An Introduction to Scalar Conservation Laws[1]

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Contents

Abstract

In hyperbolic quasi-linear equations, we cannot expect a classical solution for all times. The solution may become multi-valued at a finite time, which is not physically acceptable. So we need to include a jump discontinuity in the solution in order to make it single valued. But this class of discontinuous functions is so large that the equation may have more than one solution. To pick up a physically relevant solution, we need a condition called 'entropy condition'. The discontinuous solution in the weak sense that satisfies entropy condition indeed exists and is unique. When we are looking for solutions in the class of discontinuous functions, the quasi-linear equations will no longer be valid. So, we need to write the equation in the divergence form which is the so called conservation law, and look for the weak solution of this conservation law which also satisfies the entropy condition. In our discussion, we will start with distribution theory which is a useful concept for weak solutions. Then we will see how the characteristics method is used in solving hyperbolic equations and, finally, we will go to quasi-linear equations and discuss about scalar conservation laws, weak solutions satisfying entropy condition, existence and uniqueness.

Chapter 1

Mathematical Preliminaries

1.1 Some Definitions

The set of all numbers is denoted by $\mathbb{R}$. An open interval in $\mathbb{R}$ is defined as,

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

where $a, b \in \mathbb{R}$ and $a < b$. A closed interval in $\mathbb{R}$ is defined as,

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

where $a, b \in \mathbb{R}$ and $a \leq b$.

Definition 1.1 (Open and closed sets)**.** *(1) A set $F \subseteq \mathbb{R}$ is said to be an open set if for each $x \in \mathbb{F}$, there exists an open interval, say (a_x, b_x) such that $x \in (a_x, b_x)$ and $(a_x, b_x) \subset F$. (2) A set $G \subseteq \mathbb{R}$ is said to be closed if G^c is open. Hence c denotes the complement of a set.*

Note

1. Every open interval is an open set and every closed interval is a closed set.

2. The interval
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
is neither open nor closed.

3. Arbitrary union of open intervals is open and finite intersection of open intervals is open.

4. Arbitrary intersection of closed intervals is closed and finite union of closed intervals is closed.

5. $\mathbb{R}$ is both open as well as closed.

Definition 1.2 (Bounded set). *A set S is said to be bounded if there exists a positive integer M such that,*

$$|x| \leq M$$

for all $x \in S$.

Note

1. $(0, 1)$ is bounded.
2. $\mathbb{R}$ is unbounded (i.e. not bounded).

Definition 1.3 (Compact set). *A closed and bounded subset of $\mathbb{R}$ is called a compact set.*

Definition 1.4 (Closure of a set). *The closure of a set $S \subset \mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ which contains S, and is denoted by $\overline{S}$.*

Note

1. Closure of (a, b) is $[a, b]$.
2. Closure of a closed set is the set itself.

Definition 1.5 (Function). *Let A and B be two sets and let there be a rule which associates to each member x of A, a member y of B. Such a rule or a correspondence f under which to each element x of the set A there corresponds exactly one element y of the set B is called a function.*

Symbolically we write $f : A \to B$. That is f is a function from A into B. Suppose f takes an element $x \in A$ to an element $y \in B$ then we write,

$$f(x) = y$$

Here A is called the domain of f and $f(A) \subset B$ is called the range.

Definition 1.6 (Bounded Function). *A function $f : S \to \mathbb{R}$ is said to be bounded if there exists an integer $M > 0$ such that $|f(x)| \leq M$ for all $x \in S$.*

Definition 1.7 (Support of a function). *Let S be any subset of $\mathbb{R}$ and let $f : S \to \mathbb{R}$. Then, support of f is defined as,*

$$Supp(f) := \overline{\{x \in S : f(x) \neq 0\}}$$

Example 1.1 (Support of a function).

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as,

$$f(x) = \begin{cases} 1, & \text{if } \ x \in (-1,1) \\ 0, & \text{if } \ x \notin (-1,1) \end{cases}$$

Then

$$\begin{aligned} Supp(f) &= \overline{\{x \in \mathbb{R} : f(x) \neq 0\}} \\ &= \overline{(-1,1)} \\ &= [-1,1] \end{aligned}$$

Note $Supp(f)$ is a closed set for any function f.

Definition 1.8 (Continuous Function). *A function $f : S \to \mathbb{R}$ is said to be continuous if for each $\varepsilon > 0, \exists\, \delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $x, y \in S$. In other words, $\lim_{x \to y} f(x)$ exists and is equal to $f(y)$.*

Definition 1.9 ($C(S)$). *Let S be any non-empty subset of $\mathbb{R}$. Then $C(S)$ is defined as the set of all bounded continuous functions $f : S \to \mathbb{R}$. The norm on $C(S)$ is defined as,*

$$\|f\|_{C(S)} := \sup_{x \in S} |f(x)|$$

Definition 1.10 (Differentiable Function). *Let Ω be an open subset of $\mathbb{R}$. Then $f : \Omega \to \mathbb{R}$ is said to be differentiable if,*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists for every $x \in \Omega$ and this quantity is denoted by $f'(x)$ and is called as the first derivative of f.

The second derivative of f is defined as the first derivative of f', and so on.

Theorem 1.1. *Let $f : \Omega \to \mathbb{R}$ be differentiable. Then f is continuous.*

Proof We have to show that for any $y \in \Omega$,

$$\lim_{x \to y} f(x) = f(y)$$

But this is easy because,

$$\begin{aligned} [\lim_{x \to y} f(x)] - f(y) &= \lim_{x - y} (f(x) - f(y)) \\ &= \lim_{x \to y} \frac{f(x) - f(y)}{x - y}(x-y) \\ &= f'(y) \lim_{x \to y} (x-y) \\ &= f'(y)0 \\ &= 0 \end{aligned}$$

Remark The statement of Theorem 1.1 can contrapositively be stated as, *Every discontinuous function is non-differentiable.*

Definition 1.11 ($C^n(\Omega)$). *Let Ω be any open, non-empty subset of $\mathbb{R}$. Then $C^n(\Omega)$ is defined as the set of all bounded continuous functions $f : \Omega \to \mathbb{R}$ such that $f^{(n)}$ exists and is continuous and bounded.*

The norm on $C^n(\Omega)$ is defined as,

$$\|f\|_{C^n(\Omega)} := \sum_{i=0}^{n} \|f^{(i)}\|_{C(\Omega)}$$

where $f^{(0)} = f$

Note If $n = \infty$ then the set $C^\infty(\Omega)$ is defined as the set of all continuous, bounded functions whose derivative of any order exist, continuous and bounded. Such functions are called *smooth functions.* The norm defined on $C^n(\Omega)$ for finite n will not work for $n = \infty$ because, in this case the RHS is an infinite series which may not converge. So, what will be the "suitable" norm on $C^\infty(\Omega)$? Actually, the suitable one comes from a semi-norm which we will not discuss here, but we will just give the meaning for the convergence of a sequence in $C^\infty(\Omega)$.

Definition 1.12 (Convergence in C^∞). *A sequence $\{f_n\}$ in $C^\infty(\Omega)$ converges to zero in $C^\infty(\Omega)$ if for every m = 0,1,2,....., $\{f_n^{(m)}\}$ converges uniformly to zero for every compact subset K of Ω.*

Definition 1.13 (C_0^∞). *Let Ω be an open, non-empty subset of $\mathbb{R}$. Then, $C_0^\infty(\Omega)$ is defined as the set of all $C_0^\infty(\Omega)$ functions whose support is a compact set. That is,*

$$C_0^\infty(\Omega) := \{f \in C^\infty(\Omega) : \mathit{supp}(f) \text{ is compact } \}$$

Definition 1.14 (Convergence in C_0^∞). *A sequence $\{f_n\}$ in $C_0^\infty(\Omega)$ converges to zero in $C_0^\infty(\Omega)$ if there exists a fixed compact set $K \subset \Omega$ such that $supp(f_n) \subset K$ for all n and if for every $m = 0, 1, 2, ...$, the sequence $\{f_n^{(m)}\}$ converges uniformly in K.*

Note The functions $f \in C_0^\infty(\Omega)$ are called the *test functions.*

1.2 Distributions

To motivate the concept of distributions, let us first introduce locally integrable functions.

Definition 1.15 (Locally integrable function)**.** *A function f is said to be locally integrable if,*

$$\int\limits_K |f(x)|dx < \infty$$

for every compact set $K \subset \mathbb{R}$.

