Lagrange finite elements and interpolation error

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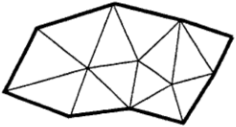
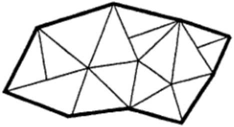
Finite element mesh

We want to approximate functions by piecewise polynomials.

We will assume that Ω is a polygonal domain.

We must first partition the domain Ω into a *finite element mesh* or *triangulation* $\mathcal{T}_h$ which is *admissible*, i.e.,

- $\mathcal{T}_h = \{K : K \text{ is a polygonal subdomain of } \Omega\}, \qquad \bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$
- If $K_1 \cap K_2$ is a single point, then it is a common vertex of K_1 and K_2.
- If $K_1 \cap K_2$ consists of more than one point, then $K_1 \cap K_2$ is a common edge or common face of K_1 and K_2.

Admissible and non-admissible triangulations

Complete Polynomials

For $d = 1, 2, 3$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$. Let us denote the space of polynomials of degree at most k by $\mathbb{P}_k$, i.e., any $p \in \mathbb{P}_k$ has the form

$$p(x) = \sum_{0<|\alpha|\leq k} a_\alpha x^\alpha, \qquad x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_d$$

$\underline{d = 1,\ k = 1}$

$$p(x) = a + bx$$

$\underline{d = 1,\ k = 2}$

$$p(x) = a + bx + cx^2$$

So for $d = 1$ we have

$$\dim(\mathbb{P}_k) = k + 1$$

$\underline{d = 2,\ k = 1}$

$$p(x, y) = a + bx + cy$$

$\underline{d = 2,\ k = 2}$

$$p(x, y) = a + bx + cy + dx^2 + exy + fy^2$$

So for $d = 2$ we have

$$\dim(\mathbb{P}_k) = \frac{1}{2}(k + 1)(k + 2)$$

1-D elements

We divide $\Omega = (a, b)$ into non-overlapping intervals

$$K_j = (x_{j-1}, x_j), \qquad h_j = x_j - x_{j-1}$$

which form the elements of our mesh. To determine a degree k polynomial on each element, we need $k+1$ function values at any $k+1$ distinct points in the element. We will use function values at the two end-points of the element. Any element $K_j = (x_{j-1}, x_j)$ can be mapped to unit interval $(0, 1)$ by an *affine map*

$$\hat{x} = \frac{1}{h_j}(x - x_{j-1}), \qquad x = x_{j-1} + h_j \hat{x}$$

<u>$p \in \mathbb{P}_1$, $k = 1$</u>

$$\hat{\varphi}_0(\hat{x}) = 1 - \hat{x}, \qquad \hat{\varphi}_1(\hat{x}) = \hat{x}$$

$$p(x) = p(x_{j-1})\hat{\varphi}_0(\hat{x}) + p(x_j)\hat{\varphi}_1(\hat{x})$$

<u>$p \in \mathbb{P}_2$, $k = 2$</u>

$$\hat{\varphi}_0(\hat{x}) = 2(\frac{1}{2} - \hat{x})(1 - \hat{x}), \qquad \hat{\varphi}_1(\hat{x}) = 4\hat{x}(1 - \hat{x}), \qquad \hat{\varphi}_2(\hat{x}) = 2\hat{x}(\hat{x} - \frac{1}{2})$$

$$p(x) = p(x_{j-1})\hat{\varphi}_0(\hat{x}) + p(x_{j-\frac{1}{2}})\hat{\varphi}_1(\hat{x}) + p(x_j)\hat{\varphi}_2(\hat{x}), \quad x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1} + x_j)$$

d-simplex elements

Simplex elements are triangles in 2-D and tetrahedra in 3-D.

A triangle K is defined by its three vertices $a_1, a_2, a_3 \subset \mathbb{R}^2$ and is the convex hull of these three vertices.

A tetrahedron K is defined by its four vertices $a_1, a_2, a_3, a_4 \subset \mathbb{R}^3$ and is the convex hull of these four vertices.

For $a_j = (a_{j,1}, \ldots, a_{j,d}) \in \mathbb{R}^d$ define the matrix

$$A = \begin{bmatrix} a_{1,1} & a_{2,1} & \ldots & a_{d+1,1} \\ \vdots & \vdots & & \vdots \\ a_{1,d} & a_{2,d} & \ldots & a_{d+1,d} \\ 1 & 1 & \ldots & 1 \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}$$

