Scalar Conservation Laws

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First Order Quasi-linear Hyperbolic Equations

$$\begin{aligned} u_t + c(u)u_x &= 0, \qquad x \in \mathbb{R}, \qquad t \in \mathbb{R}^+ \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R} \end{aligned}$$

Charactertistic equations

$$\left. \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x(t) &= c(u(x(t),t)) \\ \frac{\mathrm{d}}{\mathrm{d}t} u(x(t),t) &= 0 \end{aligned} \right\} \tag{1}$$

- u is constant along a characteristic.
- Since $c(u)$ is the slope, characteristics are straight lines.

Hence

$$x = c(u)t + x_0, \qquad u(x,t) = u_0(x_0)$$

$$x = c(u_0(x_0))t + x_0 \qquad \Longrightarrow \qquad x_0 = x_0(x,t)$$

$$u(x,t) = u_0(x_0(x,t))$$

Classical solution

Assume that c and u_0 are C^1-functions.

$$u_t = u_0'(x_0)x_{0_t}, \qquad u_x = u_0'(x_0)x_{0_x}$$

$$x_{0_t} = \frac{-c(u_0(x_0))}{1 + c'(u_0(x_0))u_0'(x_0)t}, \qquad x_{0_x} = \frac{1}{1 + c'(u_0(x_0))u_0'(x_0)t}$$

Therefore, we have

$$u_t = \frac{-c(u_0(x_0))u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} \qquad u_x = \frac{u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t}$$

Since c and u_0 are C^1-functions, the RHS of the above two equations exist and therefore u is a C^1-function. And

$$u_t + c(u)u_x = \frac{-c(u_0(x_0))u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} + c(u_0(x_0))\frac{u_0'(x_0)}{1 + c'(u_0(x_0))u_0'(x_0)t} = 0$$

Loss of regularity

Suppose that

$$c'(u_0) > 0, \qquad u_0'(x) < 0 \quad \Longrightarrow \quad c'(u_0)u_0'(x_0) < 0$$

Therefore, if we take

$$t = -\frac{1}{c'(u_0(x_0))u_0'(x_0)}$$

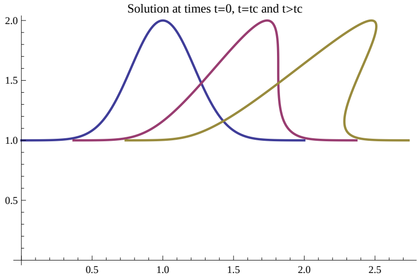
the slope of u with respect to x (of course to t also) becomes infinite. Thus, if we define

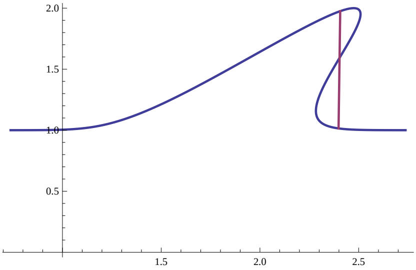
$$t_c := -\frac{1}{\min_x c'(u_0(x))u_0'(x)} \tag{2}$$

then, t_c will be minimum time at which the slope of u with respect to both x and t becomes infinite. We call t_c as the *critical time.*

For $t > t_c$, the solution becomes multi-valued.

Loss of regularity





- Solution becomes discontinuous at $t = t_c$ and then becomes multi-valued.
- To remove the multi-valuedness, we put a discontinuity in the solution, which then propagates with some speed s.
- Equal area rule: conservation principle
- Formalized through the notion of distributions and weak solutions.

Loss of regularity

- u discontinuous in x implies $c(u)$ is discontinuous in x and u_x has a Dirac mass, it does not make sense to talk of

$$c(u)\frac{\partial u}{\partial x}$$

- Assume $c(u) = f'(u)$ so the PDE is

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

  This is known as a conservation law. In practice such models arise from physical conservation principle, like mass, momentum and energy.

