Galerkin Finite Element Method

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Boundary value problem

Partial differential equation

$$
\begin{aligned}
Lu &= f \qquad \text{in} \quad \Omega \\
Bu &= 0 \qquad \text{on} \quad \partial\Omega
\end{aligned}
$$

Weak formulation

Find $u \in V$ such that

$$a(u, v) = \ell(v) \qquad \forall\ v \in V \tag{1}$$

Theorem (Lax-Milgram)

Let V be a Hilbert space with norm $\|\cdot\|_V$

- $a : V \times V \to \mathbb{R}$ be a bilinear form
- (continuity) $\exists\ \gamma > 0$ such that

$$|a(u, v)| \leq \gamma \|u\|_V \|v\|_V \quad \forall\ u, v \in V$$

- (coercivity) $\exists\ \alpha > 0$ such that

$$a(u, u) \geq \alpha \|u\|_V^2 \quad \forall\ u \in V$$

- $\ell : V \to \mathbb{R}$ is a linear continuous functional, i.e., $\ell \in V'$

Then there exists a unique $u \in V$ that solves (1) and

$$\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'}$$

Symmetric case

If in addition, the bilinear form is symmetric, i.e.,

$$a(u, v) = a(v, u) \qquad \forall\, u, v \in V$$

then $a(\cdot, \cdot)$ is an inner product on V, and the Riesz representation theorem suffices to infer existence and uniqueness for the solution of (1). Moreover, this solution is also the solution to the following minimization problem

$$\text{find } u \in V \text{ such that } J(u) \leq J(v) \quad \forall\, v \in V$$

where

$$J(u) = \frac{1}{2} a(u, u) - \ell(u)$$

This is known as *Dirichlet principle*.

Galerkin method

We want to approximate V by a finite dimensional subspace $V_h \subset V$ where $h > 0$ is a small parameter that will go to zero

$$h \to 0 \quad \Longrightarrow \quad \dim(V_h) \to \infty$$

In the finite element method, h denotes the mesh spacing. Let

$$\{V_h : h > 0\}$$

denote a family of finite dimensional subspaces of V. We assume that

$$\forall\, v \in V, \qquad \inf_{v_h \in V_h} \|v - v_h\|_V \to 0 \quad \text{as} \quad h \to 0 \tag{2}$$

Galerkin approximation

Given $\ell \in V'$, find $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \qquad \forall\, v_h \in V_h \tag{3}$$

Theorem (Galerkin method)

Under the assumptions of Lax-Milgram theorem, there exists a unique solution u_h to (3) which is stable since

$$\|u_h\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'}$$

Moreover, if u is the solution to (1), it follows that

$$\|u - u_h\|_V \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V \tag{4}$$

hence u_h converges to u due to (2).

Proof: The existence and uniqueness of u_h follows from Lax-Milgram theorem. Stability is obtained from coercivity of a and continuity of ℓ

$$\alpha \|u_h\|_V^2 \leq a(u_h, u_h) = \ell(u_h) \leq \|\ell\|_{V'} \|u_h\|_V \quad \Longrightarrow \quad \|u_h\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'}$$

Now u and u_h satisfy

$$a(u, v_h) = \ell(v_h), \qquad a(u_h, v_h) = \ell(v_h), \qquad \forall\ v_h \in V_h$$

which implies that

$$a(u - u_h, v_h) = 0, \qquad \forall\ v_h \in V_h$$

Then

$$\begin{aligned} \alpha \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u) - a(u - u_h, u_h) \\ &= a(u - u_h, u) - 0 \\ &= a(u - u_h, u) - a(u - u_h, v_h) \quad \forall\ v_h \in V_h \\ &= a(u - u_h, u - v_h) \\ &\leq \gamma \|u - u_h\|_V \|u - v_h\|_V \end{aligned}$$

which implies that

$$\|u - u_h\|_V \leq \frac{\gamma}{\alpha} \|u - v_h\|_V \qquad \forall\ v_h \in V_h$$

This shows property (4) which is known as *Cea's lemma.* Convergence of u_h to u is obtained, i.e., $\|u - u_h\|_V \to 0$ as $h \to 0$ due to the approximation property (2) of the spaces V_h.

