Finite volume method for conservation laws II

Godunov scheme

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Non-linear conservation law: monotone flux

$$u_t + f_x = 0$$

Assume $f(u)$ is monotone. If $f' \geq 0$ then use backward difference for f_x

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{f_j^n - f_{j-1}^n}{\Delta x} = 0$$

and if $f' \leq 0$ then use forward difference for f_x

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{f_{j+1}^n - f_j^n}{\Delta x} = 0$$

Numerical flux function

$$g_{j+\frac{1}{2}} = \begin{cases} f_j & f' \geq 0 \\ f_{j+1} & f' < 0 \end{cases}$$

This is the upwind scheme for the non-linear problem.

Non-linear conservation law: general flux

$$u_t + f_x = 0$$

We can generalize the previous numerical flux function

$$g_{j+\frac{1}{2}} = \begin{cases} f_j & a_{j+\frac{1}{2}} \ge 0 \\ f_{j+1} & a_{j+\frac{1}{2}} < 0 \end{cases}$$

where $a_{j+\frac{1}{2}} \approx f'(x_{j+\frac{1}{2}})$ is an estimate of wave speed at $x_{j+\frac{1}{2}}$, e.g.,

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1}-f_j}{v_{j+1}-v_j} & v_j \ne v_{j+1} \\ f'(v_j) & v_j = v_{j+1} \end{cases}$$

Then the numerical flux can be written as

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a_{j+\frac{1}{2}}|(v_{j+1} - v_j)$$

This is known as Murman-Roe scheme. It does not satisfy entropy condition.

Remark: Compare with upwind flux for linear equation.

Riemann problem

$$u_t + f_x = 0$$

Initial condition

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

Self-similar solution

$$u(x,t) = w_R(x/t; u_l, u_r)$$

comprises of shocks and rarefactions. The maximum wave speed is bounded by

$$\max\{|a(\xi)|, \ \xi \ \text{ between } u_l \text{ and } u_r\}$$

Godunov scheme

- How to compute the flux across the cell faces so that the resulting scheme is consistent, stable and satisfies entropy condition ?
- Finite volume solution is made of piece-wise constant states

$$v_\Delta(x,t) = v_j^n, \qquad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \qquad t \in [t_n, t_{n+1})$$

  There is a Riemann problem defined at each cell face $x_{j+\frac{1}{2}}$!!!
- Idea of Godunov: exactly solve this Riemann problem and compute the flux from the Riemann solution.
- Riemann solution is made of shock and rarefaction waves
- Choose Δt small so that waves from neighboring Riemann problems do not interact.
- Evolution for Δt and projection to cell averages
- Apply this technique to a linear conservation law and show that you get the upwind scheme.

Godunov scheme for linear PDE

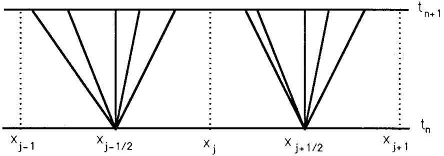
Godunov scheme

Step 1 (Evolution): Solve exactly the problem

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t \in (t_n, t_{n+1}]$$
$$w(x, t_n) = v_\Delta(x, t_n)$$

$v_\Delta \in L^\infty(\mathbb{R})$ problem has unique entropy solution which can be determined explicitly for small Δt. We have to solve the local Riemann problems

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t \in (0, \Delta t]$$
$$w(x, 0) = \begin{cases} v_j^n & x < x_{j+\frac{1}{2}} \\ v_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases}$$



Two neighboring Riemann problems do not interact before time Δt if

$$\lambda \max\{|a(v)| : v \text{ between } v_j^n \text{ and } v_{j+1}^n\} \leq \frac{1}{2}, \qquad j \in \mathbb{Z}$$

Godunov scheme

Total solution is made up of local Riemann problem solutions

$$w(x, t_{n+1}) = w_R\left(\frac{x - x_{j+\frac{1}{2}}}{\Delta t}; v_j^n, v_{j+1}^n\right), \quad x \in (x_j, x_{j+1}), \quad j \in \mathbb{Z}$$

Step 2 (Projection): Cell average at time t_{n+1}

$$v_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x, t_{n+1})\mathrm{d}x$$

which can be written as

$$v_j^{n+1} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{0} w_R(x/\Delta t; v_j^n, v_{j+1}^n)\mathrm{d}x + \frac{1}{\Delta x}\int_{0}^{\frac{1}{2}\Delta x} w_R(x/\Delta t; v_{j-1}^n, v_j^n)\mathrm{d}x$$

