Finite difference method for elliptic problems: II

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2-D elliptic BVP

$$\boxed{\begin{aligned} -\Delta u(x) + c(x)u(x) = f(x) \qquad & x \in \Omega = (0,1) \times (0,1) \\ u = 0 \qquad & x \in \partial\Omega \end{aligned}} \tag{1}$$

$c \geq 0$ is a continuous function and f is continuous or in $L^2(\Omega)$.

Finite difference mesh: Let $N \geq 2$ be an integer and let

$$\text{mesh size:} \qquad h = \frac{1}{N}$$

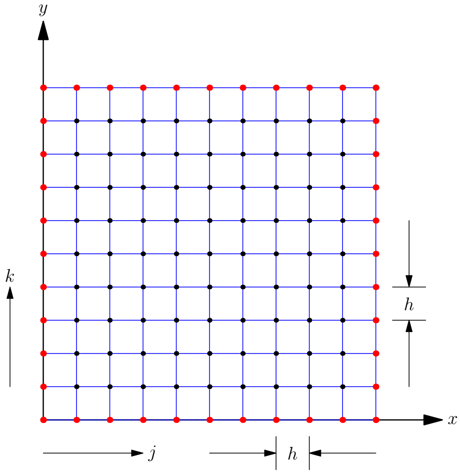
$$\text{mesh points:} \qquad (x_i, y_j) = (ih, jh) \qquad i, j = 0, 1, \ldots, N$$

$$\bar{\Omega}_h = \{(x_i, y_j) : i, j = 0, 2, \ldots, N\}$$

$$\text{Interior mesh points} \quad \Omega_h = \{(x_i, y_j) : i, j = 1, 2, \ldots, N-1\}$$

$$\text{Boundary mesh points} \quad \Gamma_h = \bar{\Omega}_h \setminus \Omega_h$$

2-D elliptic BVP



Note: Change in index notation

Finite difference scheme

Define

$$c_{ij} = c(x_i, y_j), \qquad f_{ij} = f(x_i, y_j)$$

the finite difference scheme is

$$\begin{aligned} -(\underbrace{D_x^+ D_x^- U_{ij}}_{u_{xx}} + \underbrace{D_y^+ D_y^- U_{ij}}_{u_{yy}}) + c_{ij} U_{ij} &= f_{ij} \quad \text{in} \quad \Omega_h \\ U_{ij} &= 0 \quad \text{on} \quad \Gamma_h \end{aligned} \tag{2}$$

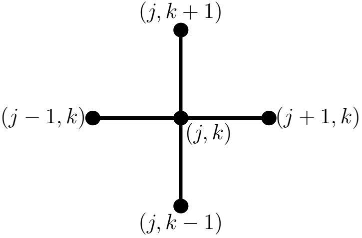
In expanded form: for $i, j = 1, 2, \ldots, N-1$

$$-\left[\frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2} + \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h^2} \right] + c_{ij} U_{ij} = f_{ij}$$

$$U_{ij} = 0 \qquad \text{if} \quad i = 0 \quad \text{or} \quad i = N \quad \text{or} \quad j = 0 \quad \text{or} \quad j = N$$

At each point $(i, j) \in \Omega_h$ the FD scheme depends on five values of U: $U_{ij}, U_{i-1,j}, U_{i+1,j}, U_{i,j-1}, U_{i,j+1}$.

Finite difference scheme



(Note: change in notation)

Let us arrange the unknown values in a vector U

$$
\begin{aligned}
U =&(U_{11}, U_{21}, \dots, U_{N-1,1}, U_{12}, U_{22}, \dots, U_{N-1,2}, \dots \\
&\dots, U_{1i}, U_{2i}, \dots, U_{N-1,i}, \dots, U_{1,N-1}, U_{2,N-1}, \dots, U_{N-1,N-1})^\top
\end{aligned}
$$

and similarly the right hand side

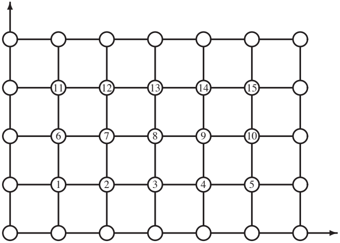
$$
\begin{aligned}
F =&(f_{11}, f_{21}, \dots, f_{N-1,1}, f_{12}, f_{22}, \dots, f_{N-1,2}, \dots \\
&\dots, f_{1i}, f_{2i}, \dots, f_{N-1,i}, \dots, f_{1,N-1}, f_{2,N-1}, \dots, f_{N-1,N-1})^\top
\end{aligned}
$$