Symmetric case: Ritz method

When $a(\cdot,\cdot)$ is symmetric, Galerkin method is also known as *Ritz method*. In this case existence and uniqueness still follows from Riesz representation theorem. As $a(\cdot,\cdot)$ is a inner product, we have the *Galerkin orthogonality* property

$$a(u - u_h, v_h) = 0 \qquad \forall\ v_h \in V_h$$

the error $u - u_h$ of the Galerkin solution is orthogonal to the space V_h. Then we say that u_h is the *Ritz projecton* of u onto V_h.
Defining the *energy norm*

$$\|u\|_a = \sqrt{a(u,u)}$$

the error in energy norm is

$$\begin{aligned}
\|u - u_h\|_a^2 &= a(u - u_h, u - u_h) = a(u - u_h, u) - a(u - u_h, u_h) \\
&= a(u - u_h) - a(u - u_h, v_h), \qquad \forall\ v_h \in V_h \\
&= a(u - u_h, u - v_h) \leq \|u - u_h\|_a \|u - v_h\|_a
\end{aligned}$$

Hence

$$\|u - u_h\|_a \leq \|u - v_h\|_a \qquad \forall\ v_h \in V_h$$

Symmetric case: Ritz method

which implies that

$$\|u - u_h\|_a = \min_{v_h \in V_h} \|u - v_h\|_a$$

Thus u_h is the best approximation to u in the energy norm.
Moreover, u_h also solves the following minimization problem

$$J(u_h) = \min_{v_h \in V_h} J(v_h), \qquad J(u) = \frac{1}{2} a(u,u) - \ell(u)$$

In the symmetric case, we can also improve the Cea's lemma.

$$\begin{aligned}
\alpha \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) \\
&\leq \|u - u_h\|_a^2 \leq \|u - v_h\|_a^2 \qquad \forall\, v_h \in V_h \\
&= a(u - v_h, u - v_h) \leq \gamma \|u - v_h\|_V^2
\end{aligned}$$

which implies

$$\|u - u_h\|_V \leq \left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2}} \|u - v_h\|_V \qquad \forall\, v_h \in V_h$$

Symmetric case: Ritz method

Since

$$\alpha \|u\|_V^2 \leq a(u,u) \leq \gamma \|u\|_V^2$$

we get $\alpha < \gamma$ and hence $(\gamma/\alpha)^{\frac{1}{2}} < \gamma/\alpha$.

Remark: The problem of estimating the error in the Galerkin solution is reduced to estimating the approximation error

$$\inf_{v_h \in V_h} \|u - v_h\|_V$$

Remark: If $\alpha \ll \gamma$ then the Galerkin solution will have large error. This usually happens in convection-dominated situation. A very fine mesh $h \ll 0$ will be required to reduce the error to acceptable levels.

Galerkin method summary

- Write the weak formulation of the problem: find $u \in V$ such that $a(u, v) = \ell(v)$ for all $v \in V$. Existence, uniqueness and stability follow from Lax-Milgram theorem.
- Choose a family of finite dimensional spaces $V_h \subset V$ such that for

$$\forall\ v \in V, \qquad \inf_{v_h \in V_h} \|v - v_h\|_V \to 0 \quad \text{as} \quad h \to 0$$

- Find the Galerkin approximation: $u_h \in V_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in V_h$. Again use Lax-Milgram theorem.
- Convergence follows from Cea's lemma

$$\|u - u_h\|_V \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V \to 0 \qquad \text{as} \quad h \to 0$$

- Let $I_h : V \to V_h$ be the interpolation operator and show an error estimate

$$\forall\ u \in V, \qquad \|u - I_h u\|_V \leq C(u) h^p \qquad \text{for some } p > 0$$

