundary fluxes

$$u_{\frac{1}{2}} = a, \quad u_{n+\frac{1}{2}} = u_n, \quad u'_{\frac{1}{2}} = \frac{u_1 - a}{h/2}, \quad u'_{n+\frac{1}{2}} = \frac{b - u_n}{h/2}$$

Example of upwind scheme: $-\frac{1}{\mathrm{Pe}}u'' + u' = 0$

System of finite volume equations

$$\begin{aligned}
(1+\frac{2}{P})u_1 - \frac{1}{P}u_2 &= (1+\frac{1}{P})a \\
-(1+\frac{1}{P})u_{j-1} + (1+\frac{2}{P})u_j - \frac{1}{P}u_{j+1} &= 0, \quad j = 2, 3, \ldots, n-1 \\
-(1+\frac{1}{P})u_{n-1} + (1+\frac{3}{P})u_n &= \frac{2}{P}b
\end{aligned}$$

with the mesh Peclet number $\boxed{P = h\mathrm{Pe}}$.

Matlab code: `convdiff_upwind_ccfvm.m`

Schemes of positive type

Lemma (Discrete maximum principle)

Let c_n and x_n, $n = 1, 2, \ldots, N$ be reals satisfying

$$\sum_{n=1}^{N} c_n = 0, \qquad c_1 > 0 \quad \text{and} \quad c_n < 0 \quad \text{for} \quad n > 1$$

and

$$\sum_{n=1}^{N} c_n x_n \leq 0$$

Then

$$\boxed{x_n = x_1, \quad n = 1, \ldots, N} \qquad \text{or} \qquad \boxed{x_1 < \max_{n>1}\{x_n\}}$$

Schemes of positive type

Proof: We have $c_1 > 0$. Wlog we may take $c_1 = 1$, so that

$$x_1 \leq \sum_{n=2}^{N} (-c_n) x_n \leq \max_{n>1}\{x_n\}$$

Property of convex combinations:

$$\sum_{n=2}^{N} (-c_n) x_n = \max_{n>1}\{x_n\} \qquad \text{iff} \qquad x_n = \text{const} = \max_{n>1}\{x_n\}$$

If

$$x_1 = \max_{n>1}\{x_n\} \leq \sum_{n=2}^{N} (-c_n) x_n \leq \max_{n>1}\{x_n\}$$

then it implies that $x_1 = x_n$, $n = 2, 3, \ldots, N$. Otherwise we have strict inequality. □

Schemes of positive type

FD operator: Let L_h be a linear discrete operator arising from a numerical scheme

$$L_h\psi_j = \sum_{k\in K} \alpha(j,k)\psi_{j+k}$$

where K is some index set, and $\alpha(j,k)$ are coefficients.

Example: Central difference scheme for $Lu = -u''$

$$\begin{aligned} L_h\psi_j &= -\frac{1}{h^2}\psi_{j-1} + \frac{2}{h^2}\psi_j - \frac{1}{h^2}\psi_{j+1}, \qquad K = \{-1, 0, +1\} \\ &= c_2\psi_{j-1} + c_1\psi_j + c_3\psi_{j+1} \\ &= \alpha(j,-1)\psi_{j-1} + \alpha(j,0)\psi_j + \alpha(j,+1)\psi_{j+1} \end{aligned}$$

Definition

The operator L_h is of *positive* type if

$$\sum_{k \in K} \alpha(j,k) = 0, \quad j = 2, 3, \ldots, J-1$$

and

$$\alpha(j,k) < 0, \quad k \neq 0, \quad j = 2, 3, \ldots, J-1$$

The above two conditions imply that

$$\alpha(j,0) > 0$$

Remark: This is related to the characterization of the scheme to be an M-matrix.

Theorem (Discrete maximum principle)

If L_h is of positive type and

$$L_h\psi_j \leq 0, \qquad j = 2, 3, \ldots, J-1$$

then local maxima of ψ can occur only for $j = 1$ or $j = J$, or ψ_j =constant.

Proof: For any $j \in \{2, 3, \ldots, J-1\}$ let

$$c_1 = \alpha(j, 0) > 0 \qquad \text{and} \qquad c_k = \alpha(j, k) \leq 0, \quad k \neq 0$$

Then Lemma applies after suitable re-indexing of c_k. Considering each interior grid point in turn, the theorem follows. □

If the scheme L_h is applied at the boundary point $j = J$ also (e.g. Neumann condition) then

Corollary

If the scheme L_h is of positive type and $L_h \psi_j \leq 0$ for $j = 2, 3, \ldots, J$, then a local maximum of ψ can occur only for $j = 1$.

Remarks:

- If $L_h \psi_j \geq 0$ for $j = 2, 3, \ldots, J - 1$ and L_h is of positive type then it can be shown that local minima can occur only for $j = 1$ and $j = J$.
- Hence, if $L_h \psi_j = 0$ then local extrema can only occur for $j = 1$ or $j = J$.
- We see that if the discretization is of positive type, then it obeys a similar maximum principle as the differential equation. Hence, spurious wiggles cannot occur.

Order barrier We would like to have accurate scheme which is free of wiggles. Since convection term is what causes wiggles in numerical solution, we would like to have accurate, wiggle-free schemes for the case of $\mathrm{Pe} = \infty$, i.e., for pure convection problem. But

Theorem (Order barrier)

Linear discretization schemes of positive type for the following differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = q(x)$$

are at most first order accurate.