Conservation laws

Scalar conservation law

$$u_t + f(u)_x = 0$$

Integrate over any spatial domain $[a, b]$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x &= \int_a^b u_t \mathrm{d}x = -\int_a^b f_x \mathrm{d}x \\ &= f(u(a,t)) - f(u(b,t)) \\ &= \text{[inflow at a] - [outflow at b]}\end{aligned}$$

Conservation principle

u is neither created nor destroyed; the total amount of u contained inside any given interval $[a, b]$ can change only due to flux of u across the two end points.

- u is called a conserved quantity
- $f(u)$ is its flux
- Conservation principle is more fundamental; does not require derivatives.
- Under smoothness assumption, we get the PDE

Burgers equation

Famous non-linear hyperbolic PDE

$$u_t + uu_x = 0$$

or in conservation form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

The flux function

$$f(u) = \frac{u^2}{2}$$

is smooth and convex.

Remark: The viscous Burgers equation

$$u_t + uu_x = \nu u_{xx}, \qquad \nu > 0$$

is the simplest model for incompressible Navier-Stokes equations, since it has the quadratic non-linearity and dissipation.

Burgers equation: Shock solution

Initial condition

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Method of characteristics ($c(u) = f'(u) = u$)

$$u(x,t) = u_0(x_0), \qquad x_0 = x - u(x,t)t = x - u_0(x_0)t$$

which leads to

$$x_0 = \begin{cases} x - t & \text{if } x < t < 1 \\ \frac{x-t}{1-t} & \text{if } t \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$$

and the solution is

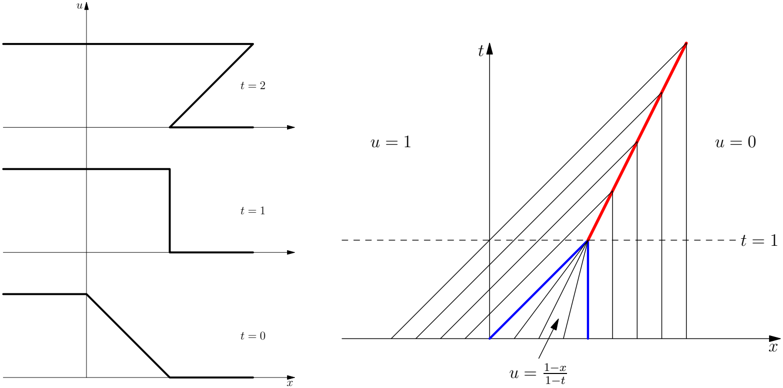
$$u(x,t) = \begin{cases} 1 & \text{if } x < t < 1 \\ \dfrac{1-x}{1-t} & \text{if } t \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$t \le x \le 1 \quad \Longrightarrow \quad u(x,t) = u_0(x_0) = 1 - x_0 = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}$$

Burgers equation: Shock solution

Regularity of solution: Observe that $u_0' < 0$ for $0 \le x \le 1$, so the critical time

$$t_c = -\frac{1}{\min_x u_0'(x)} = -\frac{1}{(-1)} = 1$$



Burgers equation: Shock solution

At $t = 1$ a shock is formed at $x = 1$ with $u_l = 1$ and $u_r = 0$, and this must propagate at a speed given by RH condition (we will see this soon)

$$s = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_l^2}{u_r - u_l} = \frac{1}{2}(u_l + u_r) = \frac{1}{2}$$

Equation of shock line

$$x = \frac{t}{2} + \mathrm{const} = \frac{1}{2}(t+1) \qquad x(1) = 1$$

For $t > 1$ the solution is given by

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}(t+1) \\ 0 & \text{if } x > \frac{1}{2}(t+1) \end{cases}$$

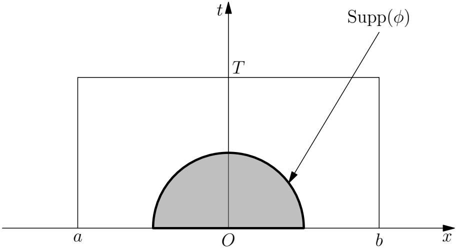
Weak solution

Initial value problem

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \qquad u(x,0) = u_0(x)$$

Test functions compactly supported in $\mathbb{R} \times \mathbb{R}^+$

$$\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$$



$$\int_0^\infty \int_{\mathbb{R}} (u_t + f_x)\phi \mathrm{d}x\mathrm{d}t = 0$$

We can find $D = [a,b] \times [0,T]$ such that $supp(\phi) \subset D$

Remark: ϕ need not vanish on $t = 0$.