This matrix is non-singular if all the points do not lie on the same hyperplane since

$$\det(A) = \det \begin{bmatrix} a_{1,1} - a_{d+1,1} & \ldots & a_{d,1} - a_{d+1,1} \\ \vdots & & \vdots \\ a_{1,d} - a_{d+1,d} & \ldots & a_{d,d} - a_{d+1,d} \end{bmatrix} \neq 0$$

d-simplex elements

Any $x \in K$ can be written in terms of *barycentric coordinates*

$$x = \sum_{j=1}^{d+1} \lambda_j a_j, \qquad \sum_{j=1}^{d+1} \lambda_j = 1$$

These coordinates exist because they are just the solution of

$$A\lambda = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Thus $\lambda = \lambda(x) \in \mathbb{R}^{d+1}$ are affine functions of $x \in \mathbb{R}^d$. Moreover, they satisfy the interpolation property

$$\lambda_j(x_k) = \delta_{jk}$$

$\mathbb{P}_1$ on d-simplex

Let K be a d-simplex and suppose for $p \in \mathbb{P}_1$ we are given the information

$$\Sigma_K = \{p(a_i) : 1 \leq i \leq d+1\} = \text{Degrees of freedom}$$

Now the barycentric coordinate functions $\lambda_1(x), \ldots, \lambda_{d+1}(x) \in \mathbb{P}_1$ are linearly independent since

$$\sum_{j=1}^{d+1} c_j \lambda_j(x) = 0, \quad \forall\, x \in K \quad \Longrightarrow \quad c_j = 0 \quad \text{since} \quad \lambda_j(a_k) = \delta_{jk}$$

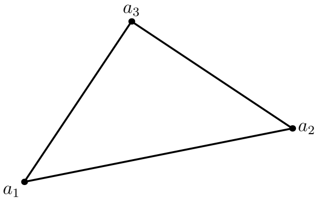
Thus the functions $\lambda_j(x)$ form a basis for $\mathbb{P}_1$ and we can write any $p \in \mathbb{P}_1$ as

$$p(x) = \sum_{j=1}^{d+1} \alpha_j \lambda_j(x)$$

But $$p(a_k) = \sum_{j=1}^{d+1} \alpha_j \lambda_j(a_k) = \sum_{j=1}^{d+1} \alpha_j \delta_{jk} = \alpha_k$$

Thus

$$p(x) = \sum_{j=1}^{d+1} p(a_j) \lambda_j(x)$$



$\mathbb{P}_1$ on d-simplex

The barycentric coordinates also have the geometric interpretation

$$\lambda_1(x) = \frac{\text{Area of triangle } (a_2, a_3, x)}{\text{Area of } K}, \qquad \text{etc.}$$

The barycentric functions λ_j are the basis functions.

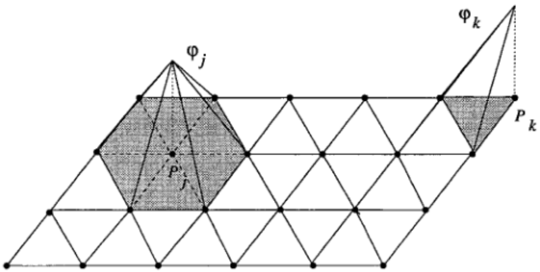
$$I_h v(x) = \sum_{j=1}^{d+1} v(a_j^K)\lambda_j^K(x), \qquad x \in K$$

or in terms of basis functions

$$I_h v(x) = \sum_j v_j \phi_j(x), \qquad x \in \Omega$$

Each ϕ_j is obtained by patching up the barycentric functions corresponding to vertex j.

$\mathbb{P}_1$ on d-simplex



In general, the support of each basis function involves at most the adjacent elements.

Remark: We constructed the interpolant on each element K individually. Show that the interpolant is continuous across the elements.

$\mathbb{P}_2$ on d-simplex

Let a_{ij} be the mid-points of the line joining a_i and a_j. Given $p \in \mathbb{P}_2$ let us use the information, which are the degrees of freedom,

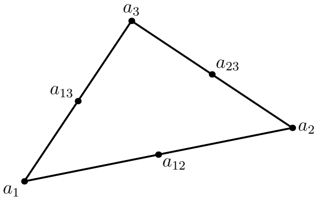
$$\Sigma_K = \{p(a_i),\ 1 \le i \le d+1; \quad p(a_{ij}),\ 1 \le i < j \le d+1\}$$

and we claim that this uniquely determines $p \in \mathbb{P}_2$. Firstly we note that we have as much information as the dimension of $\mathbb{P}_2$. Then the set of functions

$$\{\lambda_i(2\lambda_i - 1),\ 1 \le i \le d+1\} \cup \{\lambda_i \lambda_j,\ 1 \le i < j \le d+1\} \subset \mathbb{P}_2$$

are linearly independent since

$$\lambda_i(a_j) = \delta_{ij}, \quad \lambda_i(a_{jk}) = \begin{cases} \frac{1}{2} & \text{if } i = k \text{ or } i = j \\ 0 & \text{otherwise} \end{cases}$$



Let us write

$$p(x) = \sum_{i=1}^{d+1} \alpha_i \lambda_i(x)(2\lambda_i(x) - 1) + \sum_{1 \le i < j \le d+1} \beta_{ij} \lambda_i(x) \lambda_j(x)$$

$\mathbb{P}_2$ on d-simplex

Then

$$p(a_k) = \sum_{i=1}^{d+1} \alpha_i \delta_{ik}(2\delta_{ik} - 1) + 0 = \alpha_k$$