  Then

$$\|u - u_h\|_V \leq \frac{\gamma}{\alpha} \|u - I_h u\|_V \leq \frac{\gamma}{\alpha} C(u) h^p \to 0 \qquad \text{as} \quad h \to 0$$

Laplace equation: Homogeneous BC

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 and given $f \in L^2(\Omega)$, consider

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

The weak formulation of this problem is:

$$\text{find} \quad u \in H_0^1(\Omega) \quad \text{such that} \quad a(u, v) = \ell(v) \qquad \forall\, v \in H_0^1(\Omega)$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \qquad \ell(v) = \int_\Omega f v \mathrm{d}x$$

By Poincare inequality, we have

$$\|u\|_0 \leq c(\Omega)|u|_1 \qquad \forall\, u \in H_0^1(\Omega)$$

Laplace equation: Homogeneous BC

so that $\|\cdot\|_1$ and $|\cdot|_1$ are equivalent norms. We verify the conditions of Lax-Milgram theorem in the norm $|\cdot|_1$. Continuity follows from Cauchy-Schwarz inequality

$$|a(u,v)| \leq |u|_1 |v|_1$$

while coercivity is trivial

$$a(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x = |u|_1^2$$

Also

$$|\ell(v)| \leq \|f\|_0 \|v\|_0 \leq c(\Omega) \|f\|_0 |v|_1$$

Thus existence and uniqueness of solution follows from Lax-Milgram theorem.

Galerkin method:
For $k \geq 1$, the approximating space is taken to be

$$V_h = X_h^k := \{v_h \in C^0(\bar{\Omega}) : v_h|_K \in \mathbb{P}_k,\ v_h|_{\partial\Omega} = 0\} \subset H_0^1(\Omega)$$

Laplace equation: Homogeneous BC

From Cea's lemma, we get the error estimate for the Galerkin solution u_h

$$|u - u_h|_1 \leq \inf_{v_h \in V_h} |u - v_h|_1 \leq |u - I_h^k u|_1$$

The interpolation error estimate tells us that

$$u \in H^s(\Omega),\ s \geq 2 \quad \Longrightarrow \quad |u - I_h^k u|_1 \leq C h^l |u|_{l+1}, \qquad 1 \leq l \leq \min(k, s-1)$$

which implies convergence of the Galerkin method

$$|u - u_h|_1 \leq |u - I_h^k u|_1 \leq C h^l |u|_{l+1} \to 0 \qquad \text{as} \quad h \to 0$$

Regularity theorem

Let a be an $H_0^1(\Omega)$ elliptic bilinear form with sufficiently smooth coefficient functions.

1. If Ω is convex, then the Dirichlet problem is H^2-regular.
2. If Ω has a C^s boundary with $s \geq 2$, then the Dirichlet problem is H^s-regular.

Laplace equation: Homogeneous BC

Theorem (Error in H^1-norm)

Suppose $\mathcal{T}_h$ is a regular family of triangulations of Ω which is a convex polygonal domain, then the finite element approximation $u_h \in X_h^k$ $(k \geq 1)$ satisfies

$$\|u - u_h\|_1 \leq Ch \|u\|_2 \leq Ch \|f\|_0$$

Proof: Since Ω is convex, we have $u \in H^2(\Omega)$ and $\|u\|_2 \leq C \|f\|_0$. Since the semi-norm $|\cdot|_1$ is an equivalent norm on $H_0^1(\Omega)$ we have

$$\|u - u_h\|_1 \leq C|u - u_h|_1 \leq C|u - I_h^k u|_1 \leq Ch|u|_2 \leq Ch \|u\|_2 \leq Ch \|f\|_0$$

Remark: We would also like to obtain error estimates in L^2-norm corresponding to interpolation error estimate $\|u - I_h^k u\|_0 \leq Ch^2|u|_2$. Consider the weak formulation

$$\text{find} \quad u \in V \quad \text{such that} \quad a(u, v) = \ell(v) \quad \forall\, v \in V$$

and its Galerkin approximation

$$\text{find} \quad u_h \in V_h \quad \text{such that} \quad a(u_h, v) = \ell(v_h) \qquad \forall\, v_h \in V_h$$