Weak solution

Applying integration by parts, we get,

$$
\begin{aligned}
\int_a^b \int_0^T u_t \phi \mathrm{d}t \mathrm{d}x &= \int_a^b u(x,t)\phi(x,t)\Big|_{t=0}^{t=T} \mathrm{d}x - \int_a^b \int_0^T u(x,t)\phi_t(x,t) \mathrm{d}x \mathrm{d}t \\
&= -\int_a^b u(x,0)\phi(x,0)\mathrm{d}x - \int_a^b \int_0^T u\phi_t \mathrm{d}x \mathrm{d}t
\end{aligned}
$$

since $\phi(x,T) = 0$ and

$$
\int_0^T \int_a^b f_x \phi \mathrm{d}t \mathrm{d}x = \int_0^T f\phi\Big|_{x=a}^{x=b} \mathrm{d}t - \int_0^T \int_a^b f\phi_x \mathrm{d}x \mathrm{d}t = -\int_0^T \int_a^b f\phi_x \mathrm{d}x \mathrm{d}t
$$

Therefore, we have

$$
\int_0^\infty \int_{-\infty}^\infty (u\phi_t + f(u)\phi_x) \mathrm{d}x \mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0
$$

Weak solution

Definition: Weak solution

A function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a weak solution of the IVP

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \qquad u(x,0) = u_0(x)$$

together with locally integrable initial data u_0 if u is locally integrable and satisfies

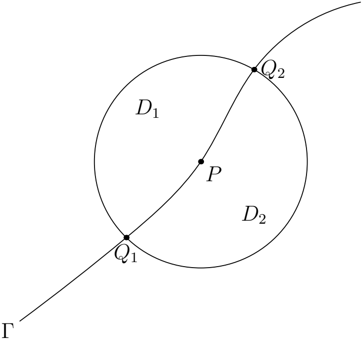
$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + f(u)\phi_x)\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(x,0)\mathrm{d}x = 0, \qquad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$$

Lemma: Classical solution

Let $u \in C^1(\mathbb{R} \times \mathbb{R}^+)$ be a weak solution. Then it is a classical solution.

Rankine-Hugoniot (RH) condition

Weak solutions can be discontinuous. However not every type of discontinuity is admissible. The solution across the curve of discontinuity Γ must satisfy certain **compatibility condition**. Let P be any point on Γ and let D be a small ball centered at P. We assume that in D, Γ is given by $x = x(t)$. Let D_1 and D_2 be the components of D which are determined by Γ. Let $\phi \in C_0^1(D)$,



$$0 \;=\; \int\limits_D (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t = \int\limits_{D_1} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t + \int\limits_{D_2} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t$$

Since u is smooth in D_1 and D_2, we have, for $i = 1, 2$

$$\int\limits_{D_i} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t = \int\limits_{D_i} [(u\phi)_t + (f\phi)_x]\mathrm{d}x\mathrm{d}t - \int\limits_{D_i} (u_t + f_x)\phi\mathrm{d}x\mathrm{d}t$$

Rankine-Hugoniot (RH) condition

From Green's theorem

$$\int_{D_i} [(u\phi)_t + (f\phi)_x]\mathrm{d}x\mathrm{d}t = \int_{\partial D_i} (un_t + fn_x)\phi \mathrm{d}\sigma$$

$$\text{But} \qquad n_t = -\frac{\mathrm{d}x}{\mathrm{d}\sigma}, \qquad n_x = \frac{\mathrm{d}t}{\mathrm{d}\sigma}$$

we get,

$$\int_{D_i} (u\phi_t + f\phi_x)\mathrm{d}x\mathrm{d}t = \int_{\partial D_i} (-u\mathrm{d}x + f\mathrm{d}t)\phi$$

