FEM for convection-dominated and hyperbolic problems

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Convection-diffusion-reaction equation

$$\begin{aligned} -\varepsilon\Delta u + \beta\cdot\nabla u + \sigma u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \Gamma \end{aligned}$$

Assume that

$$\varepsilon > 0, \qquad \nabla\cdot\beta = 0, \qquad \sigma \ge 0$$

The bilinear form and linear functional are

$$B(u,v) = \int_\Omega (\varepsilon\nabla u\cdot\nabla v + (\beta\cdot\nabla u)v + \sigma uv), \qquad L(v) = \int_\Omega fv$$

The weak formulation is:

$$\text{find} \quad u \in H_0^1(\Omega) \quad \text{such that} \quad B(u,v) = L(v) \quad \forall\ v \in H_0^1(\Omega)$$

Recall the Poincare inequality:

$$\int_\Omega |v|^2 \le C_\Omega \int_\Omega |\nabla v|^2, \qquad \forall\ v \in H_0^1(\Omega)$$

Convection-diffusion-reaction equation

Continuity of B:

$$|B(u,v)| \leq \gamma |u|_1 |v|_1, \qquad \gamma = \varepsilon + C_\Omega^{\frac{1}{2}} \|\beta\|_\infty + C_\Omega \|\sigma\|_\infty$$

Coercivity of B:

$$B(u,u) \geq \varepsilon |u|_1^2$$

The existence and uniqueness of the solution follows from Lax-Milgram lemma with the stability estimate

$$|u|_1 \leq \frac{C_\Omega^{\frac{1}{2}}}{\varepsilon} \|f\|$$

Galerkin method

Choose V_h to be a finite dimensional space of continuous, piecewise polynomials so that $V_h \subset H_0^1(\Omega)$. Then the Galerkin method is:

$$\text{find} \quad u^h \in V_h \quad \text{such that} \quad B(u^h, v^h) = L(v^h) \quad \forall\, v^h \in V_h$$

Cea's lemma together with interpolation error estimates gives us the error estimate for u^h as

$$|u^h - u|_1 \le \frac{\gamma}{\varepsilon} \inf_{v^h \in V_h} |v^h - u|_1 \le \left(1 + C_\Omega^{\frac{1}{2}} \frac{\|\beta\|_\infty}{\varepsilon} + C_\Omega \frac{\|\sigma\|_\infty}{\varepsilon}\right) C h^k \|u\|_{k+1}$$

and guarantees convergence of the solution u^h to u as $h \to 0$. However, if $\varepsilon \ll \gamma$ which can happen if $\varepsilon \ll \|\beta\|_\infty$ or $\varepsilon \ll \|\sigma\|_\infty$, then the Galerkin solution can have large error for finite h. We would have to use a very fine mesh (small h) in order to reduce the error to acceptable levels.

Model problems

Let Ω be a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary Γ and let $\beta = (\beta_1, \beta_2)$ be a constant vector with $|\beta| = 1$. Consider the following stationary boundary value problem

$$\begin{aligned} -\varepsilon \Delta u + u_\beta + u &= f \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \Gamma \end{aligned} \tag{1}$$

where ε is a positive constant and

$$u_\beta := \beta \cdot \nabla u$$

denotes derivative in the β-direction.

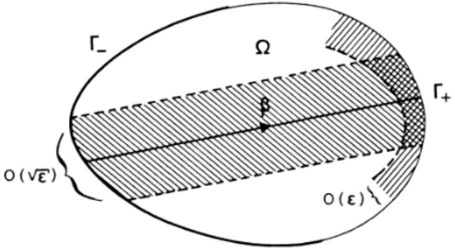
Simplified, hyperbolic model: If $\varepsilon \ll 1$ we can choose to ignore the elliptic term

$$\begin{aligned} u_\beta + u &= f \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \Gamma_- \end{aligned} \tag{2}$$

where now the boundary condition can be specified only on the inflow portion

$$\Gamma_- = \{x \in \Gamma : n \cdot \beta \leq 0\}$$

Model problems



- If the boundary data g is discontinous on Γ_-, then the diffusion model creates an internal layer whose width is $\mathcal{O}\left(\sqrt{\varepsilon}\right)$.
- If $\varepsilon \ll 1$ then the data g on Γ_- is carried towards Γ_+ along the streamlines β. But if this solution is different from the data $g_{|\Gamma_+}$ then a boundary layer of width $\mathcal{O}\left(\varepsilon\right)$ is created.

