Finite Element Method for Elliptic equations

1-D boundary value problem

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Two-point boundary value problem

$$Au := -(au')' + cu = f \quad \text{in} \quad \Omega = (0,1)$$
$$u(0) = u(1) = 0$$

where $a = a(x)$, $c = c(x)$ are given bounded functions, with

$$a(x) \geq a_0 > 0, \qquad c(x) \geq 0 \quad \text{in} \quad \bar{\Omega}$$

and $f \in L_2(\Omega)$.

Variational formulation

Find $u \in H_0^1(\Omega)$ such that

$$a(u, \phi) = (f, \phi), \quad \forall \phi \in H_0^1(\Omega) \tag{1}$$

where

$$a(v, w) = \int_\Omega (av'w' + cvw)\mathrm{d}x \quad \text{and} \quad (f, w) = \int_\Omega fw\mathrm{d}x$$

From Lax-Milgram Theorem, we know that this problem has a unique solution $u \in H_0^1(\Omega)$ which satisfies

$$\|u\|_1 \leq C \|f\|$$

Moreover, u is actually more regular, i.e., $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\|u\|_2 \leq C \|f\|$$

The essential properties required to apply Lax-Milgram Theorem are given next.

Properties of $a(u, v)$

Poincare inequality: For all $u \in H_0^1(\Omega)$

$$\|u\| \leq |u|_1 = \left\|u'\right\|$$

Hence $\|u'\|$ is an equivalent norm on $H_0^1(\Omega)$. For all $u, v, w \in H_0^1(\Omega)$

1. Bilinear

$$a(u, \alpha_1 v + \alpha_2 w) = \alpha_1 a(u, v) + \alpha_2 a(u, w), \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

2. Symmetric

$$a(u, v) = a(v, u)$$

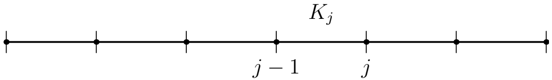
3. Coercive

$$a(v, v) = \int_\Omega [a(x)(v')^2 + c(x)v^2]\mathrm{d}x \geq a_0 \|v'\|^2$$

4. Continuous

$$|a(u, v)| \leq C\|u'\| \; \|v'\|$$

Finite element grid and approximation space

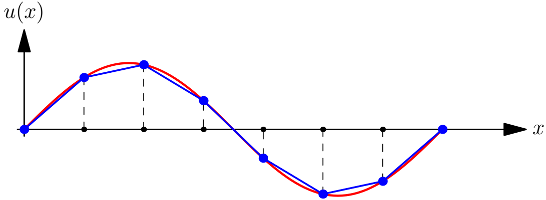


Consider a partition of $\Omega = (0,1)$ by the vertices

$$0 = x_0 < x_1 < \ldots < x_M = 1$$

and define for $j = 1, 2, \ldots, M$

$$h_j = x_j - x_{j-1}, \quad K_j = (x_{j-1}, x_j), \quad h = \max_j h_j, \quad \Omega = \cup_{j=1}^{M} K_j$$



Finite element grid and approximation space

Finite dimensional space approximating the space $H_0^1(\Omega)$: Space of continuous, piecewise polynomials of degree one

$$V_h = \{v \in C(\bar{\Omega}) : v|_{K_j} \in P_1(K_j),\ v(0) = v(1) = 0\} \subset H_0^1(\Omega)$$

Let us find a set of basis functions for V_h so that we can write for $v \in H_0^1(\Omega)$ its piecewise linear interpolant $I_h v \in V_h$

$$I_h v(x) = \sum_{j=1}^{M-1} v(x_j)\phi_j(x)$$

i.e.,

$$I_h v(x_i) = v(x_i), \qquad i = 1, 2, \ldots, M-1$$

Finite element basis

$\phi_j \in P_1$ should have the interpolation property:

$$\phi_j(x) = \begin{cases} 1 & x = x_j \\ 0 & x = x_k, \quad k \neq j \end{cases}$$

This condition is satisfied by

$$\phi_j(x) = \begin{cases} 1 + \frac{x - x_j}{h_j} & x_{j-1} \leq x \leq x_j \\ 1 - \frac{x - x_j}{h_{j+1}} & x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

Any function $v_h \in V_h$ can be written in terms of the basis functions

$$v_h(x) = \sum_{j=1}^{M-1} v_j \phi_j(x), \qquad v_j = v_h(x_j)$$

Note that we do not need ϕ_0, ϕ_M since all functions $v_h \in V_h$ are such that $v_h(0) = v_h(1) = 0$.