Further

$$p(a_{kl}) = \sum_{i=1}^{d+1} \alpha_i \lambda_i(a_{kl})(2\lambda_i(a_{kl}) - 1) + \sum_{1 \le i < j \le d+1} \beta_{ij} \lambda_i(a_{kl}) \lambda_j(a_{kl})$$

and the first sum is zero since all terms in the sum are zero. Now

$$\lambda_i(a_{kl})\lambda_j(a_{kl}) = \begin{cases} \frac{1}{4} & \text{if } (i,j) = (k,l) \text{ or } (i,j) = (l,k) \\ 0 & \text{otherwise} \end{cases}$$

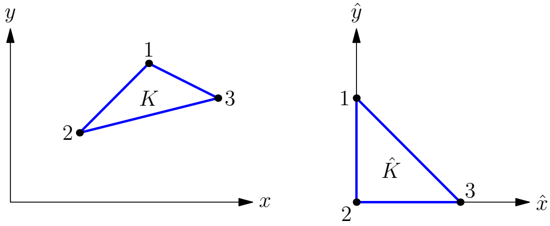
Hence $\beta_{kl} = 4p(a_{kl})$ and the polynomial is given by

$$p(x) = \sum_{i=1}^{d+1} p(a_i) \lambda_i(x)(2\lambda_i(x) - 1) + \sum_{1 \le i < j \le d+1} 4p(a_{ij}) \lambda_i(x) \lambda_j(x)$$

Affine elements

Let us select a reference d-simplex denoted $\hat{K}$.

For example in 2-D, we can choose the right-angled triangle with unit side length.



Any d-simplex K can be mapped to this reference d-simplex by a bijective, affine transformation

$$x = F_K(\hat{x}) = B_K \hat{x} + b_K \qquad \text{and} \qquad K = F_K(\hat{K})$$

where $B_K \in \mathbb{R}^{d \times d}$ is a non-singular matrix and $b_K \in \mathbb{R}^d$.

Affine elements

For example, in 2-D the affine map would be given by

$$\begin{aligned} x &= \hat{y}(x_1 - x_2) + \hat{x}(x_3 - x_2) + x_2 \\ y &= \hat{y}(y_1 - y_2) + \hat{x}(y_3 - y_2) + y_2 \end{aligned}$$

We can construct the interpolating polynomial $\hat{p} \in \mathbb{P}_k$ on the reference element $\hat{K}$ and then map it to the original element

$$p(x) = \hat{p}(\hat{x}) = \hat{p} \circ F_K^{-1}(x) = p \circ F_K(\hat{x})$$

Since F_K is affine, we note that $p \in \mathbb{P}_k$.

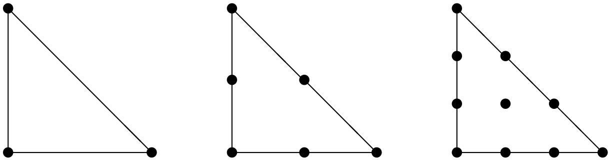
Theorem (Interpolation on triangles)

Let $k \geq 1$. Given a triangle K, suppose $z_1, z_2, \ldots, z_s$ with $z_i = (x_i, y_i)$ are the

$$s = 1 + 2 + \ldots + (k+1) = \frac{1}{2}(k+1)(k+2)$$

points in K which lie on $k+1$ lines as in figure. Then for every $f \in C^0(K)$, there is a unique polynomial p of degree $\leq k$ satisfying the interpolation conditions

$$p(z_i) = f(z_i), \qquad i = 1, 2, \ldots, s$$



Location of dofs for degree $k = 1, 2, 3$ polynomials

Proof: For $k = 1$ the result is true. Assume that it is true for $k - 1$ and we will show it is true for k. Let $z_1, z_2, \ldots, z_{k+1}$ be the points on the edge of the triangle lying on the x-axis. Then we can construct a unique univariate polynomial $p_0(x)$ solving the interpolation problem

$$p_0(x_i) = f(z_i), \qquad i = 1, 2, \ldots, k+1$$

By induction hypothesis, we know that there is a polynomial $q(x, y) \in \mathbb{P}_{k-1}$ which solves the interpolation problem

$$q(x_i, y_i) = \frac{1}{y_i}[f(z_i) - p_0(x_i)], \qquad i = k+2, \ldots, s$$

Then define

$$p(x, y) = p_0(x) + yq(x, y) \in \mathbb{P}_k$$

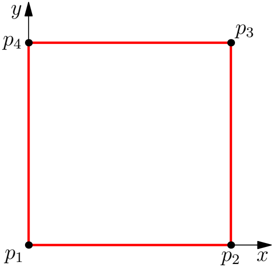
which solves the desired interpolation problem.

Rectangular elements

We cannot use the polynomials $\mathbb{P}_k$ to construct interpolation on rectangles because continuity cannot be maintained at the element boundaries. Instead we consider *tensor product polynomials* like

$$p(x, y) = a + bx + cy + dxy \in \mathbb{Q}_1$$

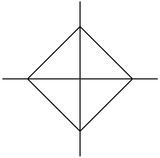
The coefficients (a, b, c, d) can be uniquely determined if we know the values of p at the four vertices. For example on the unit square we get

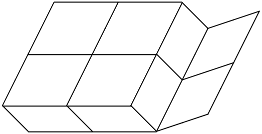


$$p(x, y) = p_1(1-x)(1-y) + p_2x(1-y) + p_3xy + p_4(1-x)y$$

The restriction of p to any edge is in $\mathbb{P}_1$ and because of this we get global continuity since neighbouring elements share the same nodal information.