Norms, interpolation error

$$\|v\| = \|v\|_{L_2(\Omega)}\,, \qquad \|v\|_s = \|v\|_{H^s(\Omega)}$$

$$\langle v, w\rangle = \int_\Gamma (n\cdot\beta) vw, \qquad \langle v, w\rangle_\pm = \int_{\Gamma_\pm} (n\cdot\beta) vw$$

$$|v| = \left(\int_\Gamma |n\cdot\beta| v^2\right)^{\frac{1}{2}}$$

Greens' formula:

$$(v_\beta, w) = \langle v, w\rangle - (v, w_\beta)$$

Let $\{\mathcal{T}_h\}$ be a family of shape regular, quasi-uniform triangulations of Ω with mesh size h. For a given positive integer k we introduce the finite element space

$$V_h = \{v \in \mathcal{C}^0(\bar{\Omega}) : v|_K \in \mathbb{P}_k \ \forall\ K \in \mathcal{T}_h\}$$

From the theory of interpolation error estimates we have

$$\|u - I_h u\| \leq Ch^{k+1}\|u\|_{k+1} \tag{3}$$

$$|u - I_h u|_1 \leq Ch^{k}\|u\|_{k+1} \tag{4}$$

Norms, interpolation error

Moreover, if the derivatives of u of order $k+1$ are bounded on $\bar{\Omega}$, then

$$|u - I_h u| \leq C h^{k+1}$$

and with less stringent regularity requirements (see Ciarlet)

$$|u - I_h u| \leq C h^{k+\frac{1}{2}} \|u\|_{k+1} \tag{5}$$

These results imply that

$$\|u - I_h u\| + |u - I_h u|_1 + |u - I_h u| \leq C h^k \|u\|_{k+1}$$

In the sequel we will frequently use the following inequality: for real numbers a, b and for any $\gamma > 0$

$$2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2$$

Standard Galerkin method

For simplicity, assume the boundary data $g \equiv 0$. Then the weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$\varepsilon(\nabla u, \nabla v) + (u_\beta + u, v) = (f, v) \qquad \forall\ v \in H_0^1(\Omega)$$

With the finite element space

$$\mathring{V}_h = \{v \in V_h : v|_\Gamma = 0\}$$

the Galerkin method is: find $u^h \in \mathring{V}_h$ such that

$$\varepsilon(\nabla u^h, \nabla v^h) + (u_\beta^h + u^h, v^h) = (f, v^h) \qquad \forall\ v^h \in \mathring{V}_h$$

A 1-D Example

Consider the boundary value problem

$$\begin{aligned} -\varepsilon u_{xx} + u_x =& 0, \qquad 0 < x < 1 \\ u(0) = 1,& \qquad u(1) = 0 \end{aligned}$$

with $0 < \varepsilon \ll 1$. The exact solution is

$$u(x) = \frac{1 - \mathrm{e}^{-\frac{1-x}{\varepsilon}}}{1 - \mathrm{e}^{-\frac{1}{\varepsilon}}} \qquad \text{has boundary layer near } x = 1$$

If we apply the Galerkin method with piecewise linear basis functions on a uniform mesh of size h, then the resulting set of equations can be written as

$$-\frac{\varepsilon}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + \frac{1}{2h}(u_{i+1} - u_{i-1}) = 0$$

This is identical to applying central finite difference scheme, and we know that this scheme produces oscillatory solutions in the boundary layer if the cell Peclet number is large

$$\mathrm{Pe}_h = \frac{h}{\varepsilon} > 2$$

Galerkin method for hyperbolic problem

Galerkin method with strongly imposed boundary conditions
Find $u^h \in V_h$ with $u^h = g$ at the nodes on Γ_- such that

$$(u^h_\beta + u^h, v^h) = (f, v^h) \qquad \forall\ v^h \in V_h \quad \text{with} \quad v^h = 0 \quad \text{on} \quad \Gamma_-$$

Galerkin method with weakly imposed boundary conditions
Find $u^h \in V_h$ such that

$$(u^h_\beta + u^h, v^h) - \left\langle u^h, v^h \right\rangle_- = (f, v^h) - \left\langle g, v^h \right\rangle_- \qquad \forall\ v^h \in V_h$$

Introduce the notation

$$a(v, w) = (v_\beta + v, w) - \langle v, w \rangle_-\,, \qquad \ell(w) = (f, w) - \langle g, w \rangle_-$$

The Galerkin method reads: find $u^h \in V_h$ such that

$$a(u^h, v^h) = \ell(v^h) \qquad \forall\ v^h \in V_h \tag{6}$$

The exact solution also satisfies the above equation, i.e.,

$$a(u, v^h) = \ell(v^h) \qquad \forall\ v^h \in V_h$$

Galerkin method for hyperbolic problem

Subtracting the two, we get for the error $e = u - u^h$

$$a(e, v^h) = 0 \qquad \forall\ v^h \in V_h$$

To show stability, we need the following property of the bilinear form.