Note Every continuous function is locally integrable. For, since a continuous function on a compact set is bounded, we have for every compact interval (here we took a compact interval just for convenience) $I - [a, b]$,

$$\begin{aligned} \int\limits_I |f(x)|dx &\leq \max_{x\in I} |f(x)| \int\limits_I dx \\ &= \max_{x\in I} |f(x)|(b-a) \\ &< \infty \end{aligned}$$

Now define a function $T_f : C_0^\infty(\Omega) \to \mathbb{R}$ as,

$$T_f(\phi) := \int\limits_\Omega f(x)\phi(x)dx$$

Then, if $\phi_1, \phi_2 \in C_0^\infty(\Omega)$, we have,

$$T_f(\phi_1 + \phi_2) = T_f(\phi_1) + T_f(\phi_2)$$

Also, if $\alpha \in \mathbb{R}$, then,

$$T_f(\alpha\phi) = \alpha T_f(\phi)$$

for every $\phi \in C_0^\infty(\Omega)$. Finally, let $\{\phi_n\}$ be a sequence in $C_0^\infty(\Omega)$ and let $\phi_n \to 0$ in $C_0^\infty(\Omega)$ (meaning is given in def 14). Then,

$$\begin{aligned} |T_f(\phi_n)| &= |\int\limits_\Omega f(x)\phi_n(x)\mathrm{d}x| \\ &\leq \int\limits_\Omega |f(x)||\phi_n(x)|\mathrm{d}x \\ &= \int\limits_K |f(x)||\phi_n(x)|\mathrm{d}x \to 0 \end{aligned}$$

since $\phi_n \to 0$, where K is as given in definition 1.14. Such a function T_f is called a distribution. In general, we have the following definition.

Definition 1.16 (Distribution). *$T : C_0^\infty(\Omega) \to \mathbb{R}$ is said to be a distribution on Ω if,*

1. *$T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2)$ for all $\phi_1, \phi_2 \in C_0^\infty(\Omega)$.*
2. *$T(\alpha\phi) = \alpha T(\phi)$, for all $\alpha \in \mathbb{R}$*
3. *$\phi_n \to 0$ in $C_0^\infty(\Omega) \Rightarrow T(\phi_n) \to 0$ in $\mathbb{R}$.*

Note From the above argument, we can conclude that with the help of every locally integrable function, we can define a distribution. Here is another important example of a distribution.

Example 1.2 (Dirac distribution).

Define $\delta : C_0^\infty(\Omega) \to \mathbb{R}$ as,

$$\delta(\phi) = \phi(0), \text{ for all } \phi \in C_0^\infty(\Omega)$$

It is easy to prove that δ is a distribution. This is called as the *Dirac mass.*

1.2.1 Differentiation of distributions

If $\phi \in C_0^\infty((a,b))$, then by integration by parts, we have,

$$\begin{aligned}\int_a^b f'(x)\phi(x)\mathrm{d}x &= f(x)\phi(x)|_a^b - \int_a^b f(x)\phi'(x)\mathrm{d}x \\ &= f(b)\phi(b) - f(a)\phi(a) - \int_a^b f(x)\phi'(x)\mathrm{d}x\end{aligned}$$

since supp $(\phi) \subset (a,b)$, we have

$$\phi(a) = \phi(b) = 0$$

Therefore, we obtain,

$$\int_a^b f'(x)\phi(x)\mathrm{d}x = -\int_a^b f(x)\phi'(x)\mathrm{d}x$$

The interesting thing is that on the RHS of the above equation the derivative of f is not involved. Thus, if f is a locally integrable function, then let us define the derivative of the corresponding distribution as,

$$T_f'(\phi) = -T_f(\phi') = -\int_a^b f(x)\phi'(x)\mathrm{d}x$$

Similarly, we define,

$$T_f^{(n)}(\phi) = (-1)^n T_f(\phi^{(n)})$$

In general we have the following definition.

Definition 1.17 (n^{th} derivative of a Distribution)**.** *Let T be any distribution. Then the n^{th} derivative of T is defined as,*

$$T^{(n)}(\phi) = (-1)^n T(\phi^{(n)})$$

for all $\phi \in C_0^\infty(\Omega)$.

Note Since for every locally integrable function f, we can define a corresponding distribution T_f, we will denote the distribution T_f as f itself. If we say that (for a locally integrable function f) "differentiating f in the sense of distribution", we mean that f is considered as a distribution (i.e., T_f) and differentiating that distribution as in the definition 1.17.

Example 1.3 (Heaviside function)**.**

Let

$$H(x) = \begin{cases} 1, & \text{if} \quad x \geq 0 \\ 0, & \text{if} \quad x < 0 \end{cases}$$

This function is called the *Heaviside* function on $\mathbb{R}$. Clearly, H is locally integrable. Thus, H can be considered as a distribution. Now, for all $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} H'(\phi) &= -H(\phi') \\ &= -\int_0^\infty \phi'(x)\mathrm{d}x \\ &= -[\phi(x)]_0^\infty \\ &= -(\phi(\infty) - \phi(0)) \\ &= -(0 - \phi(0)), \qquad (\phi(\infty) = 0 \text{ since } \phi \in C_0^\infty(\mathbb{R})) \\ &= \phi(0) \\ &= \delta(\phi), \qquad \text{(see Example 1.2)} \end{aligned}$$

Note In theorem 1.1, we have seen that discontinuous functions are not differentiable. Therefore, being a discontinuous function, H is not differentiable. But, we have seen in the above example that, we can still talk about the differentiability of H in the sense of distribution.

Definition 1.18 (Product of a Distribution with a C^∞ - function)**.** *Let $f \in C^\infty(\Omega)$ and T be any distribution. Then the product fT is defined as*

$$(fT)(\phi) = T(f\phi), \qquad \forall \phi \in C_0^\infty(\Omega)$$

Note

1. The reader should be very careful with the notations. Since f is basically a continuous function, as mentioned in the note of Definition 1.15, it is locally integrable. Thus, the f on the LHS denotes the distribution. On the other hand the f on the RHS is function itself. Therefore, the LHS is the product of two distributions acting on ϕ and the RHS is the distribution T acting on the product of two functions.

2. Since $f \in C^\infty(\Omega), f\phi \in C_0^\infty(\Omega)$. Thus, the RHS is meaningful. But, if f is not a $C^\infty(\Omega)$ function, i.e., $f \notin C^\infty(\Omega)$, then $f\phi \notin C_0^\infty(\Omega)$ and therefore RHS is not defined at all, since T is defined on $C_0^\infty(\Omega)$. Hence the product of any arbitrary function with a distribution is meaningless.

Remark The discussions we have made so far hold on $\mathbb{R}^n$ also.

Chapter 2

Characteristics and Initial Value Problems

Roughly speaking, characteristics are curves which carry information. These curves are very useful in solving partial differential equations (PDE) in which solution is known initially, that is, on the curve $t = 0$.

Consider the general form of the first order PDE:

$$F(x, t, u(x, t), u_x(x, t), u_t(x, t)) = 0 \tag{2.1}$$

where $(x, t) \in \Omega$, for $\Omega \subset \mathbb{R} \times R^+$ is assumed to be open and its boundary $\partial\Omega$ intersects the curve $t = 0$, and $u : \Omega \to \mathbb{R}$. Let us assume that

$$u(x, 0) = u_0(x) \tag{2.2}$$

and $(x, 0) \in \overline{\Omega}$. The problem of finding the solution u for equation (2.1) which satisfies the initial condition (2.2) is called the Initial Value Problem (IVP).

The aim of the characteristic method is to solve the IVP (2.1)-(2.2) by converting the PDE into an appropriate system of ODEs. First, let us assume that $u \in C^1(\Omega)$ satisfies IVP (2.1)-(2.2). Such a solution u is called a *smooth solution* or a *classical solution* of the IVP (2.1)-(2.2). Let $(x, t) \in \Omega$ be fixed. The idea is to calculate $u(x, t)$ by finding some curve in Ω, connecting with a point $(x_0, 0)$ on the x-axis along which we can compute u, see figure 2.1. Since we know the value of u at $(x_0, 0)$ from the initial condition (2.2), we can hope to be able to calculate u all along the curve, and in particular at x.

2.1 Construction of the characteristic curve

Now, the question is how to choose such a curve ? Suppose that the parametric representation of our curve is

$$X(\eta) = (x(\eta), t(\eta)), \quad \eta \in \mathbb{R}$$

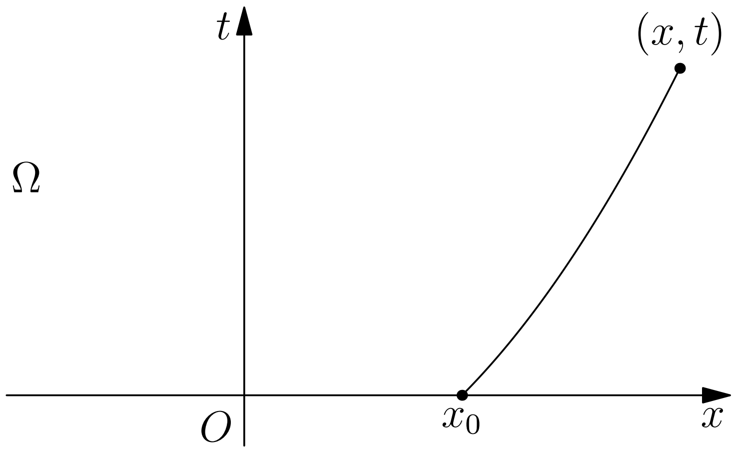

Figure 2.1: Method of characteristics

Assuming u to be a smooth solution of the IVP (2.1)-(2.2), let us define

$$Z(\eta) := u(X(\eta)) \tag{2.3}$$

and also define

$$\left.\begin{array}{lll} P_1(\eta) & := & u_x(X(\eta)) \\ P_2(\eta) & := & u_t(X(\eta)) \end{array}\right\} \tag{2.4}$$

Thus, $Z(\cdot)$ gives the value of u along the curve $X(\cdot)$ and $P(\cdot) := (P_1(\cdot), P_2(\cdot))$ gives the value of u_x and u_t along $X(\cdot)$.