Godunov scheme

To get simpler expression, we integrate conservation law over space-time slab $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (t_n, t_{n+1})$

$$\begin{aligned}
0 &= \int_0^{\Delta t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} \right) \mathrm{d}x\mathrm{d}t \\
&= \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (w(x, \Delta t) - w(x, 0))\mathrm{d}x + \int\limits_0^{\Delta t} [f(w(x_{j+\frac{1}{2}} - 0, t)) - f(w(x_{j-\frac{1}{2}} + 0, t))]\mathrm{d}t \\
&= \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (w(x, \Delta t) - w(x, 0))\mathrm{d}x + \Delta t [f(w_R(0-; v_j^n, v_{j+1}^n)) - f(w_R(0+; v_{j-1}^n, v_j^n))]
\end{aligned}$$

Now the mapping $\xi \to f(w_R(\xi; u_l, u_r))$ is continuous at $\xi = 0$. If $w_R(\xi; u_l, u_r)$ is discontinuous at $\xi = 0$, then there is a stationary discontinuity; hence by RH condition

$$f(w_R(0+; u_l, u_r)) = f(w_R(0-; u_l, u_r))$$

Godunov scheme

Thus the Godunov scheme simplifies to

$$v_j^{n+1} = v_j^n - \lambda[f(w_R(0; v_j^n, v_{j+1}^n)) - f(w_R(0; v_{j-1}^n, v_j^n))]$$

This scheme is in conservation form and the numerical flux is given by

$$g_{j+\frac{1}{2}} = g^G(v_j, v_{j+1}), \qquad g^G(u, v) = f(w_R(0; u, v))$$

Observe that only $f(w(x_{j+\frac{1}{2}}, t))$ is needed; so the flux formula is valid (but not the Riemann solution) provided the waves from $x_{j-\frac{1}{2}}$ do not reach $x_{j+\frac{1}{2}}$ in time Δt. This leads to CFL condition

$$\lambda \max\{|a(v)| : v \text{ between } v_j^n \text{ and } v_{j+1}^n\} \le 1, \qquad \forall\, j \in \mathbb{Z}$$

which is less restrictive. If $f(u)$ is convex, this condition becomes

$$\lambda \max_j |a(v_j)| \le 1$$

Godunov flux: linear case

f **linear or monotone**: Riemann solution is

$$w_R(0; u, v) = \begin{cases} u & \text{if } a = f' > 0 \\ v & \text{if } a = f' < 0 \end{cases}$$

so that

$$g^G(u, v) = \begin{cases} f(u) & \text{if } a > 0 \\ f(v) & \text{if } a < 0 \end{cases}$$

This leads to the upwind finite difference scheme.

Godunov flux: non-linear case

$\underline{f \textbf{ is convex}}$: Define $u^* = w_R(0; u_l, u_r)$. Then $g^G(u_l, u_r) = f(u^*)$

(1) $f'(u_l),\ f'(u_r) \geq 0$

$$u^* = u_l$$

(2) $f'(u_l),\ f'(u_r) \leq 0$

$$u^* = u_r$$

(3) $f'(u_l) \geq 0 \geq f'(u_r)$, we have a shock with speed $s = [f]/[u]$

$$u^* = \begin{cases} u_l & \text{if } \frac{[f]}{[u]} > 0 \\ \\ u_r & \text{if } \frac{[f]}{[u]} < 0 \end{cases}$$

(4) $f'(u_l) < 0 < f'(u_r)$, we have a rarefaction

$$u^* = \bar{u}, \qquad a(\bar{u}) = f'(\bar{u}) = 0$$

Godunov flux: non-linear case

Verify that the resulting Godunov flux can be written as

$$g^G(u_l, u_r) = \begin{cases} \min\limits_{u \in [u_l, u_r]} f(u) & \text{if } u_l \leq u_r \\ \max\limits_{u \in [u_r, u_l]} f(u) & \text{if } u_l > u_r \end{cases}$$

The previous formula is valid even in the case of nonconvex fluxes and leads to the solution satisfying Oleiniks entropy condition. Verify this claim using the convex hull construction of the entropy satisfying Riemann solution.