Finite dimensional approximation

Finite element approximation: Find $u_h \in V_h$ such that

$$a(u_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h \tag{2}$$

This is also known as *Galerkin finite element method.* We can write u_h in terms of the basis functions

$$u_h(x) = \sum_{j=1}^{M-1} U_j \phi_j(x)$$

Then the finite element problem is

$$a(u_h, \phi_i) = (f, \phi_i) \qquad \text{for} \qquad i = 1, 2, \ldots, M-1$$

Since a is linear in the first argument also

$$\sum_{j=1}^{M-1} U_j a(\phi_j, \phi_i) = (f, \phi_i), \quad i = 1, 2, \ldots, M-1$$

Finite dimensional approximation

This can be written in matrix form as:

$$AU = b$$

where

$$U = [U_i] \in \mathbb{R}^{M-1} \qquad A = [a_{ij}] \in \mathbb{R}^{(M-1)\times(M-1)} \qquad b = [b_i] \in \mathbb{R}^{M-1}$$

with

$$\begin{aligned} a_{ij} &= a(\phi_j, \phi_i) = \int_\Omega [a(x)\phi_i'(x)\phi_j'(x) + c(x)\phi_i(x)\phi_j(x)]\mathrm{d}x \\ b_i &= (f, \phi_i) = \int_\Omega f(x)\phi_i(x)\mathrm{d}x \end{aligned}$$

We already know the existence and uniqueness of the solution u_h, but we can see this at the level of linear algebra also.

Existence of finite element solution

Note that A is a symmetric matrix. It is also positive definite since if $v_h(x) = \sum_{j=1}^{M-1} V_i \phi_i(x)$, $V = [V_i]$ we have

$$\begin{aligned} V^T A V &= \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} V_i a_{ij} V_j \\ &= a\left(\sum_{i=1}^{M-1} V_i \phi_i, \sum_{j=1}^{M-1} V_j \phi_j \right) \\ &= a(v_h, v_h) \geq a_0 \|v_h'\|^2 \geq 0 \end{aligned}$$

But $\|v_h'\| = 0 \implies v_h'(x) = 0 \implies v_h(x) = 0 \implies V = 0$. Hence

$$V^T A V > 0 \quad \forall V \in \mathbb{R}^{M-1}, V \neq 0$$

and the matrix A is positive definite and invertible. Thus the solution exists. In finite dimension, linearity + existence $\implies$ uniqueness.

Energy norm

The bilinear form $a(\cdot, \cdot)$ is symmetric, positive definite and hence is an inner product on H_0^1. The norm induced by this inner product is the *energy norm*

$$\|v\|_a = \sqrt{a(v, v)} = \left(\int_0^1 [a(v')^2 + cv^2] \mathrm{d}x \right)^{1/2}$$

Best approximation property

Let u and u_h be the solutions of (1) and (2). Then

$$\|u_h - u\|_a = \min_{\phi_h \in V_h} \|\phi_h - u\|_a$$

Since $V_h \subset H_0^1$, we have $\forall \phi_h \in V_h$

$$a(u, \phi_h) = (f, \phi_h) \quad \text{and} \quad a(u_h, \phi_h) = (f, \phi_h)$$

and subtracting

$$a(u_h - u, \phi_h) = 0 \quad \forall \phi_h \in V_h \tag{3}$$

Now

$$\begin{aligned} \|u_h - u\|_a^2 &= a(u_h - u, u_h - u) = a(u_h - u, u_h) - a(u_h - u, u) \\ &= 0 - a(u_h - u, u) = a(u_h - u, \phi_h) - a(u_h - u, u) \\ &= a(u_h - u, \phi_h - u) \leq \|u_h - u\|_a \|\phi_h - u\|_a \end{aligned}$$

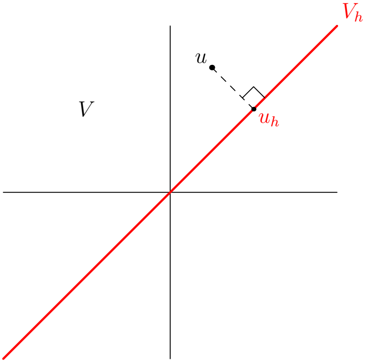
Best approximation property

which gives

$$\|u_h - u\|_a \leq \|\phi_h - u\|_a \quad \forall \phi_h \in V_h$$

This proves the best approximation property; the Galerkin approximation u_h is the best approximation to the exact solution u when measured in the energy norm.

Remark: Property (3) is known as *Galerkin orthogonality*; it states that the error $u_h - u$ is orthogonal to V_h and that u_h is obtained by making an orthogonal projection of u onto V_h using the inner product $a(\cdot, \cdot)$.