Lemma

For any $v \in H^1(\Omega)$ we have

$$a(v, v) = \|v\|^2 + \frac{1}{2}|v|^2$$

Proof: By Green's formula

$$(v_\beta, v) = -(v, v_\beta) + \langle v, v \rangle$$

so that

$$(v_\beta, v) = \frac{1}{2}\langle v, v \rangle = \frac{1}{2}\langle v, v \rangle_+ + \frac{1}{2}\langle v, v \rangle_-$$

Galerkin method for hyperbolic problem

Hence

$$\begin{aligned} a(v,v) &= (v_\beta, v) + (v,v) - \langle v,v\rangle_- \\ &= \frac{1}{2}\langle v,v\rangle_+ + \frac{1}{2}\langle v,v\rangle_- + \|v\|^2 - \langle v,v\rangle_- \\ &= \|v\|^2 + \frac{1}{2}\langle v,v\rangle_+ - \frac{1}{2}\langle v,v\rangle_- \\ &= \|v\|^2 + \frac{1}{2}|v|^2 \end{aligned}$$

where the last step follows because $n\cdot\beta \geq 0$ on Γ_+ and $n\cdot\beta < 0$ on Γ_-.

Remark: We obtain the stability estimate

$$\left\|u^h\right\|^2 + \frac{1}{2}|u^h|^2 = a(u^h,u^h) = \ell(u^h) \leq \|f\|\left\|u^h\right\| \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\left\|u^h\right\|^2$$

and hence

$$\left\|u^h\right\|^2 + |u^h|^2 \leq \|f\|^2$$

Galerkin method for hyperbolic problem

Theorem (Error estimate)

Let u and u^h be solutions of (2) and (6). Then

$$\left\|u - u^h\right\| + |u - u^h| \le Ch^k \left\|u\right\|_{k+1}$$

Proof: Write the error

$$e = u - u^h = (u - I_h u) + (I_h u - u^h) = \rho + \theta$$

Using previous Lemma and noting that $\theta \in V_h$

$$\begin{aligned}
\left\|\theta\right\|^2 + \frac{1}{2}|\theta|^2 &= a(\theta, \theta) = a(e - \rho, \theta) = a(e, \theta) - a(\rho, \theta) \\
&= -a(\rho, \theta) = -(\rho_\beta, \theta) - (\rho, \theta) + \langle \rho, \theta \rangle_- \\
&\le \left\|\rho_\beta\right\|^2 + \frac{1}{4}\left\|\theta\right\|^2 + \left\|\rho\right\|^2 + \frac{1}{4}\left\|\theta\right\|^2 + |\rho|^2 + \frac{1}{4}|\theta|^2 \\
&= \left\|\rho_\beta\right\|^2 + \left\|\rho\right\|^2 + |\rho|^2 + \frac{1}{2}\left(\left\|\theta\right\|^2 + \frac{1}{2}|\theta|^2\right)
\end{aligned}$$

Galerkin method for hyperbolic problem

and hence

$$\|\theta\|^2 + \frac{1}{2}|\theta|^2 \leq 2(\|\rho_\beta\|^2 + \|\rho\|^2 + |\rho|^2) \leq Ch^{2k} \|u\|_{k+1}$$

where the last inequality follows using (3), (4), (5). Now

$$(\|\theta\|+|\theta|)^2 = \|\theta\|^2+|\theta|^2+2\|\theta\|\,|\theta| \leq \|\theta\|^2+|\theta|^2+2\|\theta\|^2 + \frac{1}{2}|\theta|^2 = 3(\|\theta\|^2+\frac{1}{2}|\theta|^2)$$