Aubin-Nitsche lemma

Let H be a Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let V be a subspace which is also a Hilbert space with norm $\|\cdot\|_V$. In addition let $V \hookrightarrow H$ be continuous. Then the finite element solution $u_h \in V_h$ satisfies

$$\|u - u_h\|_H \leq \gamma \|u - u_h\|_V \sup_{g \in H} \left\{ \frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right\} \tag{5}$$

where for every $g \in H$, $\varphi_g \in V$ denotes the corresponding unique weak solution of the dual problem

$$a(w, \varphi_g) = (g, w)_H \qquad \forall\, w \in V \tag{6}$$

Proof: Due to continuity of $V \hookrightarrow H$, we have for any $g, w \in H$

$$|(g, w)_H| \leq \|g\|_H \|w\|_H \leq C \|g\|_V \|w\|_V$$

By Lax-Milgram lemma, problem (6) has a unique solution. The Galerkin solution satisfies

$$a(u - u_h, v_h) = 0 \qquad \forall\, v_h \in V_h$$

Take $w = u - u_h$ in (6)

$$(g, u - u_h) = a(u - u_h, \varphi_g) = a(u - u_h, \varphi_g - v_h) \leq \gamma \|u - u_h\|_V \|\varphi_g - v_h\|_V$$

Since this is true for any $v_h \in V_h$ we obtain

$$(g, u - u_h) = a(u - u_h, \varphi_g) \leq \gamma \|u - u_h\|_V \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V$$

Now the error in H is given by

$$\begin{aligned} \|u - u_h\|_H &= \sup_{g \in H} \frac{(g, u - u_h)}{\|g\|_H} \\ &\leq \gamma \|u - u_h\|_V \sup_{g \in H} \left\{ \frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right\} \end{aligned}$$

Laplace equation: Homogeneous BC

L^2 error for Dirichlet problem

Under the conditions of previous theorem, we have

$$\|u - u_h\|_0 \leq Ch^2 \|f\|_0$$

Proof: Take $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. Then $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous since $\|\cdot\|_0 \leq \|\cdot\|_1$. Let $\varphi_{g,h} \in V_h$ be the Galerkin solution of problem (6). Then

$$\|\varphi_g - \varphi_{g,h}\|_1 \leq Ch \|g\|_0$$

and

$$\inf_{v_h \in V_h} \|\varphi_g - v_h\|_1 \leq \|\varphi_g - \varphi_{g,h}\|_1 \leq Ch \|g\|_0$$

and the Aubin-Nitsche lemma yields

$$\|u - u_h\|_0 \leq Ch \|u - u_h\|_1 \leq Ch^2 \|f\|_0$$

Laplace equation: Homogeneous BC

Numerical implementation:
Arrange the dofs so that all interior dofs are in the range $i = 1, 2, \ldots, M_h$ while the boundary dofs are $i = M_h + 1, \ldots, N_h$. Note that

$$\varphi_i(x) = 0, \quad x \in \partial\Omega, \quad i = 1, 2, \ldots, M_h$$

and

$$V_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_{M_h}\}$$

Then the Galerkin solution $u_h \in V_h$ can be written as

$$u_h = \sum_{j=1}^{M_h} u_j \varphi_j$$

The Galerkin formulation is

$$a(u_h, \varphi_i) = \ell(\varphi_i) \qquad i = 1, 2, \ldots, M_h$$

Trace theorem

The space $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$ for domains with Lipschitz continuous boundary.

Consequently we have the trace operator

$$\gamma : H^1(\Omega) \to L^2(\partial\Omega)$$

Trace theorem

Let Ω be a bounded open set of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$ and let $s > \frac{1}{2}$.