This leads to,

$$\int_{\partial D_1} (-u\mathrm{d}x + f\mathrm{d}t)\phi + \int_{\partial D_2} (-u\mathrm{d}x + f\mathrm{d}t)\phi = 0$$

But, $\phi \equiv 0$ on ∂D, and may not be zero on Γ. Thus, if we denote,

$$\begin{aligned} u_l &= u_l(x(t), t) = u(x(t) - 0, t) \\ u_r &= u_r(x(t), t) = u(x(t) + 0, t) \end{aligned}$$

Rankine-Hugoniot (RH) condition

then we have,

$$\int_{Q_1}^{Q_2} (-u_l \mathrm{d}x + f_l \mathrm{d}t)\phi - \int_{Q_1}^{Q_2} (-u_r \mathrm{d}x + f_r \mathrm{d}t)\phi = 0$$

$$\int_{Q_1}^{Q_2} \left[(u_r - u_l)\frac{\mathrm{d}x}{\mathrm{d}t} - (f_r - f_l)\right]\phi \mathrm{d}t = 0$$

Since this is true $\forall \phi \in C_0^1(D)$, we have

$$s := \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f_r - f_l}{u_r - u_l}, \qquad \text{speed of propagation of discontinuity}$$

A discontinuity in the solution is admissible provided it satisfies the RH condition

$$f(u_r) - f(u_l) = s \cdot (u_r - u_l)$$

Rankine-Hugoniot (RH) condition

Remark: If the discontinuity is stationary $(s = 0)$ then $f_l = f_r$

Remark: The shock speed is same for the discontinuity from $u_l \to u_r$ and for $u_r \to u_l$ since the RH condition is symmetric. Both discontinuities are admissible weak solutions. However we will see that uniquess requires us to restrict the possible discontinuities.

Weak solution

A weak solution u is a piecewise smooth solution which satisfies the RH condition at the points of discontinuity of u.

Conservation law: Important remark

For smooth solutions, Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \Longrightarrow \quad u_t + u u_x = 0$$

can also be written as

$$u u_t + u^2 u_x = 0, \qquad (u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0$$

which looks like another conservation law. For the second equation, the RH condition is

$$s = \frac{\frac{2}{3}u_r^3 - \frac{2}{3}u_l^3}{u_r^2 - u_l^2} = \frac{2(u_l^2 + u_l u_r + u_r^2)}{3(u_l + u_r)} \neq \frac{1}{2}(u_l + u_r)$$

The shock speeds are different for the two conservation laws and hence they are not equivalent for weak solutions. The physical origin of conservation is important and it should not be mathematically manipulated (unless sufficient care is taken).

Burgers equation: Rarefaction solution

Initial condition

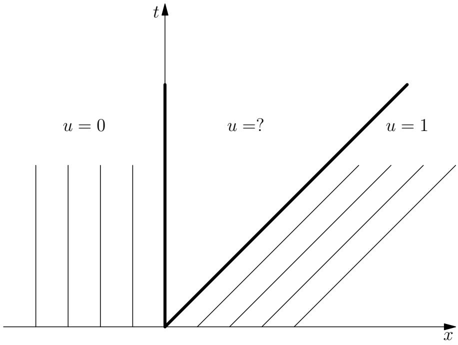
$$u_0(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Foot of characteristic

$$x_0 = \begin{cases} x & \text{if } x < 0 \\ x - t & \text{if } x > t \end{cases}$$

so that

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > t \end{cases}$$



For $x \in (0, t)$, the characteristic can be drawn such that it intersects at $x_0 = 0$