and hence

$$\|\theta\| + |\theta| \leq Ch^k \|u\|_{k+1}$$

Finally, since $e = \theta + \rho$

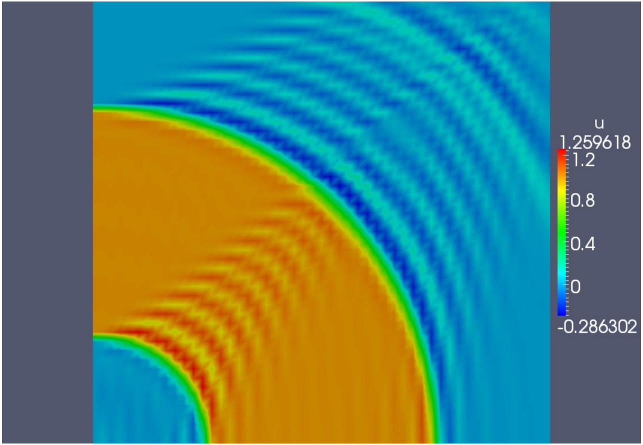
$$\|e\| + |e| \leq \|\theta\| + \|\rho\| + |\theta| + |\rho| \leq Ch^k \|u\|_{k+1}$$

where we used interpolation error estimates (3) and (5) for $\|\rho\|$ and $|\rho|$.

Remark: This shows that if the solution u is smooth enough, then the Galerkin method converges at the rate h^k, which is one order less than the rate observed for elliptic equations. If the solution is not smooth, then the Galerkin method leads to large error.

Solution of $\beta \cdot \nabla u = 0$ using Galerkin Method

$$\beta = (y, -x), \qquad u(0,y) = \begin{cases} 0 & \text{if} \quad |y - \frac{1}{2}| > \frac{1}{4} \\ 1 & \text{otherwise} \end{cases}, \qquad u(x,1) = 0$$



Degree = 1

Classical artificial diffusion

Instead of solving the hyperbolic equation, we add an elliptic term

$$h(\nabla u^h, \nabla v^h) + (u^h_\beta + u^h, v^h) = (f, v^h) \qquad \forall\ v^h \in \mathring{V}_h$$

Mesh Peclet number is

$$\mathrm{Pe}_h = \frac{h}{h} = 1$$

- This scheme leads to non-oscillating solutions even for discontinouos data, but the discontinuities are highly smeared, even in the direction perpendicular to the streamlines.
- The solutions will only be first order accurate due to the presence of the $\mathcal{O}(h)$ term.
- Note the the scheme is not strongly consistent, i.e., the true solution u does not satisfy the numerical scheme.

Motivation for streamline diffusion method

For the equation

$$\beta \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

add an artificial diffusion term with $\mu \geq 0$

$$\beta \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \mu \frac{\mathrm{d}u}{\mathrm{d}x}$$

Multiply by a test function v and integrate

$$\int \beta \frac{\mathrm{d}u}{\mathrm{d}x} v = \int v \frac{\mathrm{d}}{\mathrm{d}x} \mu \frac{\mathrm{d}u}{\mathrm{d}x} = -\int \mu \frac{\mathrm{d}u}{\mathrm{d}x} \frac{\mathrm{d}v}{\mathrm{d}x}$$

Choosing

$$\mu = \beta \tau \beta$$

we can write

$$\int \beta \frac{\mathrm{d}u}{\mathrm{d}x} \underbrace{\left(v + \tau \beta \frac{\mathrm{d}v}{\mathrm{d}x} \right)}_{\text{test function}} = 0$$

Motivation for streamline diffusion method

The test function is modified with the addition of the derivative in the direction of β. The scheme is strongly consistent since the exact solution satisfies the numerical scheme. It can be shown that with piecewise linear polynomials, the solution of the above scheme is nodally exact provided we choose

$$\tau = \frac{h}{2|\beta|}$$

and the resulting system of discrete equations is

$$\frac{\beta + |\beta|}{2} \frac{u_j - u_{j-1}}{h} + \frac{\beta - |\beta|}{2} \frac{u_{j+1} - u_h}{h} = 0$$

which is precisely the upwind scheme.

Remark: The above method has test functions which lie in a different space compared to trial functions. Such finite element schemes are said to be *Petrov-Galerkin*, and the above scheme is also called as *Streamline Upwind Petrov-Galerkin* (SUPG) method.