Rectangular elements





But in the case of a rotated square as shown in figure, we cannot determine the above polynomial since the xy term vanishes at all the four vertices. This problem can be overcome by working with a reference element.

In 2-D (3-D), any parallelogram (parallelepiped) can be mapped to unit square (cube) by an affine transformation.

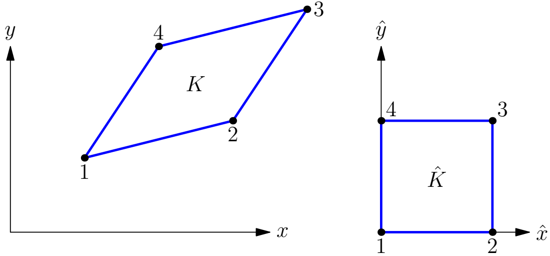
Let the sides 14 and 23 be given by the parallel lines

$$\alpha_1 x + \beta_1 y = \gamma_1, \qquad \alpha_1 x + \beta_1 y = \delta_1$$

and the sides 12 and 43 are given by

$$\alpha_2 x + \beta_2 y = \gamma_2, \qquad \alpha_2 x + \beta_2 y = \delta_2$$

Rectangular elements



Define the affine mapping to reference element (unit square) by

$$\hat{x} = \frac{\alpha_1 x + \beta_1 y - \gamma_1}{\delta_1 - \gamma_1}, \qquad \hat{y} = \frac{\alpha_2 x + \beta_2 y - \gamma_2}{\delta_2 - \gamma_2}$$

Then the polynomial $p \circ F_K \in \mathbb{Q}_1(\hat{K})$ interpolating the vertex values is

$$p(x,y) = \hat{p}(\hat{x},\hat{y}) = p_1(1-\hat{x})(1-\hat{y}) + p_2\hat{x}(1-\hat{y}) + p_3\hat{x}\hat{y} + p_4(1-\hat{x})\hat{y}$$

Quadrilateral element

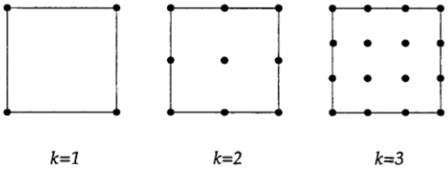
An arbitrary quadrilateral can be mapped to the unit square by the *non-affine map*

$$\begin{aligned} x &= x_1(1-\hat{x})(1-\hat{y}) + x_2\hat{x}(1-\hat{y}) + x_3\hat{x}\hat{y} + x_4(1-\hat{x})\hat{y} \\ y &= y_1(1-\hat{x})(1-\hat{y}) + y_2\hat{x}(1-\hat{y}) + y_3\hat{x}\hat{y} + y_4(1-\hat{x})\hat{y} \end{aligned}$$

Then the polynomial $p \circ F_K \in \mathbb{Q}_1(\hat{K})$ interpolating the vertex values is

$$p(x,y) = \hat{p}(\hat{x},\hat{y}) = p_1(1-\hat{x})(1-\hat{y}) + p_2\hat{x}(1-\hat{y}) + p_3\hat{x}\hat{y} + p_4(1-\hat{x})\hat{y}$$

The polynomials are *always* constructed on the reference element and then mapped to the actual element by the above non-affine map. The dofs required for $\mathbb{Q}_1$, $\mathbb{Q}_2$, $\mathbb{Q}_3$ are shown in figure



Quadrilateral element

The tensor product polynomials of degree k have the form

$$\hat{p}(\hat{x}) = \sum_{0\leq\alpha_1,\ldots,\alpha_d\leq k} a_\alpha \hat{x}^\alpha$$

These polynomials can also be characterized by the fact that their restriction to any side of the element K leads to a polynomial in $\mathbb{P}_k(K)$.

Examples:

$$\mathbb{Q}_1 = \text{span}\{1, \hat{x}, \hat{y}, \hat{x}\hat{y}\}, \qquad \mathbb{Q}_2 = \text{span}\{1, \hat{x}, \hat{y}, \hat{x}\hat{y}, \hat{x}^2, \hat{y}^2, \hat{x}\hat{y}^2, \hat{x}^2\hat{y}, \hat{x}^2\hat{y}^2\}$$

Example: Consider the case $d = 2$ and $k = 1$ and consider the restriction to the side $\hat{y} = 0$. The map F_K restricted to $\hat{y} = 0$ becomes affine

$$\begin{aligned} x &= x_1(1-\hat{x}) + x_2\hat{x} \\ y &= y_1(1-\hat{x}) + y_2\hat{x} \end{aligned}$$

and the polynomial restricted to $\hat{y} = 0$ is

$$p(x, y) = \hat{p}(\hat{x}, \hat{y}) = p_1(1-\hat{x}) + p_2\hat{x} \in \mathbb{P}_1$$

This ensures the continuity of the interpolation across elements.