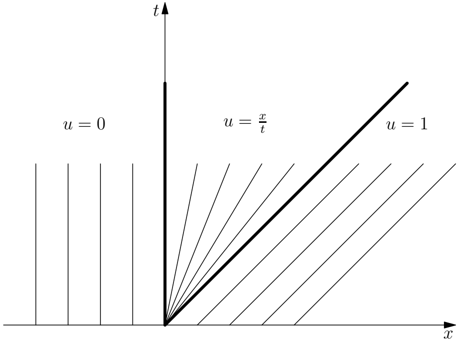
$$x = ut + x_0, \quad \text{and} \quad x_0 = 0, \quad \Longrightarrow \quad u(x,t) = \frac{x}{t}$$

This satisfies the PDE, $u_t + uu_x = -\frac{x}{t^2} + \frac{x}{t} \cdot \frac{1}{t} = 0$

Burgers equation: Rarefaction solution

Thus the solution can be completed as

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 \le x \le t \\ 1 & \text{if } x > t \end{cases}$$



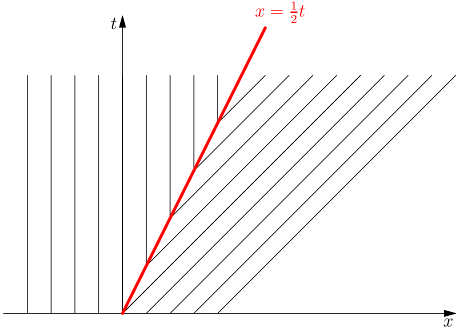
Plot the solution at any time $t > 0$. This solution is continuous but has some corners where derivatives are not defined. So this is still a weak solution of the conservation law.

Burgers equation: Non-uniqueness

We can introduce a discontinuous solution satisfying the RH condition

$$u(x,t) = \begin{cases} 0 & \text{if } x < \frac{1}{2}t \\ 1 & \text{if } x > \frac{1}{2}t \end{cases}$$

This is also a weak solution. Thus there seem to be multiple weak solutions, and this is a general feature of non-linear conservation laws.



- The characteristics show that causality is violated by this solution; characteristics are emanating from the shock line but they do not determine the future solution since we do not have data on the shock line.
- The shock solution is also unstable as is clear by smoothening the initial condition. Consider a smoothened initial condition and draw characteristics.

Entropy condition I

A shock should have characteristics going into the shock as time advances. A discontinuity propagating with speed s given by the RH condition satisfies the entropy condition if

$$f'(u_l) > s > f'(u_r)$$

Remark: For a convex flux $f(u)$, the entropy condition becomes

$$f'(u_l) > f'(u_r) \quad \Longrightarrow \quad u_l > u_r$$

Entropy condition II (Oleinik)

$u(x,t)$ is the entropy solution if all discontinuities have the property that

$$\frac{f(u) - f(u_l)}{u - u_l} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r}$$

for all u between u_l and u_r.

Entropy condition III (Oleinik)

$u(x,t)$ is the entropy solution if there is a constant $E > 0$ such that for all $a > 0$, $t > 0$ and $x \in \mathbb{R}$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}$$

Check this condition for a shock and rarefaction.

Remark: This condition is useful to check the entropy condition for numerical schemes. The numerical solution is known only at the grid points. Taking $a = \Delta x$, we must ensure that there is a constant $E > 0$ such that

$$U_{j+1}^n - U_j^n < \left(\frac{E}{t^n}\right) \Delta x$$

Weak solutions of non-linear conservation laws are not unique; by imposing an additional entropy condition, we obtain uniqueness of weak solutions. Such a solution is also refered to as an **entropy solution**.

Riemann problem for Burgers equation

Initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

If $u_l > u_r$, solution is a shock with speed $s = \frac{1}{2}(u_l + u_r)$

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}$$

If $u_l < u_r$, solution is a rarefaction

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ \frac{x}{t} & u_l t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$$

These solutions are self-similar, i.e, we can write them as

$$u(x,t) = u(x/t)$$

Riemann problem for general convex flux

Initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

If $u_l > u_r$, solution is a shock with speed $s = \frac{f(u_r)-f(u_l)}{u_r-u_l}$

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}$$

If $u_l < u_r$, solution is a rarefaction

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ (f')^{-1}(\frac{x}{t}) & f'(u_l)t \le x \le f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases}$$

These solutions are self-similar, i.e, we can write them as

$$u(x,t) = u(x/t)$$

There is always viscosity

Viscous Burgers equation

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon, \qquad \epsilon > 0$$

has smooth solutions for all time. These solutions can be obtained using Cole-Hopf transformation.