Now, our problem is to choose the function $X(\cdot)$ in such a way that we can compute $Z(\cdot)$ and $P(\cdot)$. For this, let us first differentiate (2.4) with respect to η :

$$\left.\begin{array}{lll} \dfrac{\mathrm{d}P_1}{\mathrm{d}\eta} & = & u_{xx}(X(\eta))\dfrac{\mathrm{d}x}{\mathrm{d}\eta} + u_{xt}(X(\eta))\dfrac{\mathrm{d}t}{\mathrm{d}\eta} \\ \\ \dfrac{\mathrm{d}P_2}{d\eta} & = & u_{tx}(X(\eta))\dfrac{\mathrm{d}x}{\mathrm{d}\eta} + u_{tt}(X(\eta))\dfrac{\mathrm{d}t}{\mathrm{d}\eta} \end{array}\right\} \tag{2.5}$$

RHS of the above two equations involves the second derivatives of u. But according to our assumption, u is only a C^1-function. Thus, we have to somehow get rid of these second derivates of u.

Differentiating (2.1) with respect to x and t, we get

$$\left.\begin{array}{lll} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial Z}u_x + \dfrac{\partial F}{\partial P_1}u_{xx} + \dfrac{\partial F}{\partial P_2}u_{tx} & = & 0 \\ \\ \dfrac{\partial F}{\partial t} + \dfrac{\partial F}{\partial Z}u_t + \dfrac{\partial F}{\partial P_1}u_{xt} + \dfrac{\partial F}{\partial P_2}u_{tt} & = & 0 \end{array}\right\} \tag{2.6}$$

Now we let

$$\frac{\mathrm{d}x}{\mathrm{d}\eta} = \frac{\partial F}{\partial P_1}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\eta} = \frac{\partial F}{\partial P_2} \tag{2.7}$$

Substituting (2.7) in (2.6), we get

$$\left.\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial Z}u_x + \frac{\mathrm{d}x}{\mathrm{d}\eta}u_{xx} + \frac{\mathrm{d}t}{\mathrm{d}\eta}u_{tx} &= 0 \\ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial Z}u_t + \frac{\mathrm{d}x}{\mathrm{d}\eta}u_{xt} + \frac{\mathrm{d}t}{\mathrm{d}\eta}u_{tt} &= 0 \end{aligned}\right\} \tag{2.8}$$

Substituting (2.8) in (2.5), we get

$$\left.\begin{aligned} \frac{\mathrm{d}P_1}{\mathrm{d}\eta} &= -\frac{\partial F}{\partial x} + \frac{\partial F}{\partial Z}u_x = -\frac{\partial F}{\partial x} + \frac{\partial F}{\partial Z}P_1 \\ \frac{\mathrm{d}P_2}{d\eta} &= -\frac{\partial F}{\partial t} + \frac{\partial F}{\partial Z}u_t = -\frac{\partial F}{\partial t} + \frac{\partial F}{\partial Z}P_2 \end{aligned}\right\} \tag{2.9}$$

Since u is a C^1 -function, we have $u_{xt} = u_{tx}$. We have thus got rid of the second derivatives appearing in (2.5). Finally, differentiating (2.3) with respect to η, we get

$$\left.\begin{aligned} \frac{\mathrm{d}Z}{\mathrm{d}\eta} &= u_x(X(\eta))\frac{\mathrm{d}x}{\mathrm{d}\eta} + u_t(X(\eta))\frac{\mathrm{d}t}{\mathrm{d}\eta} \\ &= P_1(\eta)\frac{\partial F}{\partial P_1} + P_2(\eta)\frac{\partial F}{\partial P_2} \end{aligned}\right\} \tag{2.10}$$

where the second equality follows from (2.4) and (2.7). Thus we get the following system of five first order ODE:

$$\left.\begin{aligned} \frac{dP_1}{\mathrm{d}\eta} &= -\left(\frac{\partial F}{\partial X} + \frac{\partial F}{\partial Z}P_1\right) \\ \frac{\mathrm{d}P_2}{\mathrm{d}\eta} &= -\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial Z}P_2\right) \end{aligned}\right\} \tag{2.11}$$

$$\frac{\mathrm{d}Z}{\mathrm{d}\eta} = P_1(\eta)\frac{\partial F}{\partial P_1} + P_2(\eta)\frac{\partial F}{\partial P_2} \tag{2.12}$$

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\eta} &= \frac{\partial F}{\partial P_1} \\ \frac{\mathrm{d}t}{\mathrm{d}\eta} &= \frac{\partial F}{\partial P_2} \end{aligned}\right\} \tag{2.13}$$

These equations are called *characteristics equations* of the PDE (2.1). The function $P(\cdot) = (P_1, P_2), Z(\cdot), X(\cdot)$ are called the *characteristics.*

We still need to discover appropriate initial conditions for the ODE (2.11)- (2.13). Before continuing our investigation of the characteristics equations (2.11) -(2.13), we will see an example.

Example 2.1 (Linear PDE).

Let
$$F(x, t, u, u_x, u_t) = u_t + cu_x, \quad x \in \Omega$$
where $c > 0$ is a constant. Then we have from (2.13)
$$\frac{\mathrm{d}x}{\mathrm{d}\eta} = \frac{\partial F}{\partial P_1} = c, \qquad \frac{\mathrm{d}t}{\mathrm{d}\eta} = \frac{\partial F}{\partial P_2} = 1$$
Further, from (2.12) we have
$$\frac{\mathrm{d}Z}{\mathrm{d}\eta} = u_x c + u_t 1 = u_t + cu_x$$
Thus, for the PDE
$$F(x, t, u, u_x, u_t) = 0$$
we have
$$\frac{\mathrm{d}Z}{\mathrm{d}\eta} = 0$$
The characteristics are given by,
$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\eta} &= c \\ \frac{\mathrm{d}t}{\mathrm{d}\eta} &= 1 \\ \frac{\mathrm{d}Z}{\mathrm{d}\eta} &= 0 \end{aligned}$$

2.2 Initial Condition for the Characteristic ODE

To solve the IVP (2.1)-(2.2), we have to solve the characteristic ODE (2.11)-(2.13) and for this we must discover appropriate initial conditions
$$P(0) = P_0, \qquad Z(0) = Z_0, \qquad X(0) = X_0$$

Clearly, if the characteristic $X(\cdot)$ passes through X_0, we should require

$$Z_0 = u(X_0) = u_0(x_0)$$

where we have assumed $t(0) = 0$ and $x(0) = x_0$. Now, what should we ask for $P(0) = P_0$? Since, from (2.2), we have

$$u(x,0) = u_0(x), \quad x \in \Omega \cap \{t = 0\}$$

Differentiating with respect to x we get,

$$u_x(x_0, 0) = u_{0_x}(x_0)$$

As we also want the PDE (2.1) to hold, we should therefore *insist* that $P_0 = (P_{1_0}, P_{2_0})$ satisfies

$$\left.\begin{aligned} P_{1_0} &= u_{0_x}(x_0) \\ F(P_0, Z_0, X_0) &= 0 \end{aligned}\right\} \tag{2.14}$$

Conditions (2.2) and (2.14) are called *compatibility conditions.*

Example [contd.] From the above arguments, we can take

$$t(0) = 0, \qquad x(0) = x_0, \qquad Z(0) = u_0(x_0)$$

so that the ODE derived in Example 2.1 becomes

$$t = \eta, \qquad x = c\eta + x_0, \qquad Z = u_0(x_0)$$

Thus from (2.3) the solution for the linear PDE

$$u_t + cu_x = 0$$

is given by

$$u(x,t) = u_0(x - ct)$$

Exercise Construct the characteristic ODE for the Hamilton-Jacobi equation

$$F(x, t, u, u_x, u_t) = u_t + H(u_x, x) = 0$$

Chapter 3

First Order Quasi-linear Hyperbolic Equations

A first order quasi-linear hyperbolic equation is of the form

$$u_t + c(u)u_x = 0, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+ \tag{3.1}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$. In this chapter we will study the solution for equation (3.1) with the initial condition

$$u(x, 0) = u_0(x) \tag{3.2}$$

The characteristic ODE is given by

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= c(u) \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= 0 \end{aligned}\right\} \tag{3.3}$$

From the second equation of (3.3), it follows that u is a constant along the characteristics. Since c is a functions of u only, it is also a constant along the characteristic. Thus from the equation of (3.3), we get

$$x = c(u)t + x_0 \tag{3.4}$$

which represents a one parametric family of straight line and u is constant along these straight lines. Since the straight line (3.4) intersects the x-axis at x_0, the solution for the IVP (3.1)-(3.2) will take the form

$$u(x, t) = u_0(x_0) \tag{3.5}$$

where

$$x_0 = x - c(u_0(x_0))t \tag{3.6}$$

Conversely, if c and u_0 are C^1-functions, then we will show that u is the classical solution for the IVP (3.1)-(3.2). From (3.5), we have

$$u_t = u_0'(x_0)x_{0_t}, \qquad u_x = u_0'(x_0)x_{0_x}$$

and from (3.6), we have

$$x_{0_t} = \frac{-c(u_0(x_0))}{1 + c'(u_0(x_0))u_0'(x_0)t}$$

and

$$x_{0_x} = \frac{1}{1 + c'(u_0(x_0))u_0'(x_0)t}$$

Therefore, we have

$$u_t = \frac{-c(u_0(x_0))u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} \tag{3.7}$$

and

$$u_x = \frac{u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} \tag{3.8}$$