Approximation by piecewise polynomials

Let X be an infinite dimensional space of functions

$$v \in X, \qquad v : \Omega \to \mathbb{R}$$

which we want to approximate by a finite dimensional space X_h of piecewise polynomials on the finite element mesh $\mathcal{T}_h$ of Ω. In the case of ***triangular*** elements, we take

$$X_h = X_h^k = \{v_h \in C^0(\bar{\Omega}) : v_h|_K \in \mathbb{P}_k(K) \ \forall \ K \in \mathcal{T}_h\}$$

while in the case of ***quadrilateral*** elements

$$X_h = X_h^k = \{v_h \in C^0(\bar{\Omega}) : v_h|_K \circ F_K \in \mathbb{Q}_k(\hat{K}) \ \forall \ K \in \mathcal{T}_h\}$$

For second order PDE, the weak formulation involves first derivatives. Hence we would like $X_h \subset H^1(\Omega)$.

Theorem

A function $v : \Omega \to \mathbb{R}$ belongs to $H^1(\Omega)$ if and only if

1. $v|_K \in H^1(K)$ for each $K \in \mathcal{T}_h$
2. for each common face $F = K_1 \cap K_2$, $K_1, K_2 \in \mathcal{T}_h$, the trace on F of $v|_{K_1}$ and $v|_{K_2}$ is the same.

Proof: (a) Using (1) define the functions $w_j \in L^2(\Omega)$, $j = 1, 2, \ldots, d$ by

$$w_j|_K := D_j(v|_K) \qquad \forall\ K \in \mathcal{T}_h$$

To conclude that $v \in H^1(\Omega)$ we will show that $w_j = D_j v$. By using Greens formula we can write for each $\phi \in \mathcal{D}(\Omega)$

$$\int_\Omega w_j \phi = \sum_K \int_K w_j \phi = -\sum_K \int_K v|_K D_j \phi + \sum_K \int_{\partial K} v|_K \phi n_{K,j}$$

where n_K is unit outward normal vector on ∂K. Since ϕ vanishes on $\partial\Omega$, and $n_{K_1} = -n_{K_2} =: n$ on a common face $F = K_1 \cap K_2$, we have, after using (2)

$$\int_\Omega w_j \phi = -\int_\Omega v D_j \phi + \sum_F \int_F (v|_{K_1} - v|_{K_2}) \phi n_j = -\int_\Omega v D_j \phi$$

i.e., $w_j = D_j v$ in the sense of distributions.

(b) On the other hand if we assume that $v \in H^1(\Omega)$ it follows immediately that (1) holds. Moreover, we have $w_j = D_j v$, hence by proceeding as before one finds

$$\sum_F \int_F (v|_{K_1} - v|_{K_2}) \phi n_j = 0 \qquad \forall\, \phi \in \mathcal{D}(\Omega), \quad j = 1, 2, \ldots, d$$

i.e., (2) is satisfied.

Remark: The finite elements we have considered on triangles and quadrilaterals satisfy the continuity condition required in the above theorem.

Remark: For fourth order PDE, the weak formulation would involve second derivatives. Then to have $X_h \subset H^2(\Omega)$ we would need to construct spaces like

$$X_h = X_h^k = \{v_h \in C^1(\bar{\Omega}) : v_h|_K \in \mathbb{P}_k(K)\ \forall\ K \in \mathcal{T}_h\}$$

i.e., we need continuity of derivatives across the elements.

Interpolation operator

Let us choose a

- finite element mesh $\mathcal{T}_h$
- a set of N_h dofs and associated locations

$$A_h = \{a_i \in \mathbb{R}^d, i = 1, 2, \ldots, N_h\}$$

  which allow us to construct a polynomial of degree k uniquely on each $K \in \mathcal{T}_h$
- we also have the Lagrange basis functions associated to each dof φ_i, $i = 1, 2, \ldots, N_h$ and $\varphi_i(a_j) = \delta_{ij}$

For any $v \in C^0(\bar{\Omega})$ we define the interpolation by

$$I_h^k v := \sum_{i=1}^{N_h} v(a_i)\varphi_i$$

Note that we actually constructed I_h^k by performing element-wise interpolations. If

$$\{a_{i,K}, \quad i = 1, 2, \ldots, N_K\} \subset A_h$$

Interpolation operator

are the nodes in K, the local interpolation on K is

$$I_K^k v := \sum_{i=1}^{N_K} v(a_{i,K})\varphi_i|_K$$

We thus have the following property

$$(I_h^k v)|_K = I_K^k(v|_K) \qquad \forall\ K \in \mathcal{T}_h, \quad v \in C^0(\bar{\Omega})$$

This allows us to analyze the interpolation error on each element independently of other elements. The analysis will make use of the affine transformations to the reference element. We first state results which allow us to relate the semi-norms on K and $\hat{K}$.