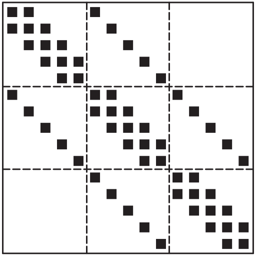
Finite difference scheme

The system of FD equations can be written in matrix form

$$AU = F$$

where A is an $(N-1)^2 \times (N-1)^2$ sparse matrix of banded structure. Any row of A has at most five non-zero entries corresponding to the 5-point stencil.





$$A = \frac{1}{h^2}\begin{pmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{pmatrix} \quad \text{with} \quad B = \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & -1 & \\ & -1 & 4 & -1 \\ & & -1 & 4 \end{pmatrix}$$

Discrete inner product

For any two grid functions V, W, define

$$(V,W)_h = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} h^2 V_{ij} W_{ij}$$

which is a discrete version of the $L^2(\Omega)$ inner product

$$(v,w) = \int_\Omega v(x) w(x) \mathrm{d}x$$

Lemma (Summation by parts)

Suppose that $V : \bar{\Omega}_h \to \mathbb{R}$ and $V = 0$ on Γ_h. Then

$$-\left(D_x^+ D_x^- V + D_y^+ D_y^- V, V\right)_h = \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{ij}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{ij}|^2$$

Proof: Apply 1-D SBP formula to $(D_x^+ D_x^- V, V)_h$ and $\left(D_y^+ D_y^- V, V\right)_h$.

Existence/uniqueness of FD solution

For any $V : \bar{\Omega}_h \to \mathbb{R}$ with $V = 0$ on Γ_h

$$
\begin{aligned}
(AV, V)_h &= -\left(D_x^+ D_x^- V + D_y^+ D_y^- V, V\right)_h + (cV, V)_h \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{ij}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{ij}|^2 + (cV, V)_h \\
&\geq \sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- V_{ij}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N} h^2 |D_y^- V_{ij}|^2 \geq 0 \qquad (3)
\end{aligned}
$$

If $AV = 0$ then

$$D_x^- V_{ij} = \frac{V_{ij} - V_{i-1,j}}{h} = 0, \qquad \begin{array}{l} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, N-1 \end{array}$$

$$D_y^- V_{ij} = \frac{V_{ij} - V_{i,j-1}}{h} = 0, \qquad \begin{array}{l} i = 1, 2, \ldots, N-1 \\ j = 1, 2, \ldots, N \end{array}$$

Since $V = 0$ on Γ_h, this implies that $V \equiv 0$. Thus $AV = 0$ iff $V = 0$ so that the matrix A is invertible. Hence (2) has a unique solution $U = A^{-1} F$.

Discrete norms

Define discrete L^2 and Sobolev norms as

$$\|V\|_h = \sqrt{(V,V)_h}, \qquad \|V\|_{1,h} = \left(\|V\|_h^2 + \|D_x^- V]|_x^2 + \|D_y^- V]|_y^2\right)^{\frac{1}{2}}$$

where

$$\|D_x^- V]|_x^2 = \sum_{i=1}^{N}\sum_{j=1}^{N-1} h^2 |D_x^- V_{ij}|^2, \qquad \|D_x^- V]|_y^2 = \sum_{i=1}^{N-1}\sum_{j=1}^{N} h^2 |D_y^- V_{ij}|^2$$

With this new notation, inequality (3) takes the form

$$\boxed{(AV,V)_h \geq \|D_x^- V]|_x^2 + \|D_y^- V]|_y^2}$$

Using a discrete Poincare-Friedrichs inequality we can show that

$$\boxed{(AV,V)_h \geq c_0 \|V\|_{1,h}^2}$$

which is a discrete coercivity property.