One can take the limit of the viscous solution u^ϵ by letting $\epsilon \to 0$. The limiting solution is the unique entropy solution.

Entropy function

Consider a convex scalar conservation law

$$u_t + f(u)_x = 0$$

Assume that there exists a convex function $\eta(u)$ and another function $\theta(u)$ such that

$$\eta'(u) f'(u) = \theta'(u)$$

Such a pair (η, θ) is called an entropy-entropy flux pair. For Burgers equation, we can choose

$$\eta(u) = u^2, \qquad \theta(u) = \frac{2}{3} u^3$$

For smooth solutions

$$u_t + f'(u) u_x = 0, \qquad \eta'(u) u_t + \eta'(u) f'(u) u_x = 0,$$

leads to another conservation law

$$\eta_t + \theta_x = 0$$

Entropy function

This equality cannot hold for discontinuous solutions; if it did, then

$$\theta(u_r) - \theta(u_l) = s \cdot (\eta(u_r) - \eta(u_l))$$

However this is in general incompatible with the RH condition for the original conservation law. In reality, the conservation law includes some dissipation

$$u_t + f_x = \epsilon u_{xx} \quad \Longrightarrow \quad \eta'(u)u_t + \eta'(u)f'(u)u_x = \epsilon\eta'(u)u_{xx}$$

leads to the entropy equation

$$\eta_t + \theta_x = \epsilon(\eta(u)u_x)_x - \epsilon\eta''(u)u_x^2 \le \epsilon(\eta(u)u_x)_x \quad \text{since} \quad \eta''(u) > 0$$

In the limit of $\epsilon \to 0$, we get

$$\eta_t + \theta_x \le 0$$

This condition must be satisfied in weak sense for all $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$, $\phi \ge 0$

$$\int_0^\infty \int_{\mathbb{R}} (\eta(u)\phi_t + \theta(u)\phi_x)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \eta(u_0(x))\phi(x,0)\mathrm{d}t \ge 0$$

Entropy function

Entropy condition IV

$u(x,t)$ is the entropy solution if for all convex entropy functions η and corresponding entropy fluxes θ, the inequality

$$\eta_t + \theta_x \leq 0$$

is satisfied in the weak sense.

Across a discontinuity, this is equivalent to

$$\theta(u_r) - \theta(u_l) \leq s \cdot (\eta(u_r) - \eta(u_l))$$

For the Burgers equation, we get

$$\frac{2}{3}(u_r^3 - u_l^3) \leq \frac{1}{2}(u_r + u_l)(u_r^2 - u_l^2) \implies \frac{1}{6}(u_r - u_l)^3 \leq 0$$

and we recover the entropy condition for a shock as $u_l > u_r$

Entropy function

GR1, Theorem 3.4

If $u(x,t)$ satisfies entropy condition for one strictly convex entropy η, then it satisfies the entropy condition for all convex entropies.

Remark: Kruzkov used the entropy pair

$$\eta(u) = |u-k|, \qquad \theta(u) = \mathrm{sign}(u-k)[f(u)-f(k)], \qquad k \in \mathbb{R}$$

Remark: To check entropy condition for numerical scheme, we verify the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \eta(u(x,t))\mathrm{d}x + \theta(u(b,t)) - \theta(u(a,t)) \leq 0$$

for the finite volume method.