Streamline diffusion method: hyperbolic problem

In this method, we replace the test function v^h by $v^h + hv^h_\beta$. The scheme is: find $u^h \in V_h$ such that

$$(u^h_\beta + u^h, v^h + hv^h_\beta) - (1+h)\left\langle u^h, v^h \right\rangle_- = (f, v^h + hv^h_\beta) - (1+h)\left\langle g, v^h \right\rangle_-$$

where for convenience of analysis, we have introduced the factor $(1+h)$ which may be ignored in the numerical computations. We notice the presence of the term $h(u^h_\beta, v^h_\beta)$ which adds some dissipation to the scheme in the direction of β, the streamline direction. Defining

$$B(w,v) = (w_\beta + w, v + hv_\beta) - (1+h)\left\langle w, v \right\rangle_-$$

and

$$L(v) = (f, v + hv_\beta) - (1+h)\left\langle g, v \right\rangle_-$$

The SUPG scheme is: find $u^h \in V_h$ such that

$$B(u^h, v^h) = L(v^h) \qquad \forall\, v^h \in V_h \tag{7}$$

Streamline diffusion method: hyperbolic problem

Note that the true solution satisfies the above equation, i.e.,

$$B(u, v^h) = L(v^h) \qquad \forall\ v^h \in V_h$$

and subtracting the previous two equations, we obtain

$$B(e, v^h) = B(u - u^h, v^h) = 0 \qquad \forall\ v^h \in V_h$$

We will prove an error estimate in the following norm

$$\|v\|_\beta = \left(h\, \|v_\beta\|^2 + \|v\|^2 + \frac{1}{2}(1+h)|v|^2 \right)^{\frac{1}{2}}$$

This choice of the norm is related to the following stability property of the bilinear form

Lemma

For any $v \in H^1(\Omega)$ we have

$$B(v, v) = \|v\|_\beta^2$$

Streamline diffusion method: hyperbolic problem

Proof: By Green's formula

$$(v_\beta, v) = \frac{1}{2} \langle v, v \rangle$$

and thus

$$\begin{aligned} B(v,v) &= \frac{1+h}{2} \langle v, v \rangle - (1+h) \langle v, v \rangle_- + \|v\|^2 + h \|v_\beta\|^2 \\ &= \frac{1+h}{2} (\langle v, v \rangle_+ - \langle v, v \rangle_-) + \|v\|^2 + h \|v_\beta\|^2 \\ &= \frac{1+h}{2} |v|^2 + \|v\|^2 + h \|v_\beta\|^2 = \|v\|_\beta^2 \end{aligned}$$

which proves the desired equality.

Theorem

Let u and u^h be solutions of (2) and (7). Then

$$\left\| u - u^h \right\|_\beta \leq C h^{k+\frac{1}{2}} \|u\|_{k+1}$$

Streamline diffusion method: hyperbolic problem

Proof: Write the error

$$e = u - u^h = (u - I_h u) + (I_h u - u^h) = \rho + \theta$$

Then

$$\begin{aligned} \|e\|_\beta^2 &= B(e,e) = B(e,\rho+\theta) = B(e,\rho) + B(e,\theta) = B(e,\rho) \\ &= (e_\beta,\rho) + h(e_\beta,\rho_\beta) + (e,\rho) + h(e,\rho_\beta) - (1+h)\langle e,\rho\rangle_- \\ &\le \frac{h}{4}\|e_\beta\|^2 + h^{-1}\|\rho\|^2 + \frac{h}{4}\|e_\beta\|^2 + h\|\rho_\beta\|^2 + \frac{1}{4}\|e\|^2 + \|\rho\|^2 + \\ &\quad \frac{1}{4}\|e\|^2 + h^2\|\rho_\beta\|^2 + \frac{1+h}{4}|e|^2 + (1+h)|\rho|^2 \\ &= \frac{1}{2}\|e\|_\beta^2 + (1+h^{-1})\|\rho\|^2 + h(1+h)\|\rho_\beta\|^2 + (1+h)|\rho|^2 \end{aligned}$$

so that

$$\|e\|_\beta^2 \le 2[(1+h^{-1})\|\rho\|^2 + h(1+h)\|\rho_\beta\|^2 + (1+h)|\rho|^2] \le Ch^{2k+1}\|u\|_{k+1}^2$$

which proves the desired result.

Streamline diffusion method: hyperbolic problem

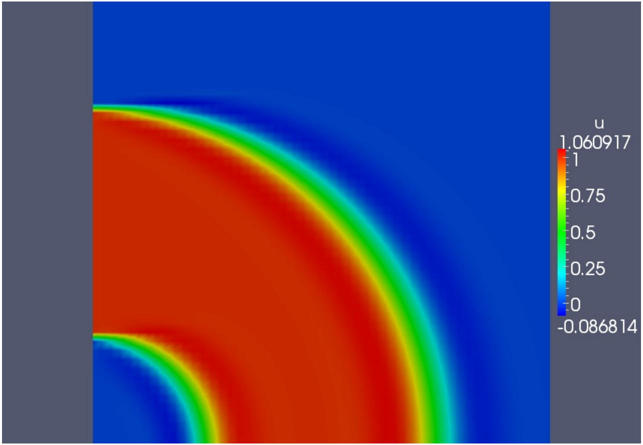
Remark: The above error estimates show that

$$\left\|u - u^h\right\| \leq Ch^{k+\frac{1}{2}} \left\|u\right\|_{k+1}, \qquad \left\|u_\beta - u_\beta^h\right\| \leq Ch^k \left\|u\right\|_{k+1}$$