Lemma (Discrete Poincare-Friedrichs inequality)

Let $V : \bar{\Omega}_h \to \mathbb{R}$ be such that $V = 0$ on Γ_h. Then $\exists\ c_* > 0$ independent of V and h such that

$$\|V\|_h^2 \leq c_* \left(\|D_x^- V]|_x^2 + \|D_y^- V]|_y^2 \right) \tag{4}$$

for all such V.

Proof: We use the 1-D SBP result.

$$\text{For every fixed } j, \quad 1 \leq j \leq N-1, \qquad \sum_{i=1}^{N-1} h|V_{ij}|^2 \leq \frac{1}{2} \sum_{i=1}^{N} h|D_x^- V_{ij}|^2$$

$$\text{For every fixed } i, \quad 1 \leq i \leq N-1, \qquad \sum_{j=1}^{N-1} h|V_{ij}|^2 \leq \frac{1}{2} \sum_{j=1}^{N} h|D_y^- V_{ij}|^2$$

Multiply by h and sum over $1 \leq j \leq N-1$ and $1 \leq i \leq N-1$ respectively

$$2\,\|V\|_h^2 \leq \frac{1}{2} \left(\|D_x^- V]|_x^2 + \|D_y^- V]|_y^2 \right)$$

which proves (4) with $c_* = \frac{1}{4}$.

From coercivity and Poincare-Friedrichs inequality

$$(AV, V)_h \geq \|D_x^- V]|_x^2 + \|D_y^- V]|_y^2 \geq \frac{1}{c_*} \|V\|_h^2$$

which implies

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2 , \qquad c_0 = (1 + c_*)^{-1} = \frac{4}{5}$$

Theorem (Stability)

The scheme (2) is stable in the sense that

$$\|U\|_{1,h} \leq \frac{1}{c_0} \|f\|_h \tag{5}$$

Proof: Similar to the 1-D case.

Theorem (Error estimate)

Let $f \in \mathcal{C}(\bar{\Omega})$, $c \in \mathcal{C}(\bar{\Omega})$ with $c \geq 0$ and suppose the corresponding solution $u \in \mathcal{C}^4(\bar{\Omega})$. Then

$$\|u - U\|_{1,h} \leq \frac{5h^2}{48}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\mathcal{C}(\bar{\Omega})}\right)$$

Proof: Define the error

$$e_{ij} = u(x_i, y_j) - U_{ij}, \qquad 0 \leq i, j \leq N$$

Then assuming that $u \in \mathcal{C}^4(\bar{\Omega})$ and using Taylor series

$$\begin{aligned}
Ae_{ij} &= \Delta u(x_i, y_j) - (D_x^+ D_x^- u(x_i, y_j) + D_y^+ D_y^- u(x_i, y_j)) \\
&= \left[\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - D_x^+ D_x^- u(x_i, y_j)\right] + \left[\frac{\partial^2 u}{\partial y^2}(x_i, y_j) - D_y^+ D_y^- u(x_i, y_j)\right] \\
&= -\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) - \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j), \quad \xi_i \in [x_{i-1}, x_{i+1}],\ \eta_j \in [y_{j-1}, y_{j+1}] \\
&=: \tau_{ij}, \qquad 1 \leq i, j \leq N-1
\end{aligned}$$

We can bound the local truncation error as

$$\|\tau\|_h \leq \frac{h^2}{12}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\mathcal{C}(\bar{\Omega})}\right)$$

The error satisfies the equation

$$\begin{aligned} Ae_{ij} &= \tau_{ij}, \quad 1 \leq i,j \leq N-1 \\ e_{ij} &= 0, \qquad \text{on} \quad \Gamma_h \end{aligned}$$

Using the stability estimate (5)

$$\|u-U\|_{1,h} = \|e\|_{1,h} \leq \frac{1}{c_0}\|\tau\|_h \leq \frac{5h^2}{48}\left(\left\|\frac{\partial^4 u}{\partial x^4}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^4 u}{\partial y^4}\right\|_{\mathcal{C}(\bar{\Omega})}\right) \quad \square$$

Problems:

1. c and f may not be continuous.
2. Boundary $\partial\Omega$ may not be smooth.
3. $u \notin \mathcal{C}^4(\bar{\Omega})$.