Kruzkov's result

The scalar Cauchy problem

$$u_t + f(u)_x = 0, \quad f \in C^1(\mathbb{R})$$

with initial condition

$$u(0,x) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R})$$

has a unique entropy solution

$$u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$$

which fulfills (important for numerics)

1. Stability: $||u(t,\cdot)||_{L^\infty} \leq ||u_0||_{L^\infty}$, a.e. in $t \in \mathbb{R}_+$
2. Monotone: if $u_0 \geq v_0$ a.e. in $\mathbb{R}$, then

$$u(t,\cdot) \geq v(t,\cdot) \quad \text{a.e. in } \mathbb{R}, \text{ a.e. in t} \in \mathbb{R}_+$$

Kruzkov's result

3. TV-diminishing: if $u_0 \in BV(\mathbb{R})$ then

$$u(t,\cdot) \in BV(\mathbb{R}) \quad \text{and} \quad TV(u(t,\cdot)) \leq TV(u_0)$$

4. Conservation: if $u_0 \in L^1(\mathbb{R})$ then

$$\int_{\mathbb{R}} u(t,x)\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\mathrm{d}x, \qquad \text{a.e. in } t \in \mathbb{R}_+$$

5. Finite domain of dependence: if u, v are two entropy solutions corresponding to $u_0, v_0 \in L^\infty$ and

$$M = \max_{\phi}\{|f'(\phi)| : |\phi| \leq \max(||u_0||_{L^\infty}, ||v_0||_{L^\infty})\}$$

then

$$\int_{|x|\leq R} |u(t,x) - v(t,x)|\mathrm{d}x \leq \int_{|x|\leq R+Mt} |u_0(x) - v_0(x)|\mathrm{d}x$$

Traffic flow

$$\rho = \text{Number of cars per unit length}$$
$$u = \text{velocity of car}$$

Number of cars is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho \mathrm{d}x = \rho(a,t)u(a,t) - \rho(b,t)u(b,t) = -\int_a^b \frac{\partial}{\partial x}(\rho u)\mathrm{d}x$$

If things are smooth, then we get the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

In order to close the model, assume a dependance of u on ρ. (1) All cars are of same type. (2) If $\rho = 0$ (there is no other car) then you drive at maximum speed u_m. (3) When $\rho = \rho_m$ (cars are bumper to bumper), your speed is zero.

$$u = u(\rho) = u_m \left(1 - \frac{\rho}{\rho_m}\right)$$

Traffic flow

Therefore flux function is

$$f(\rho) = \rho u_m \left(1 - \frac{\rho}{\rho_m}\right)$$

Characteristic speed

$$f'(\rho) = u_m \left(1 - \frac{2\rho}{\rho_m}\right)$$

Shock speed

$$s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} = u_m \left(1 - \frac{\rho_l + \rho_r}{\rho_m}\right)$$

- Entropy condition $f'(\rho_l) > f'(\rho_r)$ implies that $\rho_l < \rho_r$ which is opposite to the condition for Burgers equation, because the flux function in traffic flow model is a concave function.
- If $\rho_l > \rho_r$ we get a rarefaction solution.

Traffic flow

Example: Shock solution. Let $0 < \rho_l < \rho_r < \rho_m$

$$\rho(x,0) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases}$$

Take $\rho_r = \rho_m$ and $\rho_l < \rho_m$.

$$\begin{aligned} s &= u_m \left(1 - \frac{\rho_l + \rho_m}{\rho_m}\right) \\ &= -u_m \frac{\rho_l}{\rho_m} \\ &< 0 \end{aligned}$$

In figure, $\rho_l = \frac{1}{2}\rho_m$

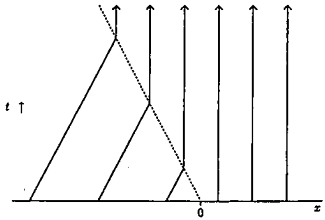


Figure 4.1. Traffic jam shock wave (vehicle trajectories), with data $\rho_l = \frac{1}{2}\rho_{\max}$, $\rho_r = \rho_{\max}$.

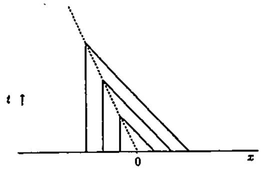


Figure 4.2. Characteristics.