The L_2 error estimate is improved by a factor of $\frac{1}{2}$ as compared to Galerkin method. Moreover we get some control on the derivative of the solution along the streamline u_β which is optimal. We had no such control on the derivative in the Galerkin method. However, we do not have any control on the derivative normal to the streamline, which can lead to oscillatory solutions with the SUPG method also.

Remark: The error estimates require the solution to be smooth but the SUPG method performs reasonably well even if the solution is not very regular. It can be shown that the effect of a source at a certain point $P \in \Omega$ decays at least as rapidly as $\exp(-Cd/\sqrt{h})$ where d is distance to P in directions perpendicular to the characteristics, and like $\exp(-Cd/h)$ in directions opposite to the characteristics (upwind direction). In particular, this means that the effect of a jump in the exact solution across a characteristic will be limited to a width of $\mathcal{O}\left(\sqrt{h}\right)$.

Solution of $\beta \cdot \nabla u = 0$ using SUPG Method



Degree $= 1$

Various schemes continue to be proposed to improve the SUPG method for better shock-capturing. Hughes, Franca, Mallet (1986) proposed a modified test function

$$v^h + \delta_h \beta \cdot \nabla v^h + \bar{\delta}_h \bar{\beta} \cdot \nabla v^h$$

where

$$\bar{\beta} = \frac{\beta \cdot \nabla u^h}{|\nabla u^h|^2} \nabla u^h$$

i.e., $\bar{\beta}$ is the projection of β onto ∇u^h. In this case the scheme is non-linear even for a linear PDE. As usual

$$\delta_h = \mathcal{O}\left(\frac{h}{|\beta|}\right) \qquad \bar{\delta}_h = \mathcal{O}\left(\frac{h}{|\bar{\beta}|}\right)$$

Another approach is to add numerical viscosity

$$(\varepsilon_h \nabla u^h, \nabla v^h)$$

where $\varepsilon_h = \varepsilon_h(u^h)$ is non-zero only in the regions where a shock is detected or the solution is oscillatory.

SUPG method for convection-diffusion problem

Let us consider problem (1) with homogeneous boundary conditions $g \equiv 0$.
For any $v \in H_0^1(\Omega)$ multiply by $v + \delta_h v_\beta$

$$(-\varepsilon \Delta u + u_\beta + u, v + \delta_h v_\beta) = (f, v + \delta_h v_\beta)$$

$$-\varepsilon(\Delta u, v) - (\varepsilon \Delta u, \delta_h v_\beta) + (u_\beta + u, v + \delta_h v_\beta) = (f, v + \delta_h v_\beta)$$

$$\varepsilon(\nabla u, \nabla v) - (\varepsilon \Delta u, \delta_h v_\beta) + (u_\beta + u, v + \delta_h v_\beta) = (f, v + \delta_h v_\beta)$$

The SUPG scheme is given by: find $u^h \in \mathring{V}_h$ such that

$$\varepsilon(\nabla u^h, \nabla v^h) - (\varepsilon \Delta u^h, \delta_h v_\beta^h) + (u_\beta^h + u^h, v^h + \delta_h v_\beta^h) = (f, v^h + \delta_h v_\beta^h) \qquad \forall\, v^h \in \mathring{V}_h$$

We make the second term meaningful by writing it as

$$(\varepsilon \Delta u^h, \delta_h v_\beta^h) = \sum_{K \in \mathcal{T}_h} (\varepsilon \Delta u^h, \delta_h v_\beta^h)_K$$

It is also useful to write the SUPG scheme in the following form

$$\varepsilon(\nabla u^h, \nabla v^h) + (u_\beta^h + u^h - f, v^h) + (-\varepsilon \Delta u^h + u_\beta^h + u^h - f, \delta_h v_\beta^h) = 0$$

SUPG method for convection-diffusion problem

The method is strongly consistent since if we plug in the exact solution, the additional SUPG terms vanish. The factor δ_h is chosen as

$$\delta_h = \begin{cases} \bar{C}h & \text{if} \quad \varepsilon < h \\ 0 & \text{otherwise} \end{cases}$$