Finite volume method

Define the cell or finite volume

$$K_{ij} = \left[x_i - \frac{h}{2}, x_i + \frac{h}{2}\right] \times \left[y_j - \frac{h}{2}, y_j + \frac{h}{2}\right] = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$$

Integrate over cell K_{ij}

$$-\int_{K_{ij}} (u_{xx} + u_{yy}) + \int_{K_{ij}} cu = \int_{K_{ij}} f$$

First term on the left becomes

$$-\int_{y_j-\frac{h}{2}}^{y_j+\frac{h}{2}} \left[\frac{\partial u}{\partial x}(x_{i+\frac{1}{2}}, y) - \frac{\partial u}{\partial x}(x_{i-\frac{1}{2}}, y)\right] \mathrm{d}y - \int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} \left[\frac{\partial u}{\partial y}(x, y_{j+\frac{1}{2}}) - \frac{\partial u}{\partial y}(x, y_{j-\frac{1}{2}})\right] \mathrm{d}x$$

Approximate by mid-point rule and central finite difference

$$\begin{aligned} -\int_{K_{ij}} u_{xx} &\approx -\left[\frac{\partial u}{\partial x}(x_{i+\frac{1}{2}}, y_j) - \frac{\partial u}{\partial x}(x_{i-\frac{1}{2}}, y_j)\right] h \\ &\approx -\left[\frac{u_{i+1,j} - u_{ij}}{h} - \frac{u_{ij} - u_{i-1,j}}{h}\right] h = -(D_x^+ D_x^- u_{ij}) h^2 \end{aligned}$$

Finite volume method

and similarly

$$-\int_{K_{ij}} u_{yy} \approx -(D_y^+ D_y^- u_{ij})h^2$$

Approximate the reaction term also with mid-point rule

$$\int_{K_{ij}} cu \approx c_{ij} u_{ij} h^2$$

Define the cell average value Tf_{ij}

$$Tf_{ij} = \frac{1}{h^2} \int_{K_{ij}} f \qquad \Longrightarrow \qquad \int_{K_{ij}} f = (Tf_{ij})h^2$$

The finite volume scheme is given by

$$\boxed{\begin{aligned} -(D_x^+ D_x^- U_{ij} + D_y^+ D_y^- U_{ij}) + c_{ij} U_{ij} = Tf_{ij} \quad & \text{in} \quad \Omega_h \\ U_{ij} = 0 \qquad & \text{on} \quad \Gamma_h \end{aligned}} \tag{6}$$

Stability of cell averaging operator T:

$$\begin{aligned} |Tf_{ij}| &= \frac{1}{h^2}\left|\int_{K_{ij}} f\right| \\ &= \frac{1}{h^2}\left|\int_{K_{ij}} (1)(f)\right| \\ &\leq \frac{1}{h^2}\left(\int_{K_{ij}} 1^2\right)^{\frac{1}{2}}\left(\int_{K_{ij}} |f|^2\right)^{\frac{1}{2}} \\ &= \frac{1}{h}\|f\|_{L^2(K_{ij})} \end{aligned}$$

and

$$\|Tf\|_h^2 = \sum_{i,j=1}^{N-1} h^2 |Tf_{ij}|^2 \leq \sum_{i,j=1}^{N-1} \|f\|_{L^2(K_{ij})}^2 = \|f\|_{L^2(\Omega)}^2$$

Hence

$$\|Tf\|_h \leq \|f\|_{L^2(\Omega)}$$

Theorem

The scheme (6) is stable in the sense that

$$\|U\|_{1,h} \leq \frac{1}{c_0} \|f\|_{L^2(\Omega)}$$

Proof: Use discrete coercivity of A and stability of T

$$\begin{aligned} c_0 \|U\|_{1,h}^2 &\leq (AU, U)_h = (Tf, U)_h \\ &\leq \|Tf\|_h \|U\|_h \leq \|f\|_{L^2(\Omega)} \|U\|_{1,h} \quad \square \end{aligned}$$

Error estimate: Define the error

$$e_{ij} = u(x_i, y_j) - U_{ij}$$

Then error satisfies the equation

$$\begin{aligned} Ae_{ij} &= Au(x_i, y_j) - AU_{ij} \\ &= -(D_x^+ D_x^- u(x_i, y_j) + D_y^+ D_y^- u(x_i, y_j) + c(x_i, y_j) u(x_i, y_j)) - Tf_{ij} \end{aligned}$$