Approximation properties of V_h

Define piecewise linear interpolant $I_h v \in V_h$ of a function $v \in C(\bar{\Omega})$ with $v(0) = v(1) = 0$ by

$$I_h v(x_j) = v(x_j), \quad j = 1, \dots, M-1$$

Since $H_0^1 \subset C^0$ in one dimension, $I_h v$ is defined for $v \in H_0^1$.

$$I_h v(x) = \sum_{j=1}^{M-1} v(x_j) \phi_j(x)$$

Using Bramble-Hilbert Lemma, we have the following error estimates on each element

$$\|I_h v - v\|_{K_j} \leq C h_j^2 |v|_{2,K_j}$$

and

$$\|(I_h v - v)'\|_{K_j} \leq C h_j |v|_{2,K_j}$$

Approximation properties of V_h

We get

$$\begin{aligned} \|I_h v - v\| &= \left(\sum_{j=1}^{M} \|I_h v - v\|_{K_j}^2 \right)^{1/2} \leq \left(\sum_{j=1}^{M} C^2 h_j^4 |v|_{2,K_j}^2 \right)^{1/2} \\ &\leq Ch^2 |v|_2, \quad \forall v \in H^2 \end{aligned}$$

and similarly

$$\|(I_h v - v)'\| \leq Ch|v|_2, \quad \forall v \in H^2 \tag{4}$$

Error estimate for $u_h' - u'$

Let u and u_h be the solutions of (1) and (2). Then

$$\|u_h' - u'\| \leq Ch\|u\|_2$$

From coercivity and continuity of $a(\cdot,\cdot)$, we have

$$\sqrt{a_0}\|v'\| \leq \|v\|_a \leq C_1\|v'\|, \quad \forall v \in H_0^1$$

This gives

$$\begin{aligned} \|u_h' - u'\| &\leq \frac{1}{\sqrt{a_0}}\|u_h - u\|_a \leq \frac{1}{\sqrt{a_0}}\|\phi_h - u\|_a \\ &\leq \frac{C_1}{\sqrt{a_0}}\|\phi_h' - u'\|, \quad \forall \phi_h \in V_h \end{aligned}$$

Taking $\phi_h = I_h u$ and using the interpolation error estimate (4) we get

$$\|u_h' - u'\| \leq \frac{C_1}{\sqrt{a_0}}\|(I_h u - u)'\| \leq Ch\|u\|_2$$

Error estimate for $u_h - u$

Let u and u_h be the solutions of (1) and (2). Then

$$\|u_h - u\| \leq Ch^2 \|u\|_2$$

Define the error $e = u_h - u$ and define the ***dual*** or ***adjoint*** problem

$$A\phi = e \quad \text{in} \quad \Omega \quad \text{with} \quad \phi(0) = \phi(1) = 0$$

Its variational formulation is: find $\phi \in H_0^1$ such that

$$a(w, \phi) = (w, e) \quad \forall w \in H_0^1$$

The regularity estimate for the PDE gives us

$$\|\phi\|_2 \leq C\|e\|$$

Error estimate for $u_h - u$

Taking $w = e$

$$\begin{aligned}
\|e\|^2 &= (e,e) = a(e,\phi) = a(e,\phi) - a(e, I_h\phi) \\
&= a(e, \phi - I_h\phi) \\
&\leq C\|e'\| \, \|(\phi - I_h\phi)'\| \\
&\leq Ch\|e'\| \, \|\phi\|_2 \\
&\leq Ch^2\|u\|_2\|e\|
\end{aligned}$$

and hence we get

$$\|e\| \leq Ch^2\|u\|_2$$

Remark: The error estimate for the finite difference method involved the fourth derivative

$$\|u - u_h\|_\infty \leq Ch^2|u|_{\mathcal{C}^4}$$

while the above FEM error estimate requires only second derivatives.

Numerical solution

We have to <u>assemble</u> the matrix A and the right hand side vector b. Note that the matrix entries a_{ij} are non-zero only if ϕ_i and ϕ_j have overlapping supports and these are

$$a_{i,i-1}, \qquad a_{i,i}, \qquad a_{i,i+1}$$

Matrix A is symmetric, tri-diagonal

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & & \\ & \beta_2 & \alpha_3 & \beta_3 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & \beta_{M-3} & \alpha_{M-2} & \beta_{M-2} \\ & & & & & \beta_{M-2} & \alpha_{M-1} \end{bmatrix}$$

$$\alpha_j = a(\phi_j, \phi_j), \qquad \beta_j = a(\phi_j, \phi_{j+1})$$

Numerical solution

Matrix is sparse. Do not compute the full matrix; compute and store only the non-zero diagonal elements. Matrix equation can be solved efficiently using Thomas Tridiagonal Algorithm.