Since c and u_0 are C^1-functions, the RHS of the above two equations exist and therefore u is a C^1-functions. So, all that remains is to show that u defined by (3.5)-(3.6) satisfies the IVP (3.1)-(3.2). But this is easy since

$$u_t + c(u)u_x = \frac{-c(u_0(x_0))u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} + c(u_0(x_0))\frac{u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} = 0$$

3.1 Loss of Regularity

A basic feature for equations of the form (3.1) is that, even for smooth initial data, the solution of the IVP (3.1)-(3.2) may develop discontinuities in finite time. Thus, we need to look for the solutions of such IVP in the class of discontinuous functions, interpreting the equation (3.1) in their distributional sense. In this section, we will see what type of initial data u_0 in (3.2) will develop discontinuities in finite time and why?

Let us begin with equation (3.8). Suppose that

$$c'(u_0) > 0 \tag{3.9}$$

and

$$u_0'(x) < 0 \tag{3.10}$$

then we have

$$c'(u_0)u_0'(x_0) < 0 \tag{3.11}$$

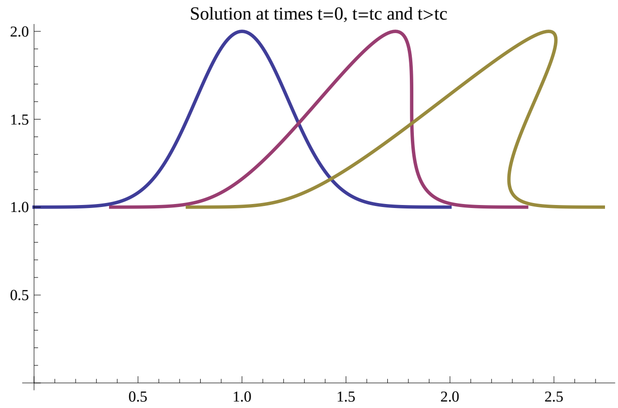

Figure 3.1: Formation of shock from a smooth initial condition. Shock is formed at time $t = t_c$. For $t > t_c$, the solution is triple valued.

Therefore, if we take

$$t = -\frac{1}{c'(u_0(x_0))u_0'(x_0)}$$

then from (3.11), $t > 0$ and for this t, the slope of u with respect to x (of course to t also) becomes infinite (this follows from (3.8)). Thus, if we define

$$t_c := -\frac{1}{\min_x c'(u_0(x))u_0'(x)} \tag{3.12}$$

then, t_c will be minimum time at which the slope of u with respect to both x and t becomes infinite. We call t_c as the *critical time.* For $t > t_c$, the solution becomes multi-valued as shown in the figure (3.1).

Let us see this physically. Equation (3.9) means that the functions c is an increasing function of u_0. Thus, since from the first equation of (3.3), c being the velocity of propagation, the larger value of u_0 propagates faster than lower values. Thus, if u_0 decreases at some interval (which is give in (3.10)), then in that interval the solution will become multi-valued.

Obviously, such multi-valued solutions are not physically acceptable. To overcome this difficulty we have to introduce a simple jump discontinuity (i.e., the limits from both sides exist but are not equal) into the multi-valued continuous solution in order to make it single-valued, as shown in figure (3.2). This requires some mathematical extension of what is meant by a *solution* to (3.1) since the derivatives of u will not exist at a discontinuity see Theorem 1.1 in Chapter 1. This can be done through the concept of a *weak solution*, which will be introduced in the next chapter. Before closing this chapter, let us make an important remark.

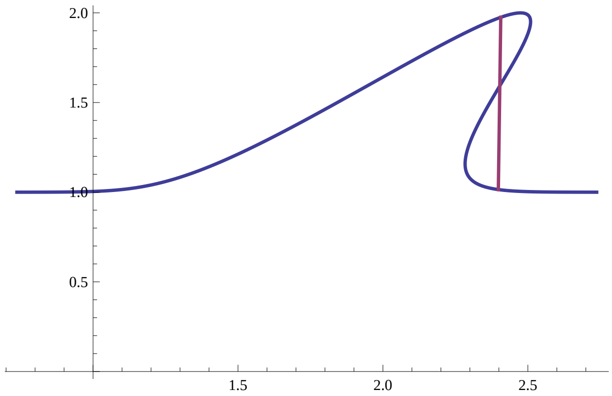

Figure 3.2: For $t > t_c$, the solution is triple valued. This is made unique by introducing a shock.

From the above discussions, it is clear that if we want to look for the solution of the IVP (3.1) - (3.2) in the class of smooth functions. i.e., in this case, in the space $C^1(\mathbb{R})$, then we can get the solution only for finite time, and if we want a global solution, then we need to look in the class of discontinuous functions. But, there is a major problem in considering such solutions. The problem is the following: If u has a jump discontinuity, then, since the LHS of (3.1) contains the product of a discontinuous function $c(u)$ with the distributional derivative u_x, which in this case contains a Dirac mass at the point of the jump, as we have seen in Chapter 1, it is in general not well defined. Thus, we can no longer work with the equation of the form (3.1) (if we are looking for the solutions in the class of discontinuous functions), and we have to rewrite equation (3.1) in the divergence form

$$u_t + f(u)_x = 0$$

where $f'(u) = c(u)$ which is the so called *conservation law.* Note that in this form, there is no such difficulty.

Chapter 4

Conservation Laws

4.1 Definition

A single conservation law in one space dimension is a first order partial differential equation of the form.

$$u_t + f(u)_x = 0 \tag{4.1}$$

here, u is the conserved quantity while f is the flux. Integrating equation (4.1) over the interval $[a, b]$ one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x &= \int_a^b u_t(x,t)\mathrm{d}x && (4.2)\\
&= -\int_a^b f(u(x,t))_x dx && (4.3)\\
&= f(u(a,t)) - f(u(b,t)) && (4.4)\\
&= \text{[inflow at } a] - \text{[outlow at } b]
\end{aligned}$$

In other words, the quantity u is neither created nor destroyed; the total amount of u contained inside any given interval $[a, b]$ can change only due to flux of u across the two end points.

We assume that the flux $f : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}$ is open, is an C^1 function, and $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, is such that range of u is contained in Ω.

If u is a classical solution of (4.1), then by chain rule (4.1) can be written as,

$$u_t + c(u)u_x = 0 \tag{4.5}$$

where $c(u) = f'(u)$, which is in the quasi-linear form. Thus, any classical solution of (4.1) is also the classical solution of the corresponding quasi-linear equation. But, we have seen

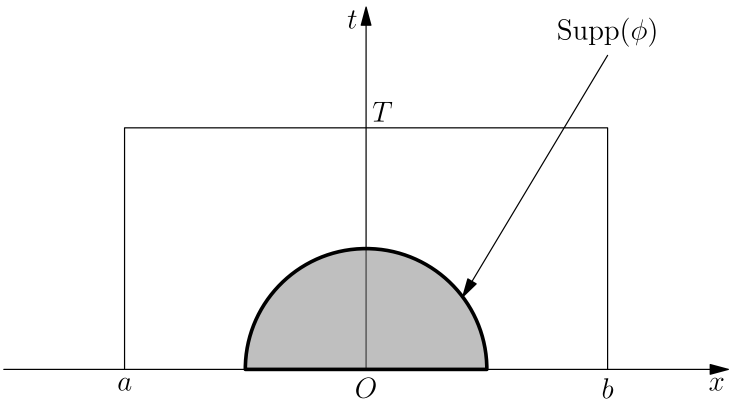

Figure 4.1: Integration for definition of weak solution

in Chapter 3 that we cannot expect classical solutions for all t, and we have to look for solutions in the class of discontinuous functions (for certain IVP's). Also, we have seen that the quasi-linear form is not well defined if u is discontinuous. On the other hand, working with the equation in the divergence form (4.1) allows us to consider discontinuous solutions as well, interpreted in distributional sense, in which case, the solution is called a weak solution. A more formal statement will be given in the next section.