From the PDE we have

$$f(x,y) = -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) + c(x,y)u(x,y)$$

so that

$$Tf_{ij} = -T\left(\frac{\partial^2 u}{\partial x^2}\right)(x_i,y_j) - T\left(\frac{\partial^2 u}{\partial y^2}\right)(x_i,y_j) + T(cu)(x_i,y_j)$$

The averages on right will be written as

$$\begin{aligned} T\left(\frac{\partial^2 u}{\partial x^2}\right)(x_i,y_j) &= \frac{1}{h}\int_{y_j-\frac{h}{2}}^{y_j+\frac{h}{2}} \frac{\frac{\partial u}{\partial x}(x_i+\frac{h}{2},y) - \frac{\partial u}{\partial x}(x_i-\frac{h}{2},y)}{h}\mathrm{d}y \\ &= D_x^+\left[\frac{1}{h}\int_{y_j-\frac{h}{2}}^{y_j+\frac{h}{2}} \frac{\partial u}{\partial x}(x_i-h/2,y)\mathrm{d}y\right] \end{aligned}$$

and similarly

$$T\left(\frac{\partial^2 u}{\partial y^2}\right)(x_i,y_j) = D_y^+\left[\frac{1}{h}\int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} \frac{\partial u}{\partial y}(x,y_j-h/2)\mathrm{d}x\right]$$

The error equation becomes

$$
\begin{aligned}
Ae = D_x^+ \phi_1 + D_y^+ \phi_2 + \psi \quad & \text{in} \quad \Omega_h \\
e = 0 \quad & \text{on} \quad \Gamma_h
\end{aligned}
$$

where

$$
\phi_1 = \frac{1}{h} \int_{y_j - \frac{h}{2}}^{y_j + \frac{h}{2}} \frac{\partial u}{\partial x}(x_i - h/2, y) \mathrm{d}y - D_x^- u(x_i, y_j) \tag{7}
$$

$$
\phi_2 = \frac{1}{h} \int_{x_i - \frac{h}{2}}^{x_i + \frac{h}{2}} \frac{\partial u}{\partial y}(x, y_j - h/2) \mathrm{d}x - D_y^- u(x_i, y_j) \tag{8}
$$

$$
\psi = (cu)(x_i, y_j) - T(cu)(x_i, y_j) \tag{9}
$$

Using discrete coercivity property

$$
c_0 \|e\|_{1,h}^2 \leq (Ae, e)_h = \left(D_x^+ \phi_1, e\right)_h + \left(D_y^+ \phi_2, e\right)_h + (\psi, e)_h \leq \|\tau\| \, \|e\|_{1,h}
$$

Consider terms on rhs one by one.

Use summation by parts to transfer D_x^+, D_y^+ onto e. Since $e = 0$ on Γ_h

$$
\begin{aligned}
\left(D_x^+\phi_1, e\right)_h &= \sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N-1} h \frac{\phi_1(x_{i+1}, y_j) - \phi_1(x_i, y_j)}{h} e_{ij} \right) \\
&= -\sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N} h\phi_1(x_i, y_j) \frac{e_{ij} - e_{i-1,j}}{h} \right) \\
&= -\sum_{j=1}^{N-1} h \left(\sum_{i=1}^{N} h\phi_1(x_i, y_j) D_x^- e_{ij} \right) \\
&= -\sum_{i=1}^{N} \sum_{j=1}^{N-1} [h\phi_1(x_i, y_j)][hD_x^- e_{ij}] \\
&\leq \left(\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |\phi_1(x_i, y_j)|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \sum_{j=1}^{N-1} h^2 |D_x^- e_{ij}|^2 \right)^{\frac{1}{2}} \\
&= \|\phi_1]|_x \|D_x^- e]|_x
\end{aligned}
$$