Traffic flow

Example: Rarefaction solution. Let $0 < \rho_r < \rho_l < \rho_m$. In particular take

$$\rho_l = \rho_m, \qquad \rho_r = \frac{1}{2}\rho_m$$

(Figures on left)

Example: Situation modeling cars stopped at a traffic signal

$$\rho_l = \rho_m, \qquad \rho_r = 0$$

(Figure not shown)

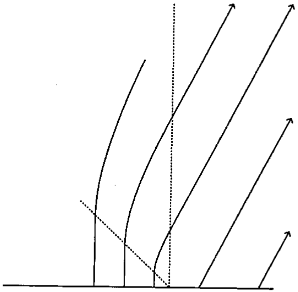


Figure 4.3. Rarefaction wave (vehicle trajectories), with data $\rho_l = \rho_{max}$, $\rho_r = \frac{1}{2}\rho_{max}$.

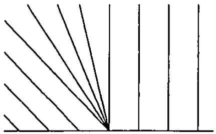


Figure 4.4. Characteristics.

Two phase flow (Buckley-Leverett equation)

Secondary oil recovery from a porous medium of rock/sand by pumping water

$$u = \text{saturation of water } \in [0, 1]$$

$u = 1$ means there is water, $u = 0$ means there is oil, $0 < u < 1$ means there is a mixture of water and oil. The flux is given by

$$f(u) = \frac{u^2}{u^2 + a(1-u)^2}$$

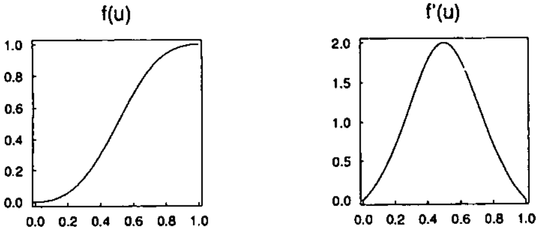


Figure 4.6. Flux function for Buckley-Leverett equation.

Two phase flow (Buckley-Leverett equation)

Example: Riemann problem with $u_l = 1$ and $u_r = 0$

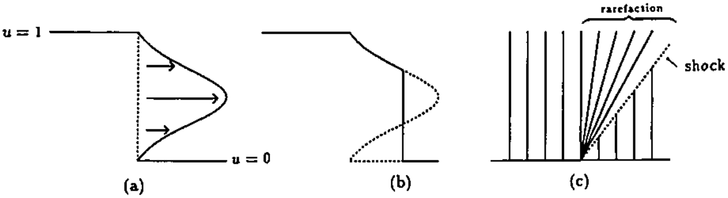


Figure 4.7. Riemann solution for Buckley-Leverett equation.

Problem: Use equal area rule to find an expression for shock location as a function of time t. Verify that RH condition is satisfied. Show that shock moves with constant speed and post shock value u^* is constant with time.

Two phase flow (Buckley-Leverett equation)

Geometric solution method: Connect u_l, u_r through shocks and rarefactions, respecting RH condition and entropy condition. Take $u_l > u_r$ and construct convex hull of the set

$$\{(x, y) : u_r \leq x \leq u_l,\ y \leq f(x)\}$$

E.g.: $u_l = 1$, $u_r = 0$

- Move from u_l to u^* through a rarefaction
- Jump from u^* to u_r through a shock

$$s = \frac{f(u_r) - f(u^*)}{u_r - u^*} = f'(u^*)$$

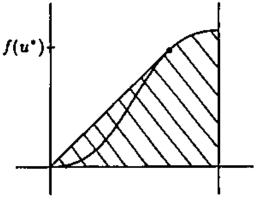


Figure 4.8. Convex hull showing shock and rarefaction.

- Intermediate state $\hat{u} < u^*$, shock speed $\frac{f(u_r)-f(\hat{u})}{u_r-\hat{u}} < f'(\hat{u})$ leads to triple valued solution.
- If $\hat{u} > u^*$ then entropy condition is violated.