4.2 Weak Solutions of Conservation Laws

We shall now generalise the notion of solution for equations of the form (4.1). We consider the IVP for (4.1) in $t > 0$:

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x) \tag{4.6}$$

Let us first suppose that u is a classical solution of (4.1) . Let D be the open rectangle $(a < x < b) \times (0 < t < T)$ (see figure 4.1), and $\phi \in C_0^1(D)$. We multiply (4.1) by ϕ and integrate over $t > 0$, we get

$$\begin{aligned} 0 &= \int\limits_{t>0} (u_t + f(u)_x)\phi \mathrm{d}x\mathrm{d}t \\ &= \int\limits_{D} (u_t + f(u)_x)\phi \mathrm{d}x\mathrm{d}t \\ &= \int\limits_a^b \int\limits_0^T (u_t + f(u)_x)\phi \mathrm{d}x\mathrm{d}t \end{aligned}$$

Now, applying integration by parts, we get,

$$\begin{aligned}\int_a^b\int_0^T u_t\phi \mathrm{d}t\mathrm{d}x &= \int_a^b u(x,t)\phi(x,t)\Big|_{t=0}^{t=T}\mathrm{d}x - \int_a^b\int_0^T u(x,t)\phi_t(x,t)\mathrm{d}x\mathrm{d}t \\ &= -\int_a^b u(x,0)\phi(x,0)\mathrm{d}x - \int_a^b\int_0^T u\phi_t \mathrm{d}x\mathrm{d}t\end{aligned}$$

since $\phi(x,T)=0$ and

$$\begin{aligned}\int_0^T\int_a^b f_x\phi \mathrm{d}t\mathrm{d}x &= \int_0^T f\phi\Big|_{x=a}^{x=b} dt - \int_0^T\int_a^b f\phi_x \mathrm{d}x\mathrm{d}t \\ &= -\int_0^T\int_a^b f\phi_x \mathrm{d}x\mathrm{d}t\end{aligned}$$

Therefore, we have

$$\int_0^\infty\int_{-\infty}^\infty (u\phi_t + f(u)\phi_x)\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0 \tag{4.7}$$

Thus, we have shown that if u is a classical solution of (4.1), then it satisfies (4.7) in which the derivatives of u are not involved, and (4.1) holds perfectly if u and u_0 are just locally integrable. We thus give the following definition for a solution of (4.1).

4.2.1 Definition of Weak solution

A function $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a weak solution of the IVP (4.1) together with locally integrable initial data u_0, if u is locally integrable and satisfies the equation (4.7), $\forall\ \phi \in C_0^1(\mathbb{R}\times\mathbb{R}^+)$.

Remark Weak solutions of (4.1) are solutions in the distributional sense.

Lemma 4.1 (Weak and classical solution)**.** *Let u be a weak solution of (4.1) and if $u \in C^1$, then u is a classical solution of (4.1).*

Proof Since u is a weak solution of (4.1), it satisfies the equation (4.7). Since u is a C^1 function, we can apply integration by parts in reverse order to get the result.

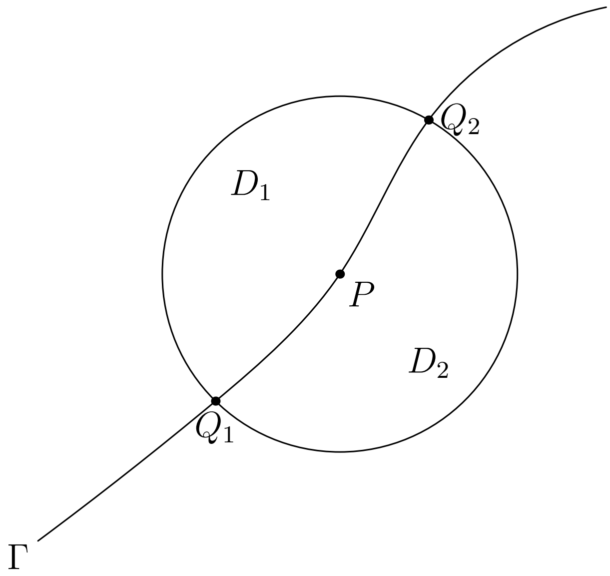

Figure 4.2: Proof of RH condition

4.2.2 Rankine-Hugoniot Condition

Our basis in removing multi-valued part in the solution is to introduce a discontinuity in the solution. In this section, we shall show that not every discontinuity is permissible.

Let Γ be a smooth curve across which u has a jump discontinuity, and u smooth away from Γ. Let P be any point on Γ and let D be a small ball centered at P. We assume that in D, Γ is given by $x = x(t)$, see figure (4.2). Let D_1 and D_2 be the components of D which are determined by Γ. Let $\phi \in C_0^1(D)$, then from (4.7),

$$\begin{aligned} 0 &= \int_D (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t \\ &= \int_{D_1} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t + \int_{D_2} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{4.8}$$

Since by our assumption, u is smooth , (i.e, C^1) in D_1 and D_2, we have, for $i = 1, 2$,

$$\int_{D_i} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t = \int_{D_i} ((u\phi)_t + (f\phi)_x)\mathrm{d}x\mathrm{d}t - \int_{D_i} (u_t\phi + f_x\phi)\mathrm{d}x\mathrm{d}t$$

since u satisfies (4.1) in D_i for $i = 1, 2$, the second term on the RHS becomes zero.

Therefore, from Green's theorem, we get,

$$\int\limits_{D_i} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t = \int\limits_{\partial D_i} (-u\mathrm{d}x + f\mathrm{d}t)\phi$$

where ∂D_i is the boundary of D_i for $i = 1, 2$. Applying these equalities in (4.8), we get,

$$\int\limits_{\partial D_1} (-u\mathrm{d}x + f\mathrm{d}t)\phi + \int\limits_{\partial D_2} (-u\mathrm{d}x + f\mathrm{d}t)\phi = 0$$

But, $\phi \equiv 0$ on ∂D, and may not be zero on Γ. Thus, if we denote,

$$\begin{aligned} u_l &= u(x(t) - 0, t) \\ u_r &= u(x(t) + 0, t) \end{aligned}$$

then we have,

$$\int\limits_{Q_1}^{Q_2} (-u_l\mathrm{d}x + f_l\mathrm{d}t)\phi - \int\limits_{Q_1}^{Q_2} (-u_r dx + f_r\mathrm{d}t)\phi = 0$$

where Q_1 and Q_2 are the points of intersection of Γ with the boundary of D. Since this is true $\forall \phi \in C_0^1(D)$, we have

$$s := \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f_l - f_r}{u_l - u_r} \tag{4.9}$$

We call s the speed of propagation of the discontinuity. Thus, we have shown that the weak solution defined by (4.7) tells us that not all the discontinuities are allowed to be introduced in the solution to avoid multi-valued parts, but only the discontinuities whose velocity is given by (4.9) is allowed. The condition (4.9) is called the *Rankine Hugoniot Condition.*

4.3 Some Examples

In this section, we will calculate weak solutions for the inviscid Burgers equation

$$u_t + uu_x = 0 \tag{4.10}$$

together with the initial condition

$$u(x, 0) = u_0(x)$$

Equation (4.10) can be put in infinitely many conservation forms,

$$(u^n)_t + \left(\frac{n}{n+1}u^{n+1}\right)_x = 0$$

where n is a positive integer. In the following examples, we will take $n = 1$, i.e. the conservation law

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{4.11}$$

Note that, in this case the flux function is given by

$$f(u) = \frac{u^2}{2}$$

which is obviously a C^1-function and

$$f''(u) = 1 > 0$$

Thus, the assumption made in the previous sections on f holds.

Example 4.1 (Shock solution)**.**

Let

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} \tag{4.12}$$

As usual, suppose that u is the classical solution for the IVP (4.11)-(4.12). Then as we have seen earlier u is also the classical solution for the IVP (4.10)-(4.12). Thus, from Chapter 3, the general form of u is given by

$$u(x,t) = u_0(x_0) \tag{4.13}$$

where

$$x_0 = x - ut = x - u_0(x_0)t$$

Therefore, in our example,

$$x_0 = \begin{cases} x - t & \text{if } x_0 < 0 \\ x - (1 - x_0)t & \text{if } 0 \le x_0 \le 1 \\ x & \text{if } 1 \le x_0 \end{cases} \tag{4.14}$$

Apply the above definition of x_0 in equation (4.13) we get the general solution of the IVP (4.11)-(4.12) as

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \dfrac{1-x}{1-t} & \text{if } t \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{4.15}$$

Thus, if u is the classical solution of (4.11)- (4.12) then u takes the form (4.15). But the question is that, can we get the classical solution u for all time t ?