and similarly

$$\left(D_y^+\phi_2, e\right)_h \le \|\phi_2]|_y \|D_y^- e]|_y$$

By Cauchy-Schwarz inequality, we also have

$$(\psi, e)_h \le \|\psi\|_h \|e\|_h$$

$$\begin{aligned} c_0 \|e\|_{1,h}^2 &\le \|\phi_1]|_x \|D_x^- e]|_x + \|\phi_2]|_y \|D_y^- e]|_y + \|\psi\|_h \|e\|_h \\ &\le \left(\|\phi_1]|_x^2 + \|\phi_2]|_y + \|\psi\|_h\right)^{\frac{1}{2}} \left(\|D_x^- e]|_x^2 + \|D_y^- e]|_y^2 + \|e\|_h^2\right)^{\frac{1}{2}} \\ &= \left(\|\phi_1]|_x^2 + \|\phi_2]|_y + \|\psi\|_h\right)^{\frac{1}{2}} \|e\|_{1,h} \end{aligned}$$

Lemma

The global error e of the scheme (6) satisfies

$$\|e\|_{1,h} \le \frac{1}{c_0} \left(\|\phi_1]|_x^2 + \|\phi_2]|_y^2 + \|\psi\|_h^2\right)^{\frac{1}{2}}$$

where ϕ_1, ϕ_2 and ψ are defined by (7), (8), (9).

To complete the error analysis, we have to estimate the norms of ϕ_1, ϕ_2, ψ appearing in the above lemma in terms of the mesh size h. Using Taylor series, show that

$$\begin{aligned}
|\phi_1(x_i, y_j)| &\leq \frac{h^2}{24}\left(\left\|\frac{\partial^3 u}{\partial x \partial y^2}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^3 u}{\partial x^3}\right\|_{\mathcal{C}(\bar{\Omega})}\right) \\
|\phi_2(x_i, y_j)| &\leq \frac{h^2}{24}\left(\left\|\frac{\partial^3 u}{\partial x^2 \partial y}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^3 u}{\partial y^3}\right\|_{\mathcal{C}(\bar{\Omega})}\right) \\
|\psi(x_i, y_j)| &\leq \frac{h^2}{24}\left(\left\|\frac{\partial^2 (cu)}{\partial x^2}\right\|_{\mathcal{C}(\bar{\Omega})} + \left\|\frac{\partial^2 (cu)}{\partial y^2}\right\|_{\mathcal{C}(\bar{\Omega})}\right)
\end{aligned}$$

Theorem

Let $f \in L^2(\Omega)$, $c \in \mathcal{C}^2(\bar{\Omega})$ with $c \geq 0$ and suppose the solution of the BVP (1) belongs to $\mathcal{C}^3(\bar{\Omega})$. Then

$$\|u - U\|_{1,h} \leq \frac{5}{96} h^2 M_3$$

where

$$M_3 = \left\{ \left(\left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{\mathcal{C}(\bar{\Omega})} + \left\| \frac{\partial^3 u}{\partial x^3} \right\|_{\mathcal{C}(\bar{\Omega})} \right)^2 + \left(\left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{\mathcal{C}(\bar{\Omega})} + \left\| \frac{\partial^3 u}{\partial y^3} \right\|_{\mathcal{C}(\bar{\Omega})} \right)^2 + \left(\left\| \frac{\partial^2 (cu)}{\partial x^2} \right\|_{\mathcal{C}(\bar{\Omega})} + \left\| \frac{\partial^2 (cu)}{\partial y^2} \right\|_{\mathcal{C}(\bar{\Omega})} \right)^2 \right\}^{\frac{1}{2}}$$

Proof: From previous lemma and using $c_0 = \frac{4}{5}$.

Relaxing smoothness requirements:
We are still able to obtain second order convergence using the fact that exact solution $u \in \mathcal{C}^3[0,1]$ which is less restrictive than previous results. We can further relax this condition by using integral representations of ϕ_1, ϕ_2 and ψ instead of Taylor series expansions. The key idea is to use Newton-Leibniz formula