Proposition

For any $v \in H^m(K)$, $m \geq 0$, define $\hat{v} := v \circ F_K$. Then $\hat{v} \in H^m(\hat{K})$ and there exists a constant $C = C(m,d)$ such that

$$|\hat{v}|_{m,\hat{K}} \leq C \, \|B_K\|^m \, |\det \, B_K|^{-\frac{1}{2}} |v|_{m,K} \qquad \forall \; v \in H^m(K) \tag{1}$$

and

$$|v|_{m,K} \leq C \, \left\|B_K^{-1}\right\|^m |\det \, B_K|^{\frac{1}{2}} |\hat{v}|_{m,\hat{K}} \qquad \forall \; \hat{v} \in H^m(\hat{K}) \tag{2}$$

where $\|\cdot\|$ is the matrix norm associated to the euclidean norm in $\mathbb{R}^d$.

Proof: Since $C^\infty(K)$ is dense in $H^m(K)$ we prove the result for a smooth v.

$$|\hat{v}|^2_{m,\hat{K}} = \sum_{|\alpha|=m} \int_{\hat{K}} |D^\alpha \hat{v}|^2$$

Take the case $m = 1$.

$$\frac{\partial \hat{v}}{\partial \hat{x}_i} = \sum_{j=1}^{d} \frac{\partial \hat{v}}{\partial x_j} \frac{\partial x_j}{\partial \hat{x}_i} = \sum_{j=1}^{d} \frac{\partial v}{\partial x_j} (B_K)_{ij} \quad \Longrightarrow \quad \hat{D}\hat{v} = B_K D v$$

Therefore

$$|\hat{v}|^2_{1,\hat{K}} \leq \|B_K\|^2 \sum_{i=1}^{d} \int_{\hat{K}} \left| \frac{\partial v}{\partial x_i} \right|^2 = \|B_K\|^2 \sum_{i=1}^{d} \int_{K} \left| \frac{\partial v}{\partial x_i} \right|^2 \det(B_K^{-1})$$

which implies that

$$|\hat{v}|_{1,\hat{K}} \leq \|B_K\| \, |\det\ B_K|^{-\frac{1}{2}} |v|_{1,K}$$

For the case $m > 1$, we see that the matrix elements B_K appear m times when we do the change of variables, which leads to the factor of $\|B_K\|^m$ in the result.

The second inequality follows along the same lines.

We have to estimate $\|B_K\|$ and $\left\|B_K^{-1}\right\|$ in terms of the element size.

Diameter of $K \in \mathcal{T}_h$

$$h_K = \text{diam}(K) = \text{length of largest edge of } K$$

It is also the diameter of the smallest circle/sphere that circumscribes K.

Diameter of $\mathcal{T}_h$

$$h = \max_{K \in \mathcal{T}_h} h_K$$

$\rho_K =$ diameter of largest circle/sphere that can be inscribed inside K.

$\hat{h}$ and $\hat{\rho}$ refer to the same quantities on the reference element.

A family of triangulations $\{\mathcal{T}_h\}$ is called *regular* or *shape regular* if $\exists\ \kappa > 0$ such that

$$\frac{h_K}{\rho_K} \le \kappa$$

This prevents the triangles from becoming too flat as $h \to 0$. It implies that the interior angles of K do not become too small or too large.

Proposition

$$\|B_K\| \le \frac{h_K}{\hat{\rho}}, \qquad \left\|B_K^{-1}\right\| \le \frac{\hat{h}}{\rho_K} \tag{3}$$

Proof: The matrix norm is by definition

$$\|B_K\| = \sup_{\xi \neq 0} \frac{|B_K \xi|}{|\xi|} = \frac{1}{\hat{\rho}} \sup\{|B_K \xi| : |\xi| = \hat{\rho}\}$$

For each ξ satisfying $|\xi| = \hat{\rho}$ find two points $\hat{x}, \hat{y} \in \hat{K}$ such that $\hat{x} - \hat{y} = \xi$ with images x, y in K. Since

$$B_K \xi = F_K \hat{x} - F_K \hat{y} = x - y \quad \Longrightarrow \quad |B_K \xi| = |x - y| \le h_K$$

which proves the first inequality. The other one is proved similarly.

Remark: We will estimate the interpolation error $|v - I_K^k v|_{m,K}$ in terms of the interpolation error on the reference element $|\hat{v} - I_{\hat{K}}^k \hat{v}|_{m,\hat{K}}$ where $x = F_K(\hat{x})$ and $\hat{v}(\hat{x}) = v(x)$.