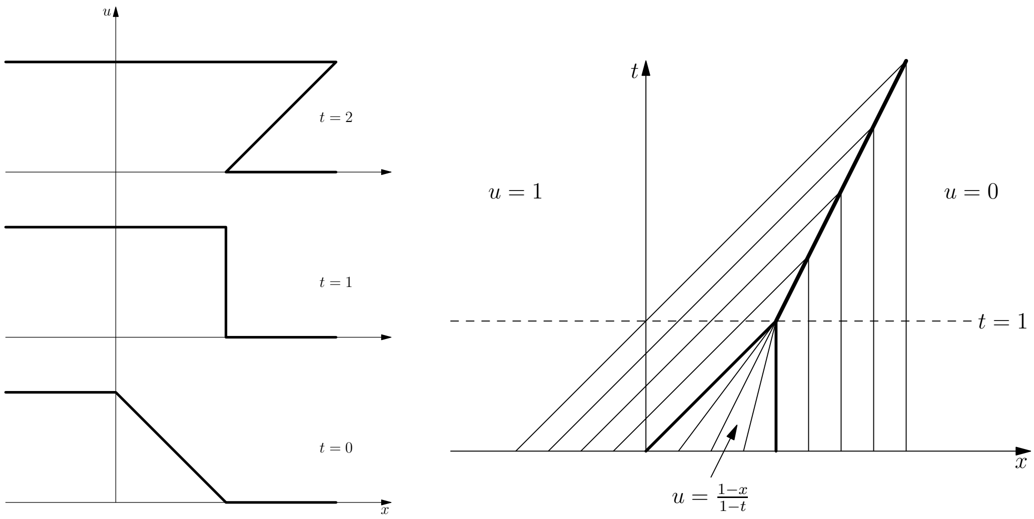

Figure 4.3: Shock solution

Observe that $u_0' = -1 < 0$ for $0 \leq x \leq 1$. Thus according to the theory discussed in Chapter 3, we cannot get the classical solution for the IVP (4.11)-(4.12), and the solution becomes multi-valued after the critical time t_c which is given by

$$t_c = -\frac{1}{\min_x u_0'(x)} = -\frac{1}{(-1)} = 1$$

That is for $t \geq 1$, the solution becomes multi-valued. This can be easily observed from (4.15).

So, we need to introduce a discontinuity into the solution (in order to make it single valued) in such a way that the RH condition

$$s = \frac{1}{2}(u_l + u_r)$$

is satisfied. Note that in our case $u_l = 1$ and $u_r = 0$. Therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = s = \frac{1}{2}$$

This gives

$$x = \frac{t}{2} + \text{const} = \frac{1}{2}(t+1)$$

Therefore the weak solution is given by

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

for $t < 1$, and

$$u(x,t) = \begin{cases} 1 & \text{if } x < \dfrac{1}{2}(t+1) \\ \\ 0 & \text{if } x > \dfrac{1}{2}(t+1) \end{cases}$$

if $t \geq 1$. The solution is $x - u$ and $x - t$ planes are shown in figure (4.3).

Example 4.2 (Rarefaction solution)**.**

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{4.16}$$

Let u be the classical solution for the IVP (4.11)-(4.16). Then we have

$$u(x,t) = u_0(x - u_0 t)$$

As in the above example we get,

$$x_0 = \begin{cases} x & \text{if } x_0 < 0 \\ x - t & \text{if } x_0 > 0 \end{cases}$$

and therefore we have

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > t \end{cases}$$

Hence, we got the solution in the intervals $(-\infty, 0)$ and (t, ∞). But if $t > 0$, we don't know the solution in the interval $[0, t]$. We can see this in the $x - t$ plane in figure (4.4).

To find u in the interval $[0, t]$. From the characteristic equation we have

$$x = ut + x_0$$

But the characteristics which enters into this region starts from $x_0 = 0$. Therefore we have

$$x = ut$$

or

$$u(x,t) = \frac{x}{t}$$

Therefore the solution is given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 \leq x \leq t \\ 1 & \text{if } x > t \end{cases}$$

Solution obtained in this example is called a rarefaction wave solution. Thus the $x - t$ plane is covered fully as shown in figure (4.5).

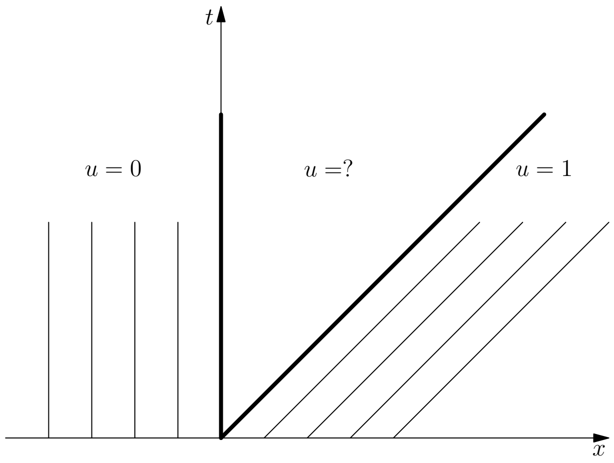

Figure 4.4: Rarefaction solution

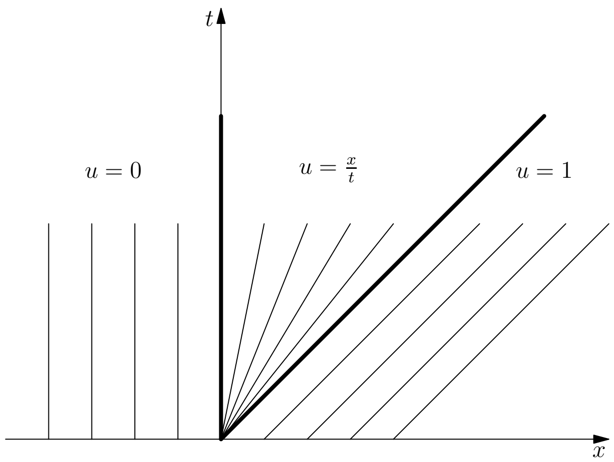

Figure 4.5: Rarefaction solution

4.4 An Important Remark

It is important to note that while u is a classical solution for the conservation law,

$$u_t + f(u)_x = 0$$

then it is also a classical solution for an equivalent equation. For example if we multiply equation (4.10) by u, we get

$$uu_t + u^2 u_x = 0 \tag{4.17}$$

which can be written in the conservation form

$$v_t + \left(\frac{2}{3}v^{3/2}\right)_x = 0 \tag{4.18}$$

where $v = u^2$. If u is the classical solution for (4.18), then it satisfies (4.17) in the ordinary sense. On the other hand if u is not a classical solution, then we will get a weak solution for (4.18) which will be different from the weak solution of the conservation law (4.11) since the RH-condition is different for these two laws. So it is important to choose the proper form of the conservation law which comes from physical conservation principles.

4.5 Non-uniqueness of Weak Solutions

We have got a solution for the IVP (4.11)-(4.16) in example (4.2) as

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 \le x \le t \\ 1 & \text{if } x > t \end{cases} \tag{4.19}$$

which is a continuous function but not differentiable. Thus, it is not a classical solution but satisfies the IVP (4.11)-(4.16) in the weak sense. Instead of filling the gap in the interval $0 < x < t$ by x/t, we could also fill it in such a way that u takes the form,

$$u(x,t) = \begin{cases} 0 & \text{if} \quad x < t/2 \\ 1 & \text{if} \quad x > t/2 \end{cases} \tag{$*$}$$

This solution in the $x - t$ plane is given in the following figure (4.6). Clearly, the curve of discontinuity satisfies the RH-condition. Only one of these solutions can have physical meaning (!!!). The question is which ? This needs an additional condition on the weak solutions called *entropy condition.*

Definition 4.1 (Entropy condition). *The condition that the characteristics starting on either side of the discontinuity curve when continued in the direction of increasing t intersect the line of discontinuity is called the Entropy condition.*

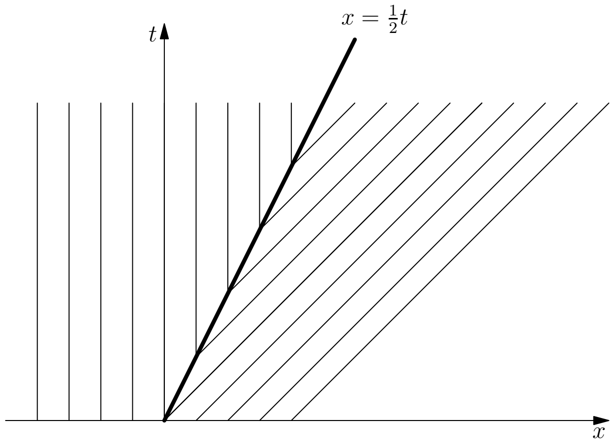

Figure 4.6: Entropy violating shock solution

This will be the case if,

$$c(u_l) > s > c(u_r) \tag{4.20}$$

where $c = f'$. Clearly, the solution given by $(*)$ will be ruled out if we impose the entropy condition on the weak solution. But since (4.19) is a continuous function, there is no question of entropy condition at all. Therefore (4.19) will be physically relevant solution.

Definition 4.2 (Shock Solution)**.** *The weak solution for the IVP*

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x) \tag{4.21}$$

is called the shock solution if it satisfies the RH-condition and the entropy condition. The curve of discontinuity is called a shock.

The task before us is to investigate whether every initial value problem (4.21) has exactly one shock solution for *all* $t > 0$. This question leads to the problem of existence and uniqueness of shock solution which will be discussed in the next chapter.

Chapter 5

Existence and Uniqueness of Shock Solution

5.1 Existence of Shock Solution

There are atleast four ways to prove the existence of a shock solution. They are

1. Vanishing viscosity method
2. Finite Difference method
3. Hamilton-Jacobi theory
4. Non-linear semi-group theory

Out of these we will take the Hamilton-Jacobi theory to prove the existence .

Let us consider the following IVP

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad (5.1) \\ u(x,0) &= u_0(x) \qquad (5.2) \end{aligned}$$

where f is strictly convex, i.e $f'' > 0$. Since divergence of f alone is involved in (5.1), we can without loss of generality, assume that

$$f(0) = 0 \qquad (5.3)$$

Let us first state a basic property of convex functions in the form of a theorem.