Proof: For an interior grid point, the scheme is of the form

$$L_h\psi_j = \sum_{k\in K} \alpha(k)\psi_{j+k}, \qquad \alpha(k) = \mathcal{O}\left(1/h\right)$$

The local truncation error

$$\begin{aligned}\tau_j &= L_h\phi(x_j) - q_j = \sum_{k\in K}\alpha(k)\phi(x_j+kh) - q_j \\ &= \sum_{k\in K}\alpha(k)\left[\phi(x_j) + kh\frac{\mathrm{d}\phi}{\mathrm{d}x}(x_j) + \frac{1}{2}k^2h^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x_j) + \mathcal{O}\left(h^3\right)\right] - \frac{\mathrm{d}\phi}{\mathrm{d}x}(x_j)\end{aligned}$$

For τ_j to be $\mathcal{O}(h)$ it is necessary that

$$\sum_{k\in K}\alpha(k) = 0, \qquad \sum_{k\in K}k\alpha(k) = \frac{1}{h}$$

For the scheme to be second order accurate, it is necessary that in addition

$$\sum_{k\in K}k^2\alpha(k) = 0$$

However, since L_h is of positive type, $\alpha(k) < 0$ for $k \neq 0$, so that the above condition cannot be satisfied.

Remark: A similar order barrier theorem holds for time-dependent case. A way to get around this is to use nonlinear schemes even for linear problems. In practice, it is preferable to use high order accurate schemes, atleast second order ($\tau_j = \mathcal{O}\left(h^2\right)$), in order to get reliable results at small computational cost.

Remark: The upwind scheme satisfies the conditions of a positive scheme.

Monotone schemes

A scheme for which ($\psi = 0$ on the boundary)

$$L_h \psi \geq 0 \quad \Longrightarrow \quad \psi \geq 0$$

is called a monotone scheme. For a monotone scheme, the solution decreases (or increases) with the right hand side. Schemes of positive type are monotone, but the converse is not true.

Monotone scheme $\Leftrightarrow$ Monotone matrix, Positive scheme $\Leftrightarrow$ M-matrix
M-matrix $\Rightarrow$ Monotone matrix

Central FV scheme for $-(Du')' + (Uu)' = q$

$$\begin{aligned}
\alpha(j,-1) &= -\frac{1}{2}U_{j-\frac{1}{2}} - \frac{D_{j-\frac{1}{2}}}{h} \\
\alpha(j,+1) &= +\frac{1}{2}U_{j+\frac{1}{2}} - \frac{D_{j+\frac{1}{2}}}{h} \\
\alpha(j,0) &= -\alpha(j,-1) - \alpha(j,+1)
\end{aligned}$$

Application of positive scheme condition gives

$$-2 < \frac{U_{j+\frac{1}{2}}h}{D_{j+\frac{1}{2}}} < +2$$

In terms of a local mesh Peclet number

$$P_{j+\frac{1}{2}} = \frac{|U_{j+\frac{1}{2}}|h}{D_{j+\frac{1}{2}}}$$

the condition for positive scheme is

$$P_{j+\frac{1}{2}} < 2$$

Artificial viscosity

Construct a central scheme for modified PDE, $\mu =$ artificial viscosity

$$\frac{\mathrm{d}}{\mathrm{d}x}(Uu) - \frac{\mathrm{d}}{\mathrm{d}x}(D + \mu)\frac{\mathrm{d}u}{\mathrm{d}x} = q$$

Scheme is of positive type if

$$\mu_{j+\frac{1}{2}} \geq \frac{1}{2}|U_{j+\frac{1}{2}}|h - D_{j+\frac{1}{2}}$$

We can choose $\mu = \mathcal{O}(h)$; the scheme is first order accurate. If we choose

$$\mu_{j+\frac{1}{2}} = \frac{1}{2}|U_{j+\frac{1}{2}}|h$$

then we obtain the upwind scheme. Thus upwind scheme implicitly adds numerical dissipation. Upwind scheme = central scheme + dissipation

$$U^{+}_{j+\frac{1}{2}}u_j + U^{-}_{j-\frac{1}{2}}u_{j+1} = \frac{1}{2}U_{j+\frac{1}{2}}(u_j + u_{j+1}) - \frac{1}{2}|U_{j+\frac{1}{2}}|(u_{j+1} - u_j)$$

Hybrid scheme

For convective flux $F = Uu$, use blending of upwind and central scheme; switch locally based on local mesh Peclet number

$$F^c_{j+\frac{1}{2}} = \frac{1}{2} U_{j+\frac{1}{2}} (u_j + u_{j+1}), \qquad F^u_{j+\frac{1}{2}} = U^+_{j+\frac{1}{2}} u_j + U^-_{j-\frac{1}{2}} u_{j+1}$$

Switching function

$$s(P) = \begin{cases} 0, & P < 2 \\ 1, & P \ge 2 \end{cases}$$

Hybrid scheme

$$F_{j+\frac{1}{2}} = s(P_{j+\frac{1}{2}}) F^u_{j+\frac{1}{2}} + [1 - s(P_{j+\frac{1}{2}})] F^c_{j+\frac{1}{2}}$$

Preferable to use a smooth switching function. Hybrid schemes introduced by Spalding (1972) and Patankar (1980).