Bramble-Hilbert Lemma

Let $L : H^s(\hat{K}) \to H^m(\hat{K})$, $m \geq 0$, $s \geq 0$ be a linear and continuous mapping such that

$$L(\hat{p}) = 0 \qquad \forall\, \hat{p} \in \mathbb{P}_l, \quad l \geq 0$$

Then for each $\hat{v} \in H^s(\hat{K})$

$$|L(\hat{v})|_{m,\hat{K}} \leq \|L\| \inf_{\hat{p} \in \mathbb{P}_l} \|\hat{v} + \hat{p}\|_{s,\hat{K}} \tag{4}$$

In particular, for each $\hat{v} \in H^s(\hat{K})$, $s \geq 2$, we have

$$|\hat{v} - I^k_{\hat{K}} \hat{v}|_{m,\hat{K}} \leq \left\| I - I^k_{\hat{K}} \right\| \inf_{\hat{p} \in \mathbb{P}_l} \|\hat{v} + \hat{p}\|_{s,\hat{K}} \quad 0 \leq m \leq s, \quad 0 \leq l \leq k \tag{5}$$

Proof: Let $\hat{v} \in H^s(\hat{K})$. For any $\hat{p} \in \mathbb{P}_l$ we have

$$|L(\hat{v})|_{m,\hat{K}} = |L(\hat{v} + \hat{p})|_{m,\hat{K}} \leq \|L\| \, \|\hat{v} + \hat{p}\|_{s,\hat{K}}$$

from which (4) follows since the above holds for any $\hat{p} \in \mathbb{P}_l$.

For $s \geq 2$, we have $H^s(\hat{K}) \subset C^0(\hat{K})$ and the interpolation operator is defined on $H^s(\hat{K})$. Then

$$L := I - I_{\hat{K}}^k, \qquad L(\hat{p}) = 0, \qquad \forall\, \hat{p} \in \mathbb{P}_l, \quad 0 \leq l \leq k$$

so that L satisfies the conditions of the proposition and (5) follows.

Remark: We can show continuity of $I - I_{\hat{K}}^k$ in following way. Since $I_{\hat{K}}^k \hat{v}$ depends linearly on the function values $\hat{v}$, i.e.,

$$I_{\hat{K}}^k \hat{v}(\hat{x}) = \sum_i \hat{v}(\hat{a}_i)\hat{\varphi}_i(\hat{x}) \quad \Longrightarrow \quad |D^\alpha(I_{\hat{K}}^k \hat{v})| \leq C \sum_i |\hat{v}(\hat{a}_i)|$$

Since the inclusion $H^s \hookrightarrow H^2 \hookrightarrow C^0$ is continuous

$$|\hat{v}(\hat{a}_i)| \leq C \left\|\hat{v}\right\|_{s,\hat{K}}$$

we obtain

$$\left\|I_{\hat{K}}^k \hat{v}\right\|_{s,\hat{K}} \leq C \left\|\hat{v}\right\|_{s,\hat{K}}$$

Then

$$\left\|\hat{v} - I_{\hat{K}}^k \hat{v}\right\|_{s,\hat{K}} \leq \left\|\hat{v}\right\|_{s,\hat{K}} + \left\|I_{\hat{K}}^k \hat{v}\right\|_{s,\hat{K}} \leq C \left\|\hat{v}\right\|_{s,\hat{K}} \qquad \forall\, \hat{v} \in H^s(\hat{K})$$

Deny-Lions Lemma

For each $l \geq 0$ there exists a constant $C = C(l, d, \hat{K})$ such that

$$\inf_{\hat{p} \in \mathbb{P}_l} \|\hat{v} + \hat{p}\|_{l+1,\hat{K}} \leq C|\hat{v}|_{l+1,\hat{K}}, \qquad \forall\ \hat{v} \in H^{l+1}(\hat{K}) \tag{6}$$

Proof: We first show that there exists a constant $C = C(l, d, \hat{K})$ such that

$$\|\hat{v}\|_{l+1,\hat{K}} \leq C \left\{ |\hat{v}|^2_{l+1,\hat{K}} + \sum_{|\alpha| \leq l} \left(\int_{\hat{K}} D^\alpha \hat{v} \right)^2 \right\}^{\frac{1}{2}} \qquad \forall\ \hat{v} \in H^{l+1}(\hat{K}) \tag{7}$$

We proceed by contradiction. If the above did not hold, then we can find a sequence $\hat{v}_j \in H^{l+1}(\hat{K})$ such that

$$(a) \quad \|\hat{v}_j\|_{l+1,\hat{K}} = 1 \qquad \text{and} \qquad (b) \quad |\hat{v}_j|^2_{l+1,\hat{K}} + \sum_{|\alpha| \leq l} \left(\int_{\hat{K}} D^\alpha \hat{v}_j \right)^2 < \frac{1}{j^2}$$

Since the imbedding $H^{l+1}(\hat{K}) \hookrightarrow H^l(\hat{K})$ is compact, we can select a subsequence, still denoted $\hat{v}_j$, which strongly converges in $H^l(\hat{K})$. As a consequence of (b), $\hat{v}_j$ is a Cauchy sequence in $H^{l+1}(\hat{K})$ with limit $\hat{w}$ such that $\|\hat{w}\|_{l+1,\hat{K}} = 1$. Moreover

$$(c) \quad \int_{\hat{K}} D^\alpha \hat{w} = 0, \quad 0 \leq |\alpha| \leq l \qquad \text{and} \qquad (d) \quad D^\alpha \hat{w} = 0, \quad |\alpha| = l+1$$

(d) implies that $\hat{w} \in \mathbb{P}_l$ and (c) implies that $\hat{w} = 0$ which is a contradiction. Hence (7) is proved.