Theorem 5.1 (Convex function). *Let $g : I \to \mathbb{R}$ be convex, where $I \subset \mathbb{R}$ is any interval. Then if g' exists in the interior of I and if $c \in int(I)$, we have*

$$g(x) \geq g(c) + g'(c)(x - c) \qquad (5.4)$$

Proof See "Convex Analysis An Introductory Text" by Jan vab Tiel, page no. 4, Theorem 1.6.

Note Geometric meaning of (5.4) is that the tangent at a point $c \in (a, b)$ always lies below the graph of g.

5.2 Hamilton-Jacobi equation

Let us assume that u is a classical solution of the IVP (5.1)-(5.2). Assume for the moment that the support of u and u_0 are compact and define

$$U(x,t) = \int_{-\infty}^{x} u(y,t)dy \tag{5.5}$$

and

$$U_0(x) = \int_{-\infty}^{x} u_0(y)dy \tag{5.6}$$

Then, we have

$$\begin{aligned}
U_t(x,t) &= \frac{\partial}{\partial t}\int_{-\infty}^{x} u(y,t)dy \\
&= \int_{-\infty}^{x} u_t(y,t))dy \\
&= -\int_{-\infty}^{x} f(u(y,t))_y dy \\
&= -f(u(x,t)) + f(u(-\infty,t)) \\
&= -f(u(x,t)) + f(0) \\
&= -f(U_x)
\end{aligned}$$

since from (5.5), we have

$$U_x = u$$

Thus, U solves the Hamilton-Jacobi equation

$$\begin{aligned}
U_t + f(U_x) &= 0 &(5.7)\\
U(x,0) &= U_0(x) &(5.8)
\end{aligned}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. Since $f'' > 0, f'$ is strictly increasing and onto. Therefore $(f')^{-1}$ exists. Let us write

$$G = (f')^{-1} \tag{5.9}$$

Then from the theory of Hamilton-Jacobi equation, the weak solution of (5.7)-(5.8) is given by

$$U(x,t) = \min_{y \in \mathbb{R}} \left\{ U_0(y) + tg\left(\frac{x-y}{t}\right)\right\} \tag{5.10}$$

where g is defined by

$$\frac{\mathrm{d}g}{\mathrm{d}z} = G(z), \qquad g(f'(0)) = 0 \tag{5.11}$$

Theorem 5.2 (Lax-Oleinik Formula). *Let u be a classical solution of (5.1)-(5.2), then,*

$$u(x,t) = G\left(\frac{x-y_*}{t}\right) \tag{5.12}$$

where $y_* = y_*(x,t)$ is such that

$$U(x,t) = U_0(y_*) + tg\left(\frac{x-y_*}{t}\right) \tag{5.13}$$

Definition 5.1 (Lax-Oleinik formula). *Equation (5.12) is called the Lax-Oleinik formula.*

Proof of Theorem 5.2 Since $U_x = u$, we have from (5.4), for any $v \in \mathbb{R}$,

$$\begin{aligned} f(U_x) &\geq f(v) + f'(v)(U_x - v) \\ -U_t &\geq f(v) + f'(v)U_x - f'(v)v \end{aligned}$$

or

$$U_t + f'(v)U_x \leq f'(v)v - f(v) \tag{5.14}$$

Denoting by y the point where the line $\mathrm{d}x/\mathrm{d}t = f'(v)$ through (x,t) intersects the x-axis, then we have

$$\frac{x-y}{t} = f'(v) \tag{5.15}$$

or, by (5.9)

$$G\left(\frac{x-y}{t}\right) = v \tag{5.16}$$

Now integrating (5.14) along the line (5.15) from 0 to t we have

$$\int_0^t \frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \leq \int_0^t (f'(v)v - f(v))\mathrm{d}t$$

or

$$U(x,t) \leq U(y,0) + t[f'(v)v - f(v)] \tag{5.17}$$

Now, denote by g the function

$$g(z) = f'(v)v - f(v) \tag{5.18}$$

where

$$G(z) = v \tag{5.19}$$

We get from (5.19) and (5.9)

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}z} &= \frac{\mathrm{d}}{\mathrm{d}z}(F'(v)v - f(v)) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}(f'(G(z))G(z) - f(G(z))) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}(zG(z) - f(G(z))) \\
&= G(z) + z\frac{\mathrm{d}G}{\mathrm{d}z} - f'(v)\frac{\mathrm{d}G}{\mathrm{d}z} \\
&= G(z)
\end{aligned}$$

That is,

$$\frac{\mathrm{d}g}{\mathrm{d}z} = G(z)$$

and by (5.3), (5.18) and (5.19), we have

$$g(f'(0)) = 0$$

Thus, the function g defined by (5.18)-(5.19) satisfies (5.11), and therefore putting g in (5.17), we get

$$U(x,t) \leq U(y,0) + tg(z)$$

which from (5.16) takes the form

$$U(x,t) \leq U(y,0) + tg\left(\frac{x-y}{t}\right) \tag{5.20}$$

This inequality is true for all choices of y. In particular it is also true for that value of y, say y_*, for which

$$\frac{x-y_*}{t} = f'(u)$$

i.e, for $v = u(x,t)$ in (5.15) and therefore from (5.16)

$$u(x,t) = G\left(\frac{x-y_*}{t}\right)$$

So, all that remains is to show that this y_* satisfies (5.13). For y_*, we have from (5.15),

$$\frac{x-y_*}{t} = f'(u)$$

Therefore, we have

$$\begin{aligned} U_t + f'(u)U_x &= -f(u) + f'(u)u \\ &= f'(v)v - f(v) \end{aligned}$$

Thus, the equality holds in (5.14) and therefore in (5.20) which completes the proof.

Before going to the actual existence theorem, let us prove the following lemma which is useful for deriving entropy condition in the proof of the existence theorem.

Lemma 5.1. *For a fixed t, the mapping $x \mapsto y_*(x,t)$ is non-decreasing, where y_* is as in Theorem 5.2.*

Proof Let us denote the RHS of equation (5.13) by $U(x,y)$ which means that for a fixed t, at a point x, y is a value for which U defined by equation (5.10) attains it's minimum. From the theory of Hamilton-Jacobi Equation, such a y always exists. Let $x_1 < x_2$. Then we have to prove that $y_1 \leq y_2$ where $y_1 = y_*(x_1,t)$, $y_2 = y_*(x_2,t)$. (Remember that t is fixed). This is equivalent to saying that,

$$U(x_2,y_1) < U(x_2,y) \tag{5.21}$$

for all $y < y_1$. Since y_1 is a point at which $U(x_1,y)$ attains it's minimum, we have

$$U(x_1,y_1) < U(x_1,y) \tag{5.22}$$

We now claim that a

$$g\left(\frac{x_2-y_1}{t}\right) + g\left(\frac{x_1-y}{t}\right) < g\left(\frac{x_1-y_1}{t}\right) + g\left(\frac{x_2-y}{t}\right) \tag{5.23}$$

if $y < y_1$, where g is given by equation (5.18).

For this, we first claim that g is a strictly convex function. But this is obvious since g satisfies equation (5.11). That is

$$\frac{\mathrm{d}g}{\mathrm{d}z} = G(z)$$

and therefore

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} = G'(Z) > 0$$

Now for any $\lambda < \infty$, we can write

$$x_2 - y_1 = \lambda(x_1 - y_1) + (1 - \lambda)(x_2 - y)$$

and

$$x_2 - y = (1 - \lambda)(x_1 - y_1) + \lambda(x_2 - y)$$

In particular, if we take

$$\lambda = \frac{y_1 - y}{x_2 - x_1 + y_1 - y}$$

then $0 < \lambda < 1$ and therefore from the definition of strictly convex functions, we have

$$\begin{aligned} g\left(\frac{x_2 - y_1}{t}\right) &= g\left(\lambda\left(\frac{x_1 - y_1}{t}\right) + (1-\lambda)\left(\frac{x_2 - y}{t}\right)\right) \\ &< \lambda g\left(\frac{x_1 - y_1}{t}\right) + (1-\lambda) g\left(\frac{x_2 - y}{t}\right) \end{aligned}$$

and similarly,

$$g\left(\frac{x_1 - y}{t}\right) < (1-\lambda) g\left(\frac{x_1 - y_1}{t}\right) + \lambda g\left(\frac{x_2 - y}{t}\right)$$

Adding these two inequalities, we get equation (5.23) . Now, multiplying (5.23) by t and adding to (5.22), we get (5.21), which is that we wish to show.

Theorem 5.3 (Existence of shock solution)**.** *A function u is a shock solution for equation (5.1) if and only if it satisfies the Lax-Olenik formula (5.12)-(5.13) in the sense of distributions for arbitrary integrable data u_0.*

Proof First suppose that u is a shock solution. That is, u is a weak solution which satisfies the entropy condition. Since

$$\begin{aligned} U_t + f(U_x) &= \int_{-\infty}^{x} u_t dy + f(u) \qquad (\text{since } u = U_x) \\ &= \int_{-\infty}^{x} (u_t + f(u)_y) dy \end{aligned}$$

The relation (5.7) is the integral form of the conservation law (5.1) . Therefore, when f is convex, following the same steps as in the proof of Theorem 5.2, the inequality (5.20) holds in the sense of distributions. Moreover, since u satisfies the entropy condition, every point (x,t) can be connected to a point y on the initial line by a backward characteristic. For that value of y the sign of equality holds in (5.20) . Thus, Theorem 5.2 applies for shock solutions as well. That is, a shock solution satisfies the Lax-Olenik formula (5.12)-(5.13) in the sense of distributions.