Remark: The above Godunov flux formula also follows from a relation due to Osher[1], who showed that the entropy solution $u(x,t) = w(x/t)$ of a general nonconvex scalar Riemann problem with data u_l, u_r satisfies the implicit relation

$$f(w(\xi)) - \xi w(\xi) = g(\xi) = \begin{cases} \min\limits_{u \in [u_l, u_r]} [f(u) - \xi u] & \text{if } u_l \leq u_r \\ \max\limits_{u \in [u_r, u_l]} [f(u) - \xi u] & \text{if } u_l > u_r \end{cases}$$

Godunov flux: non-linear case

Setting $\xi = 0$ leads to Godunov flux. Also, the Riemann solution is given as

$$w(\xi) = -g'(\xi)$$

Remark: In strictly convex case, if $\bar{u}$ is the only sonic point, then a simpler expression for the Godunov flux is given by

$$g^G(u_l, u_r) = \max(f(\max(\bar{u}, u_l)), f(\min(\bar{u}, u_r)))$$

Verify this expression.

[1] Osher, SIAM J. Num. Anal., 21 (1984), pp. 217-235

Murman-Roe scheme (Approximate Riemann solver)

Exact Riemann solution could be difficult to compute. Replace non-linear problem with a simplified (possibly linear) problem, which still retains some hyperbolic properties of the non-linear problem. Then solve the Riemann problem for the simplified model.

$$\frac{\partial w}{\partial t} + a(v_j^n, v_{j+1}^n)\frac{\partial w}{\partial x} = 0$$

$$w(x,0) = \begin{cases} v_j^n & x < x_{j+\frac{1}{2}} \\ v_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases}$$

where $a = f'(u)$ should approximate the wave speed in the Riemann problem. We can take

$$a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & u \neq v \\ f'(u) & u = v \end{cases}$$

The solution of the linearized problem is made of a shock

$$w(x,t) = w_R^{Roe}(\xi; v_j^n, v_{j+1}^n) = \begin{cases} v_j^n & \xi < a(v_j^n, v_{j+1}^n) \\ v_{j+1}^n & \xi > a(v_j^n, v_{j+1}^n) \end{cases}$$

Murman-Roe scheme (Approximate Riemann solver)

The associated numerical flux is

$$g^R(u,v) = f(w_R^{Roe}(0;u,v)) = \begin{cases} f(u) & a(u,v) > 0 \\ f(v) & a(u,v) < 0 \end{cases}$$

or

$$g^R(u,v) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}|a(u,v)|(v-u)$$

Entropy violating solution: Take $u_l < u_r$ for a strictly convex flux with $f(u_l) = f(u_r)$ (e.g. $f(u) = u^2/2$, $u_l = -1$, $u_r = 1$). The Murman-Roe scheme with initial condition

$$v_j^0 = \begin{cases} u_l & j \leq -1 \\ u_r & j \geq 0 \end{cases}$$

leads to the entropy violating solution

$$v_j^n = v_j^0$$

The correct solution contains a rarefaction wave. Entropy violation occurs because the numerical dissipation vanishes.

Engquist-Osher flux

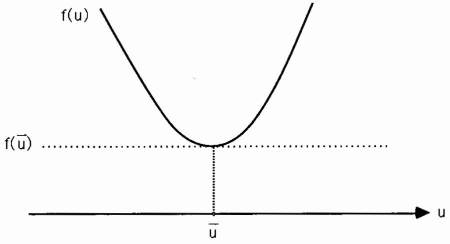
$$g^{EO}(u,v) = \frac{1}{2}(f(u)+f(v)) - \frac{1}{2}\int_u^v |a(\xi)|\mathrm{d}\xi$$

Strictly convex case: Define

$$f^+(u) = f(\max(u,\bar{u}))$$
$$f^-(u) = f(\min(u,\bar{u}))$$

where $\bar{u}$ is the only stagnation (sonic) point, i.e.,

$$a(\bar{u}) = f'(\bar{u}) = 0$$



Then

$$f(u) = f^+(u) + f^-(u) - f(\bar{u}), \qquad |f'(u)| = f^{+'}(u) - f^{-'}(u)$$

Draw graph of f^+ and f^-.

Engquist-Osher flux

Then it is easy to show that

$$g^{EO}(u,v) = f^+(u) + f^-(v) \qquad \text{(neglecting constant term } f(\bar{u})\text{)}$$

This is also referred to as a flux splitting scheme. The split fluxes f^+, f^- represent right moving and left moving waves since

$$f^{+'}(u) \geq 0, \qquad f^{-'}(u) \leq 0$$

In the domains where f' is constant, it reduces to the upwind finite difference scheme.

Remark: If we consider the entropy violating case of Murman-Roe scheme, the EO scheme does not give the entropy violating shock. Later we will show that EO scheme gives entropy satisfying solutions.

Numerical examples

Try the matlab program `conlaw.m`