Now, for each $\hat{v} \in H^{l+1}(\hat{K})$ we can construct a unique $\hat{q} \in \mathbb{P}_l$ such that

$$\int_{\hat{K}} D^\alpha \hat{q} = -\int_{\hat{K}} D^\alpha \hat{v}, \qquad \forall\ |\alpha| \leq l$$

Applying (7) to $\hat{v} + \hat{q}$ we obtain

$$\inf_{\hat{p} \in \mathbb{P}_l} \|\hat{v} + \hat{p}\|_{l+1,\hat{K}} \leq \|\hat{v} + \hat{q}\|_{l+1,\hat{K}} \leq C|\hat{v} + \hat{q}|_{l+1,\hat{K}} = C|\hat{v}|_{l+1,\hat{K}}$$

Theorem

Let $0 \le m \le l+1$, where $l = \min(k, s-1) \ge 1$. Then there exists a constant $C = C(\hat{K}, I_{\hat{K}}^k, k, m, s, d)$ such that

$$|v - I_{\hat{K}}^k v|_{m,K} \le C \frac{h_K^{l+1}}{\rho_K^m} |v|_{l+1,K} \qquad \forall\, v \in H^s(K)$$

<u>Proof</u>: Since

$$(I - I_{\hat{K}}^k)(\hat{p}) = 0 \quad \text{for each} \quad \hat{p} \in \mathbb{P}_l, \quad 0 \le l \le k$$

and

$$I - I_{\hat{K}}^k : H^s(\hat{K}) \to H^s(\hat{K})$$

is continuous, we get from Bramble-Hilbert Lemma

$$|\hat{v} - I_{\hat{K}}^k \hat{v}|_{m,\hat{K}} \le C \inf_{\hat{p} \in \mathbb{P}_l} \|\hat{v} + \hat{p}\|_{s,\hat{K}} \qquad \forall\, 0 \le m \le s$$

To apply Deny-Lions Lemma, we need $l = s-1$. Hence we get

$$|\hat{v} - I_{\hat{K}}^k \hat{v}|_{m,\hat{K}} \le C |\hat{v}|_{l+1,\hat{K}}, \quad 0 \le m \le l+1, \quad l = \min(k, s-1)$$

Using (1), (2), (3) we get

$$
\begin{aligned}
|v - I_K^k v|_{m,K} &\leq C \left\|B_K^{-1}\right\|^m |\det\, B_K|^{\frac{1}{2}} |\hat{v} - I_{\hat{K}}^k \hat{v}|_{m,\hat{K}} \\
&\leq C \left\|B_K^{-1}\right\|^m |\det\, B_K|^{\frac{1}{2}} |\hat{v}|_{l+1,\hat{K}} \\
&\leq C \left\|B_K^{-1}\right\|^m |\det\, B_K|^{\frac{1}{2}} \left\|B_K\right\|^{l+1} |\det\, B_K|^{-\frac{1}{2}} |v|_{l+1,K} \\
&\leq C \frac{h_K^{l+1}}{\rho_K^m} |v|_{l+1,K}
\end{aligned}
$$

Remark: It is not useful to take $k > s-1$ since the maximim exponent of h_K is limited by $\min(k, s-1)$; the optimal choice is $k = s-1$.

Remark: If the triangulation is regular, then $\rho_K \geq \kappa h_K$ and the error estimate becomes

$$|v - I_{\hat{K}}^k v|_{m,K} \leq C h_K^{l+1-m} |v|_{l+1,K}$$

If $s = 2$, $k = 1$, then $\left\|v - I_{\hat{K}}^k v\right\|_K \leq C h_K^2 |v|_{2,K}$ and $|v - I_{\hat{K}}^k v|_{1,K} \leq C h_K |v|_{2,K}$

Theorem

Let $\mathcal{T}_h$ be a regular family of triangulations and assume that $m = 0, 1$ and $l = \min(k, s-1) \geq 1$. Then there exists a constant C independent of h such that

$$|v - I_h^k v|_{m,\Omega} \leq C h^{l+1-m} |v|_{l+1,\Omega} \qquad \forall\ v \in H^s(\Omega)$$

<u>Proof</u>:

$$\begin{aligned}
|v - I_h^k v|_{m,\Omega}^2 &= \sum_{K \in \mathcal{T}_h} |v - I_K^k v|_{m,K}^2 \\
&\leq \sum_{K \in \mathcal{T}_h} C^2 h_K^{2(l+1-m)} |v|_{l+1,K}^2 \\
&\leq C^2 h^{2(l+1-m)} \sum_{K \in \mathcal{T}_h} |v|_{l+1,K}^2 \\
&\leq C^2 h^{2(l+1-m)} |v|_{l+1,\Omega}^2
\end{aligned}$$

which gives the desired result. The restriction of $m = 0, 1$ is needed since for Lagrange interpolation $X_h^k \subset H^m(\Omega)$ holds only if $m \leq 1$.