Conversely, suppose that a function u satisfies the Lax-Olenik formula (5.12)-(5.13) in the case of distributions with u_0 integrable. Then, we have to show that u is a shock solution.

Let $\phi \in C_0^\infty(\Omega)$. Multiplying equation (5.7) by ϕ_x and integrating on $\mathbb{R} \times (0,\infty)$, we get

$$\int_0^\infty \int_{-\infty}^\infty [U_t + f(U_x)]\phi_x \mathrm{d}x\mathrm{d}t = 0 \tag{5.24}$$

Now,

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty U_t\phi_x \mathrm{d}x\mathrm{d}t &= -\int_0^\infty \int_{-\infty}^\infty U\phi_{tx} \mathrm{d}x\mathrm{d}t - \int_{-\infty}^\infty [U\phi_x]_{t=0}\mathrm{d}x \\ &= +\int_0^\infty \int_{-\infty}^\infty U_x\phi_t \mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty [U_x\phi]_{t=0}\mathrm{d}x \end{aligned}$$

Now, since

$$U(x,0) = \int_{-\infty}^x u_0(y)dy$$

we have

$$U_x(x,0) = u_0(x) \qquad a.e.$$

Consequently,

$$\int_0^\infty \int_{-\infty}^\infty U_t\phi_x \mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{-\infty}^\infty U_x\phi_t \mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x$$

Putting this in (5.24) and noting that $u = U_x$, we get

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t| + f(u)\phi_x]\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0$$

which is the weak formulation (see Chapter 4) for the conservation law (5.1). All that remains is to show that u satisfies the entropy condition. Let $x_1 < x_2$. Then by Lemma 5.1, since $y(x,t)$ is an increasing function of x, we have,

$$y_1 = y(x_1,t) < y(x_2,t) = y_2 \tag{5.25}$$

Also since G is an increasing function and since from (5.25), we have,

$$\frac{x_1 - y_1}{t} \geq \frac{x_1 - y_2}{t}$$

we have

$$G\left(\frac{x_1 - y_1}{t}\right) \geq G\left(\frac{x_1 - y_2}{t}\right)$$

Therefore, since u satisfies (5.12), we have,

$$u(x_1,t) = G\left(\frac{x_1 - y_1}{t}\right) \geq G\left(\frac{x_1 - y_2}{t}\right) \tag{5.26}$$

From the theory of Hamilton-Jacobi equations, we may assume that G is Lipschitz function i.e., $\exists$ a constant $K > 0$ such that

$$|G(x) - G(y)| \leq K|x - y|$$

Applying this for $x = (x_1 - y_2)/t$ and $y = (x_2 - y_2)/t$ and noting that $x < y$, we get

$$G\left(\frac{x_2 - y_2}{t}\right) - G\left(\frac{x_1 - y_2}{t}\right) \leq K\left(\frac{x_2 - x_1}{t}\right)$$

or,

$$G\left(\frac{x_1 - y_2}{t}\right) \geq G\left(\frac{x_2 - y_2}{t}\right) - K\left(\frac{x_2 - x_1}{t}\right)$$

Putting this in (5.26), we get

$$\begin{aligned} u(x_1,t) &\geq G\left(\frac{x_2 - y_2}{t}\right) - K\left(\frac{x_2 - x_1}{t}\right) \\ &= u(x_2,t) - K\left(\frac{x_2 - x_1}{t}\right) \end{aligned}$$

This implies that,

$$u(x_1,t) - K\frac{x_1}{t} \geq u(x_2,t) - K\frac{x_2}{t}$$

Since we have assumed that $x_1 < x_2$, the function $x \mapsto u(x,t) - K\frac{x}{t}$ is a non-increasing function. Thus, it has both left and right-hand limits at each point. Thus, if c be any point on the discontinuity curve, and x_1 and x_2 are such that $x_1 < c < x_2$, then from the above inequality, we have

$$u_l < u_r$$

which is the entropy condition. Thus u is a shock solution.

5.3 Uniqueness Result

Lemma 5.2 (L^1 contraction)**.** *Let u and v be two weak solutions (5.1)-(5.2) such that all discontinuities of both u and v are shocks. Then*

$$\|u(t) - v(t)\|_{L^1}$$

is a decreasing function in t, where the norm is with respect to x and is defined as

$$\|u(t) - v(t)\|_{L^1} := \int\limits_{\Omega} |u(x,t) - v(x,t)|dx$$

Note Any weak solution of (1) is a piecewise smooth function. This follows from Lemma (4.1).

Proof Let us partition Ω into $y_1, y_2, \ldots y_n \ldots$ in such a way that
(i) $y_1 < y_2 < \ldots < y_n < \ldots$
(ii) $u(x,t) - v(x,t)$ has the sign $(-1)^n$ for $y_n < x < y_{n+1}$.
Observe that the y_n are function of t. Then we can write

$$\|u(t) - v(t)\|_{L_1} = \sum_{n=1}^{\infty} (-1)^n \int\limits_{y_n}^{y_{n+1}} (u(x,t) - v(x,t))\mathrm{d}x \tag{5.27}$$

Differentiating (5.27) with respect to t, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - v(t)\|_{L^1} \quad = \quad \sum_{n=1}^{\infty} \Bigg[& \int\limits_{y_n}^{y_{n+1}} \frac{\partial}{\partial t}(u(x,t) - v(x,t))\mathrm{d}x + \\ & (u-v)(y_{n+1},t)\frac{\mathrm{d}y_{n+1}}{\mathrm{d}t} - (u-v)(y_n,t)\frac{\mathrm{d}y_n}{\mathrm{d}t}\Bigg]\end{aligned}$$

Since u and v are weak solutions for (5.1)-(5.2), they satisfy $u_t = -f(u)_x$ and $v_t = -f(v)_x$ in the sense of distribution. Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - v(t)\|_{L^1} = \sum_{n=1}^{\infty} (-1)^n (f(v) - f(u))(y,t) + (u-v)(y,t)\frac{\mathrm{d}y}{\mathrm{d}t}\Bigg|_{y_n}^{y_{n+1}} \tag{5.28}$$

Now two cases arise at the point y_n.

Case 1 Both u and v are continuous at y_n. In this case, since u and v are piecewise smooth, we have

$$u(y_n,t) = v(y_n,t)$$

Since both u and v are smooth, they take the general form of the solution for (5.1) (Refer to equation (3.5), Chapter 3).

Case 2 One of the functions u or v is discontinuous at y_n. Let us consider one typical term of (5.28).

$$(-1)^n(f(v)-f(u))(y,t)+(u-v)(y,t)\frac{\mathrm{d}y}{\mathrm{d}t}\bigg|_{y_n}^{y_{n+1}} \tag{5.29}$$

In Case (1), the whole term becomes zero. So, we will consider Case (2). Suppose that u is discontinuous at $y = y_{n+1}$. Then, by the hypothesis, u has a shock at $y = y_{n+1}$. Therefore, we have $u_r < u_l$. Suppose that

$$u_r < v < u_l$$

where $v = v(y)$. Then $u - v$ is positive in (y_n, y_{n+1}), and hence n is even. According to the R-H condition, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{f(u_l)-f(u_r)}{u_l-u_r}$$

Substituting this in (5.29), we get at y_{n+1} with $u = u_l$,

$$f(v)-f(u_l)+(u_l-v)\frac{f(u_l)-f(u_r)}{u_l-u_r} = f(v) - \left[\frac{v-u_r}{u_l-u_r}f(u_l)+\frac{u_l-v}{u_l-u_r}f(u_r)\right]$$

Since f is a convex function and since $v \in (u_r, u_l)$, it follows that the RHS is negative. Similarly, we can show the same at y_n. This shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}||u(t)-v(t)\|_{L_1} \le 0$$

i.e. $\|u(t) = -v(t)\|_{L^1}$ is a decreasing function in t.

Theorem 5.4 (Uniqueness of Shock Solution)**.** *Let u and v be any two weak solutions for the IVP (5.1)-(5.2), then $u = v$, for all $t > 0$.*

Proof Since initially, i.e. at $t = 0$, $u = v = u_0$, we have

$$\|u(0)-v(0)\|_{L^1} = 0 \tag{5.30}$$

Since by Lemma (5.2), $\|u(t)-v(t)\|_{L^1}$ is a decreasing function in t, it follows from (5.30) that

$$\begin{aligned}
\|u(t)-v(t)\|_{L^1} &\le 0, \qquad \forall t \ge 0\\
\|u(t)-v(t)\|_{L^1} &= 0, \qquad \forall\, t \ge 0\\
u(t) &= v(t), \qquad \forall\, t \ge 0
\end{aligned}$$