$$w(b) - w(a) = \int_a^b w'(x)\mathrm{d}x$$

$$\begin{aligned}
\phi_1(x_i, y_j) &= \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[\frac{\partial u}{\partial x}(x_{i-\frac{1}{2}}, y) - \frac{\partial u}{\partial x}(x, y_j)\right]\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[\frac{\partial u}{\partial x}(x_{i-\frac{1}{2}}, y) - \frac{\partial u}{\partial x}(x, y)\right]\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[\frac{\partial u}{\partial x}(x, y) - \frac{\partial u}{\partial x}(x, y_j)\right]\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$= \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left[\int_{x_{i-1}}^{x_i} 1\cdot \int_x^{x_{i-\frac{1}{2}}} \frac{\partial^2 u}{\partial x^2}(\xi,y)\mathrm{d}\xi \mathrm{d}x\right]\mathrm{d}y \qquad \int_{x_{i-1}}^{x_i} a'(x)b(x)\mathrm{d}x$$

$$+\frac{1}{h^2}\int_{x_{i-1}}^{x_i}\left[\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} 1\cdot \int_{y_j}^{y} \frac{\partial^2 u}{\partial x\partial y}(x,\eta)\mathrm{d}\eta \mathrm{d}y\right]\mathrm{d}x$$

$$= \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left[x\int_x^{x_{i-\frac{1}{2}}} \frac{\partial^2 u}{\partial x^2}(\xi,y)\mathrm{d}\xi\Big|_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i} x\frac{\partial^2 u}{\partial x^2}(x,y)\mathrm{d}x\right]\mathrm{d}y$$

$$+\frac{1}{h^2}\int_{x_{i-1}}^{x_i}\left[y\int_{y_j}^{y} \frac{\partial^2 u}{\partial x\partial y}(x,\eta)\mathrm{d}\eta\Big|_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} y\frac{\partial^2 u}{\partial x\partial y}(x,y)\mathrm{d}y\right]\mathrm{d}x$$

$$= \frac{1}{h^2}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left[\int_{x_{i-1}}^{x_{i-\frac{1}{2}}} (x-x_{i-1})\frac{\partial^2 u}{\partial x^2}(x,y)\mathrm{d}x + \int_{x_{i-\frac{1}{2}}}^{x_i} (x-x_i)\frac{\partial^2 u}{\partial x^2}(x,y)\mathrm{d}x\right]\mathrm{d}y$$

$$-\frac{1}{h^2}\int_{x_{i-1}}^{x_i}\left[\int_{y_{j-\frac{1}{2}}}^{y_j} (y-y_{j-\frac{1}{2}})\frac{\partial^2 u}{\partial x\partial y}(x,y)\mathrm{d}y + \int_{y_j}^{y_{j+\frac{1}{2}}} (y-y_{j+\frac{1}{2}})\frac{\partial^2 u}{\partial x\partial y}(x,y)\mathrm{d}y\right]\mathrm{d}x$$

Define $A:[x_{i-1},x_i]\to\mathbb{R}$ and $B:[y_{j-\frac{1}{2}},y_{j+\frac{1}{2}}]\to\mathbb{R}$

$$A(x)=\begin{cases}\frac{1}{2}(x-x_{i-1})^2, & x\in[x_{i-1},x_{i-\frac{1}{2}}]\\ \frac{1}{2}(x-x_i)^2, & x\in[x_{i-\frac{1}{2}},x_i]\end{cases}$$

$$B(x) = \begin{cases} \frac{1}{2}(y - y_{j-\frac{1}{2}})^2, & y \in [y_{j-\frac{1}{2}}, y_j] \\ \frac{1}{2}(y - y_{j+\frac{1}{2}})^2, & y \in [y_j, y_{j+\frac{1}{2}}] \end{cases}$$

Note that $A(x)$, $B(x)$ are continuous and vanish at end points

$$A(x_{i-1}) = A(x_i) = 0, \qquad B(y_{j-\frac{1}{2}}) = B(y_{j+\frac{1}{2}}) = 0$$

$$\begin{aligned}
\phi_1(x_i, y_j) &= \frac{1}{h^2} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[\int_{x_{i-1}}^{x_i} A'(x) \frac{\partial^2 u}{\partial x^2}(x, y) \mathrm{d}x \right] \mathrm{d}y \\
&\quad - \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \left[\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} B'(y) \frac{\partial^2 u}{\partial x \partial y}(x, y) \mathrm{d}y \right] \mathrm{d}x \\
&= -\frac{1}{h^2} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[\int_{x_{i-1}}^{x_i} A(x) \frac{\partial^3 u}{\partial x^3}(x, y) \mathrm{d}x \right] \mathrm{d}y \\
&\quad + \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \left[\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} B(y) \frac{\partial^3 u}{\partial x \partial y^2}(x, y) \mathrm{d}y \right] \mathrm{d}x
\end{aligned}$$