1. There exists a unique linear continuous map $\gamma_0 : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ such that $\gamma_0 v = v|_{\partial\Omega}$ for each $v \in H^s(\Omega) \cap C^0(\bar{\Omega})$.
2. There exists a linear continuous map $\mathcal{R}_0 : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ such that $\gamma_0 \mathcal{R}_0 \varphi = \varphi$ for each $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$.

Analogous results also hold true if we consider the trace γ_Σ over a Lipschitz continuous subset Σ of the boundary $\partial\Omega$.

Remark: Any $\varphi \in H^{s-\frac{1}{2}}(\Sigma)$ is the trace on Σ of a function in $H^s(\Omega)$.

Remark: The above theorem also yields the existence of a constant C such that

$$\int_{\partial\Omega} (\gamma_0 v)^2 \leq C \int_\Omega (v^2 + |\nabla v|^2), \qquad \forall\, v \in H^1(\Omega)$$

Remark: The map $\mathcal{R}_0$ is said to provide a *lifting* of the boundary values.

Variant of Lax-Milgram Lemma (Necas)

Let V and W be Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively.

- $a : V \times W \to \mathbb{R}$ be a bilinear form
- $\exists\ \gamma > 0$ such that

$$|a(v,w)| \leq \gamma \|v\|_V \|w\|_W \quad \forall\ v \in V,\ w \in W$$

- $\exists\ \alpha > 0$ such that

$$\sup_{w \in W, w \neq 0} \frac{a(v,w)}{\|w\|_W} \geq \alpha \|v\|_V \quad \forall\ v \in V$$

- $\sup_{v \in V} a(v,w) > 0$, $\forall\ w \in W$, $w \neq 0$
- $\ell : W \to \mathbb{R}$ is a linear continuous functional, i.e., $\ell \in W'$

Then there exists a unique $u \in V$ that solves:

$$\text{find} \quad u \in V \quad \text{such that} \quad a(u,w) = \ell(w) \quad \forall\ w \in W$$

and

$$\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{W'}$$

Laplace equation: Non-homogeneous BC

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 and given $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$, consider

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

Define the spaces

$$V = \{v \in H^1(\Omega) : \gamma_0 v = g\}, \qquad W = \{v \in H^1(\Omega) : \gamma_0 v = 0\} = H_0^1(\Omega)$$

Then the weak formulation

$$\text{find} \quad u \in V \quad \text{such that} \quad a(u, v) = \ell(v) \quad \forall\, v \in W$$

has a unique solution due to Lax-Milgram lemma.

Another formulation:
Due to trace theorem, there exists a lifting $u_g \in H^1(\Omega)$ of g such that $\gamma_0 u_g = g$. Define

$$a(\tilde{u}, v) = \int_\Omega \nabla\tilde{u} \cdot \nabla v, \qquad \ell(v) = \int_\Omega fv - \int_\Omega \nabla u_g \cdot \nabla v$$

Laplace equation: Non-homogeneous BC

Find $\tilde{u} \in H_0^1(\Omega)$ such that

$$a(\tilde{u}, v) = \ell(v), \qquad \forall\ v \in H_0^1(\Omega)$$

Then

$$u = \tilde{u} + u_g$$

solves our problem.

Galerkin formulation:
Write $u_h = \tilde{u}_h + u_{g,h}$ where the lifting can be taken as

$$u_{g,h} = \sum_{j=M_h+1}^{N_h} g_j \varphi_j \qquad \text{and} \qquad \tilde{u}_h = \sum_{j=1}^{M_h} u_j \varphi_j \in H_0^1(\Omega)$$

and the Galerkin formulation is

$$a(\tilde{u}_h, \varphi_i) = \ell_h(\varphi_i) \qquad i = 1, 2, \ldots, M_h$$

where

$$\ell_h(v) = \int_\Omega f v - \int_\Omega \nabla u_{g,h} \cdot \nabla v$$

Poincare-Friedrich's type inequality

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then there exists a constant $C = C(\Omega)$ such that