If the constant $\bar{C}$ is small enough, we can show that the term $-(\varepsilon \Delta u^h, \delta_h v_\beta^h)$ does not degrade the extra stability introduced by the term $(u_\beta^h, \delta_h v_\beta^h)$. Using the inverse estimate for any $v^h \in \mathring{V}_h$ we get

$$\left\|\Delta v^h\right\| \leq C h^{-1} \left\|\nabla v^h\right\|$$

and hence

$$\begin{aligned} |(\varepsilon \Delta v^h, \delta_h v_\beta^h)| &\leq \varepsilon \delta_h \left\|\Delta v^h\right\| \left\|v_\beta^h\right\| \leq \sqrt{\varepsilon} \left\|\nabla v^h\right\| \cdot \sqrt{\varepsilon} \delta_h C h^{-1} \left\|v_\beta^h\right\| \\ &\leq \frac{1}{2}\varepsilon \left\|\nabla v^h\right\|^2 + \frac{1}{2}\varepsilon \delta_h^2 C^2 h^{-2} \left\|v_\beta^h\right\|^2 \end{aligned}$$

SUPG method for convection-diffusion problem

The bilinear form of the scheme is

$$B(u^h, v^h) = \varepsilon(\nabla u^h, \nabla v^h) - (\varepsilon \Delta u^h, \delta_h v^h_\beta) + (u^h_\beta + u^h, v^h + \delta_h v^h_\beta)$$

so that

$$B(v^h, v^h) \geq \frac{1}{2}\varepsilon \left\|\nabla v^h\right\| + \left\|v^h\right\|^2 + (1 - \frac{1}{2}\varepsilon\delta_h C^2 h^{-2})\delta_h \left\|v_\beta\right\|^2, \qquad \forall\ v^h \in \mathring{V}_h$$

Thus if $\bar{C}$ is so small that

$$\varepsilon\delta_h C^2 h^{-2} = C^2 \bar{C}\frac{\varepsilon}{h} \leq C^2\bar{C} < 1$$

then

$$B(v^h, v^h) \geq \frac{1}{2}(\varepsilon \left\|\nabla v^h\right\|^2 + \delta_h \left\|v^h_\beta\right\|^2 + \left\|v^h\right\|^2) \qquad \forall\ v^h \in \mathring{V}_h$$

Remark: If $\{\mathcal{T}_h\}$ is not quasi-uniform or if β is variable, then we choose

$$\delta_h|_K = \begin{cases} \frac{\bar{C} h_K}{|\beta|} & \text{if} \quad \varepsilon < h_K|\beta| \quad \text{i.e.,} \quad \mathrm{Pe}_K = \frac{h_K|\beta|}{\varepsilon} > 1 \\ 0 & \text{otherwise} \end{cases}$$

Discontinous Galerkin Method for hyperbolic problem

Hyperbolic problems can admit discontinuous solutions. Hence it is useful to allow the approximate solutions to contain discontinuities. Let us define the space of piecewise polynomials on a mesh $\mathcal{T}_h$

$$V_h = Y_h^k = \{v \in L_2(\Omega) : v|_K \in \mathbb{P}_k \ \forall \ K \in \mathcal{T}_h\}$$

Note that a $v \in V_h$ may not be continuous across ∂K. The basis functions for V_h will be taken to be polynomials whose support is contained in some element K.
In order to derive the DG scheme, let us put the PDE in conservation form

$$\nabla \cdot F(u) = 0$$

Multiply by a function $v \in L_2(\Omega)$ with $\text{supp}(v) = K$ and integrate by parts

$$\int_{\partial K} (F \cdot n)v - \int_K F \cdot \nabla v = 0$$

Since u is possibly discontinuous across ∂K it is not clear how to evaluate the flux $(F \cdot n)$. Here we can make use of ideas from Godunov finite volume

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scheme and evaluate the flux using some Godunov or approximate Riemann solver scheme

$$F \cdot n|_{\partial K} \approx H(u_+, u_-, n)$$

E.g., in the linear case, the upwind flux is given by

$$H(u_+, u_-, n) = (\beta \cdot n)^+ u_+ + (\beta \cdot n)^- u_- = \begin{cases} (\beta \cdot n) u_+ & \text{if} \quad \beta \cdot n \geq 0 \\ (\beta \cdot n) u_- & \text{if} \quad \beta \cdot n < 0 \end{cases}$$