But

$$|A(x)| \le \frac{h^2}{8}, \qquad |B(x)| \le \frac{h^2}{8}$$

and therefore

$$|\phi_1(x_i, y_j)| \le \frac{1}{8} \int_{x_{i-1}}^{x_i} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\left| \frac{\partial^3 u}{\partial x^3}(x,y) \right| + \left| \frac{\partial^3 u}{\partial x \partial y^2}(x,y) \right| \right) \mathrm{d}x\mathrm{d}y$$

Consequently

$$\|\phi_1]|_x^2 \le \frac{h^4}{32} \left(\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial^3 u}{\partial x \partial y^2} \right\|_{L^2(\Omega)}^2 \right)$$

and similarly

$$\|\phi_2]|_y^2 \le \frac{h^4}{32} \left(\left\| \frac{\partial^3 u}{\partial y^3} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{L^2(\Omega)}^2 \right)$$

To estimate ψ, let $w = cu$

$$\begin{aligned}
\psi(x_i, y_j) &= \frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(\int_x^{x_i}\frac{\partial w}{\partial x}(s,y)\mathrm{d}s + \int_y^{y_j}\frac{\partial w}{\partial y}(x,t)\mathrm{d}t\right.\\
&\quad \left. + \int_x^{x_i}\int_y^{y_j}\frac{\partial^2 w}{\partial x \partial y}(s,t)\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y\\
&= -\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(C(x)\frac{\partial^2 w}{\partial x^2}(x,y) + D(x)\frac{\partial^2 w}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y\\
&\quad +\frac{1}{h^2}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left(\int_x^{x_i}\int_y^{y_j}\frac{\partial^2 w}{\partial x \partial y}(s,t)\mathrm{d}s\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where

$$C(x) = \begin{cases} \frac{1}{2}(x - x_{i-\frac{1}{2}})^2, & x \in [x_{i-\frac{1}{2}}, x_i] \\ \frac{1}{2}(x - x_{i+\frac{1}{2}})^2, & x \in [x_i, x_{i+\frac{1}{2}}] \end{cases}, \qquad |C(x)| \le \frac{h^2}{8}$$

$$D(x) = \begin{cases} \frac{1}{2}(y - y_{j-\frac{1}{2}})^2, & y \in [y_{j-\frac{1}{2}}, y_j] \\ \frac{1}{2}(y - y_{j+\frac{1}{2}})^2, & y \in [y_j, y_{j+\frac{1}{2}}] \end{cases}, \qquad |D(x)| \le \frac{h^2}{8}$$

Hence

$$|\psi(x_i, y_j)| \leq \frac{1}{8} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left(\left| \frac{\partial^2 w}{\partial x^2} \right| + \left| \frac{\partial^2 w}{\partial y^2} \right| + 2 \left| \frac{\partial^2 w}{\partial x \partial y} \right| \right) \mathrm{d}x\mathrm{d}y$$

so that we have

$$\|\psi\|_h^2 \leq \frac{3h^4}{64} \left(\left\| \frac{\partial^2 w}{\partial x^2} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial^2 w}{\partial y^2} \right\|_{L^2(\Omega)}^2 + 4 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|_{L^2(\Omega)}^2 \right)$$

Theorem (Error estimate)

Let $f \in L^2(\Omega)$, $c \in \mathcal{C}^2(\bar{\Omega})$ with $c \geq 0$ in $\bar{\Omega}$, and suppose that the weak solution of the BVP (1) is in $H^3(\Omega)$. Then

$$\|u - U\|_{1,h} \leq Ch^2 \|u\|_{H^3(\Omega)}$$

Remark: This is the optimal error estimate in the sense that if we weaken the assumption $u \in H^3(\Omega)$ then we do not get $\mathcal{O}\left(h^2\right)$ error estimate in $\|\cdot\|_{1,h}$.