$$\|v\|_0 \leq C(|\bar{v}| + |v|_1) \qquad \forall\ v \in H^1(\Omega)$$

where

$$\bar{v} = \frac{1}{|\Omega|} \int_\Omega v(x) \mathrm{d}x$$

<u>Proof</u>: Suppose that the result is not true. Then we can find a sequence v_n such that

$$\|v_n\|_0 = 1, \qquad |\bar{v}_n| + |v_n|_1 < \frac{1}{n}$$

Since the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we can find a subsequence, still denoted v_n, which converges in $L^2(\Omega)$. This is a Cauchy sequence in $L^2(\Omega)$. By triangle inequality

$$|v_n - v_m|_1 \leq |v_n|_1 + |v_m|_1 < \frac{1}{n} + \frac{1}{m} \to 0, \quad \text{as} \quad n, m \to \infty$$

Hence v_n is also a Cauchy sequence in $H^1(\Omega)$ and hence converges to some $v \in H^1(\Omega)$ such that

$$\|v\|_0 = \lim_n \|v_n\|_0 = 1 \quad \text{and} \qquad \bar{v} = 0, \qquad |v|_1 = 0 \quad \Longrightarrow \quad v = 0$$

which leads to a contradiction.

Remark: For any $v \in V$ where

$$V = \{v \in H^1(\Omega) : \bar{v} = 0\}$$

we have the Poincare-Friedrich's inequality

$$\|v\|_0 \leq C|v|_1$$

Laplace equation: Neumann BC

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 and given $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, consider

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

Multiply by $v \in H^1(\Omega)$ and integrate by parts to get

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\partial\Omega} g v$$

If we take $v \equiv 1$ then we get the *compatibility condition* for the data

$$\int_\Omega f + \int_{\partial\Omega} g = 0$$

Define

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad \ell(v) = \int_\Omega f v + \int_{\partial\Omega} g v$$

Laplace equation: Neumann BC

But $a(\cdot,\cdot)$ is not coercive on $H^1(\Omega)$; moreover if u is a solution, then $u+c$ is also a solution for any $c \in \mathbb{R}$. Hence we can look for solutions in

$$V = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$$

Now $a(\cdot,\cdot)$ is coercive on V since we have Poincare inequality for any $v \in V$. The problem

$$\text{find} \quad u \in V \quad \text{such that} \quad a(u,v) = \ell(v) \quad \forall\ v \in V$$

has a unique solution. Because of the compatibility condition, the above equation is satisfied for $v =$ constant also, and hence for all $v \in H^1(\Omega)$.

Remark: The above formulation is not used in the Galerkin method since it is not possible to construct a finite dimensional space $V_h \subset V$. Instead we fix the value of the Galerkin solution at any point in Ω to some arbitray value, say zero. Suppose we fix the value of the last dof to zero; then

$$u_h = \sum_{j=1}^{N_h - 1} u_j \varphi_j \qquad \text{and} \qquad a(u_h, \varphi_i) = \ell(\varphi_i), \quad i = 1, 2, \ldots, N_h - 1$$

Another example

Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 and given $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, consider

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= g \qquad \text{on} \quad \partial\Omega \end{aligned}$$

The weak formulation is:

$$\text{find} \quad u \in H^1(\Omega) \quad \text{such that} \quad a(u,v) = \ell(v) \qquad \forall\ v \in H^1(\Omega)$$

where

$$a(u,v) = \int_\Omega (uv + \nabla u \cdot \nabla v), \qquad \ell(v) = \int_\Omega fv + \int_{\partial\Omega} gv$$

Remark: Dirichlet boundary conditions are built into the approximation spaces; hence they are called *essential boundary condition.* Neumann boundary condition is implemented through the weak formulation and are called *natural boundary condition.*

Another example

Galerkin formulation: There is no Dirichlet boundary condition and hence all dofs have to be determined from the Galerkin method. Find

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j$$

such that

$$a(u_h, \varphi_i) = \ell(\varphi_i), \qquad i = 1, 2, \dots, N_h$$