Then the DG scheme is

$$\int_{\partial K} H(u_+^h, u_-^h, n) v^h - \int_K F(u^h) \cdot \nabla v^h = 0$$

An alternate version is obtained if we do integration by parts on the second term

$$\int_{\partial K} [H(u_+^h, u_-^h, n) - F(u_+^h) \cdot n] v^h + \int_K v^h \nabla \cdot F(u^h) = 0$$

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For the linear case $F(u) = \beta u$, if we take the second form the scheme becomes

$$(u^h_\beta, v^h)_K - \int_{\partial K} (\beta \cdot n)^- u^h_+ v^h_+ = - \int_{\partial K} (\beta \cdot n)^- u^h_- v^h_+$$

which becomes

$$(u^h_\beta, v^h)_K - \int_{\partial K_-} (\beta \cdot n) u^h_+ v^h_+ = - \int_{\partial K_-} (\beta \cdot n) u^h_- v^h_+$$

The above equation amounts to applying the Galerkin method on element K together with weak imposition of boundary conditions at the inflow boundaries ∂K_- of K.

In the remaining part we will work with the first form of the scheme. Let

$$\Gamma_I = \{\text{interior edges of the mesh } \mathcal{T}_h\}$$

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We have to account for the boundary conditions in the scheme.

$$-\int_K F(u^h)\cdot\nabla v^h +$$

$$\int_{\partial K\cap\Gamma_I} H(u^h_+, u^h_-, n)v^h + \int_{\partial K\cap\Gamma_-} H(u^h, g, n)v^h + \int_{\partial K\cap\Gamma_+} (F(u^h)\cdot n)v^h = 0$$

The scheme is obtained by adding the equations for all elements

$$-\sum_{K\in\mathcal{T}_h}\int_K F(u^h)\cdot\nabla v^h + \sum_{e\in\Gamma_I}\int_e H(u^h_+, u^h_-, n)\left[\!\left[v^h \right]\!\right] +$$

$$\int_{\Gamma_-} H(u^h, g, n)v^h + \int_{\Gamma_+} (F(u^h)\cdot n)v^h = 0$$

Let us specialize this scheme to the linear case with the upwind flux. On inflow boundary Γ_-

$$\int_{\Gamma_-} H(u^h, g, n)v^h = \int_{\Gamma_-} (\beta\cdot n)^- g v^h = \int_{\Gamma} (\beta\cdot n)^- g v^h$$

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$$\int_{\Gamma_+} (F(u^h) \cdot n) v^h = \int_{\Gamma_+} (\beta \cdot n) u^h v^h = \int_{\Gamma} (\beta \cdot n)^+ u^h v^h$$

The scheme takes the form

$$-\sum_{K \in \mathcal{T}_h} \int_K F(u^h) \cdot \nabla v^h + \sum_{e \in \Gamma_I} \int_e H(u^h_+, u^h_-, n) \llbracket v^h \rrbracket +$$

$$\int_{\Gamma} (\beta \cdot n)^+ u^h v^h + \int_{\Gamma} (\beta \cdot n)^- g v^h = 0, \qquad \forall\ v^h \in V_h$$

The solutions are well behaved (respect maximum principle) but highly dissipated for $k = 0$. For $k \geq 1$, the solutions develop oscillations. There are broadly two cures to this problem. The first one is inspired by the limiters used in finite volume methods. The second approach involves adding artificial dissipation term to the scheme

$$(\varepsilon_h \nabla u^h, \nabla v^h)$$

where $\varepsilon_h = \varepsilon_h(u^h)$ is non-zero only in the regions where a shock is detected or the solution is oscillatory.

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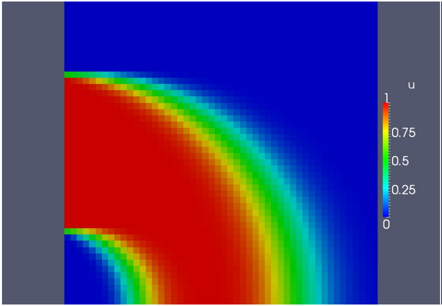

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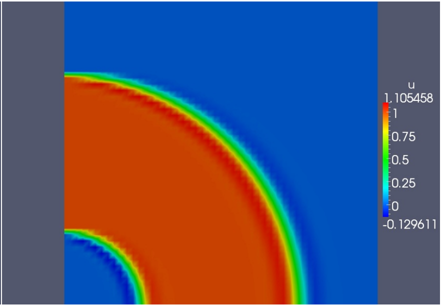

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