<u>Consider the case $c \equiv 0$</u>

$$\begin{aligned}
\alpha_j &= a(\phi_j, \phi_j) = \int_{x_{j-1}}^{x_{j+1}} \phi_j'(x)\phi_j'(x)\mathrm{d}x \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{1}{h_j}\right)^2 \mathrm{d}x + \int_{x_j}^{x_{j+1}} \left(\frac{-1}{h_{j+1}}\right)^2 \mathrm{d}x \\
&= \frac{1}{h_j} + \frac{1}{h_{j+1}} \\
\beta_j &= a(\phi_j, \phi_{j+1}) = \int_{x_j}^{x_{j+1}} \phi_j'(x)\phi_{j+1}'(x)\mathrm{d}x \\
&= \int_{x_j}^{x_{j+1}} \frac{-1}{h_{j+1}} \frac{1}{h_{j+1}} \mathrm{d}x = -\frac{1}{h_{j+1}}
\end{aligned}$$

Numerical solution

The right hand side vector is given by

$$b_j = \int_{x_{j-1}}^{x_{j+1}} f(x)\phi_j(x)\mathrm{d}x \qquad \text{(in general need numerical integration)}$$

Uniform mesh

$$\alpha_j = \alpha = \frac{2}{h}, \qquad \beta_j = \beta = -\frac{1}{h}$$

Approximate b_j by mid-point rule

$$b_j \approx f(x_j)\int_{x_{j-1}}^{x_{j+1}} \phi_j(x)\mathrm{d}x = f(x_j)h$$

The j'th equation is

$$\beta U_{j-1} + \alpha U_j + \beta U_{j+1} = f_j h \quad \Longrightarrow \quad -\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f_j$$

which is the central finite difference scheme.

Non-homogeneous boundary conditions

$$\begin{aligned} Au &= f \qquad \text{in} \qquad \Omega = (0,1) \\ u(0) &= a \\ u(1) &= b \end{aligned}$$

Write the solution as

$$u(x) = \tilde{u}(x) + u_b(x), \qquad u_b(x) = (1-x)a + xb$$

Then $\tilde{u} \in H_0^1(\Omega)$. Find $\tilde{u} \in H_0^1(\Omega)$ such that

$$a(\tilde{u}, \phi) = (f, \phi) - a(u_b, \phi), \qquad \forall\ \phi \in H_0^1(\Omega)$$

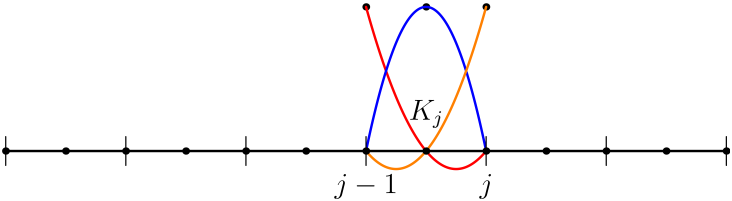
Verify that the right hand side is a continuous linear functional.
Galerkin approximation: Find $\tilde{u}_h \in V_h \subset H_0^1(\Omega)$ such that

$$a(\tilde{u}_h, \phi_h) = (f, \phi_h) - a(u_b, \phi_h), \qquad \forall\ \phi_h \in V_h$$

Degree two basis functions

$$V_h^2 = \{v \in C(\bar{\Omega}) : v|_{K_j} \in P_2(K_j)\} \subset H^1(\Omega)$$

We need three degrees of freedom in each element to uniquely define a degree two polynomial. They can be chosen as in figure. There is a dof at each end of the element which ensures continuity. Basis functions based on Lagrange polynomials are shown in figure and they have the property $\phi_i(x_k) = \delta_{ik}$ where x_k are the location of the dofs.



Note that there are more dofs than vertices in the mesh. The matrix in the Galerkin method is penta-diagonal.

Degree k basis functions

Let $k \geq 1$ be the degree of polynomial space

$$V_h^k = \{v \in C(\bar{\Omega}) : v|_{K_j} \in P_k(K_j)\} \subset H^1(\Omega)$$

We need $k+1$ dofs in each element to determine the polynomial uniquely, with one dof at each end of the element to ensure continuity.

Interpolation error: For $2 \leq r \leq k+1$

$$\begin{aligned} \|u - I_h u\|_{K_j} &\leq C h_j^r |u|_{r,K_j} \\ \left\|(u - I_h u)'\right\|_{K_j} &\leq C h_j^{r-1} |u|_{r,K_j} \end{aligned}$$

Then if $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ then the solution $u_h \in V_h^k$ satisfies

$$\begin{aligned} \|u - u_h\| &\leq C h^{k+1} |u|_{k+1} \\ \left\|u' - u_h'\right\| &\leq C h^k |u|_{k+1} \end{aligned}$$