Finite volume method for conservation laws III

Monotone and TVD schemes

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in

February 17, 2013

Definition (Monotone scheme)

The difference scheme

$$v_j^{n+1} = H(v_{j-k}^n, \ldots, v_{j+k}^n)$$

is said to be monotone if H is an increasing function of each of its arguments.

A scheme is monotone iff the operator H_Δ is monotone; i.e., for any two sequences $(v_j)_{j\in\mathbb{Z}}$ and $(w_j)_{j\in\mathbb{Z}}$

$$v \geq w \quad (i.e.,\ \forall j,\ v_j \geq w_j) \quad \Longrightarrow \quad H_\Delta(v) \geq H_\Delta(w)$$

Thus the approximate solution operator H_Δ has the same property as the continuous one (see Kruzkov theorem).

Remark: For a linear scheme

$$v_j^{n+1} = \sum_{l=-k}^{k} b_l v_{j+l}^n$$

montone scheme corresponds to the condition that all coefficients $b_l \geq 0$.

Example (Lax-Friedrich scheme)

$$H(v_{-1}, v_0, v_1) = \frac{1}{2}(v_{-1} + v_1) - \frac{1}{2}\lambda(f(v_1) - f(v_{-1}))$$

$$\frac{\partial H}{\partial v_{-1}} = \frac{1}{2} + \frac{1}{2}\lambda f'(v_{-1}) \geq 0, \qquad \frac{\partial H}{\partial v_1} = \frac{1}{2} - \frac{1}{2}\lambda f'(v_1) \geq 0$$

Scheme is monotone if CFL condition is satisfied

$$\lambda \max_j |f'(v_j)| \leq 1$$

Example (Engquist-Osher scheme)

$$\frac{\partial H}{\partial v_0} = 1 - \lambda |f'(v_0)| \geq 0, \qquad \frac{\partial H}{\partial v_{\pm 1}} = \frac{1}{2}\left(\pm f(v_{\pm 1}) + |f'(v_{\pm 1})|\right) \geq 0$$

Monotone under the same CFL condition as above.

Example: (Godunov scheme) This scheme is made of two steps. The evolution step is monotone since it involves exact solution of Riemann problems. The projection step onto piecewise constant states is also monotone. Hence the Godunov scheme is monotone.

Proposition (Monotone property of 3-point schemes)

If a 3-point conservative scheme is monotone, then the numerical flux $g(u, v)$ is an increasing function of its first argument and a decreasing function of its second argument.

Proof: A conservative scheme is of the form

$$H(v_{-1}, v_0, v_1) = v_0 - \lambda[g(v_0, v_1) - g(v_{-1}, v_0)]$$

Then the map

$$u \to g(u, v_0) = \frac{1}{\lambda}[H(u, v_0, v_1) - v_0] + g(v_0, v_1)$$

is non-decreasing iff $u \to H(u, v_0, v_1)$ is non-decreasing, and

$$v \to g(v_0, v) = \frac{1}{\lambda}[-H(v_{-1}, v_0, v) + v_0] + g(v_{-1}, v_0)$$

is non-increasing iff $v \to H(v_{-1}, v_0, v)$ is non-decreasing.

Conversely, assume that $g(u, v)$ is an increasing function in its first argument and a decreasing function in its second argument. Then it is easy to see that $H(v_{-1}, v_0, v_1)$ is increasing function of v_{-1} and v_1. The scheme is monotone if the following condition holds

$$\lambda \max_{w,z}\{|g(u,w) - g(v,w)| + |g(z,u) - g(z,v)|\} \leq |u - v|$$

This condition ensures that the function $v \to H(v_{-1}, v, v_1)$ is non-decreasing.

$$\frac{\partial H}{\partial v_0} = 1 - \lambda \left[\frac{\partial g}{\partial u_l}(v_0, v_1) - \frac{\partial g}{\partial u_r}(v_{-1}, v_0) \right] \geq 0$$

Theorem (Monotone schemes are first order accurate)

Assume that the difference scheme H can be put in conservation form, is consistent and that H is C^3. If the scheme is monotone, it is at most first order accurate.

Proof: We will show that $\beta(u,\lambda) \geq 0$ and $\beta(u,\lambda) = 0$ only in a trivial case. By monotone property, $\frac{\partial H}{\partial v_j} \geq 0$

$$-\lambda f'(u) = \sum_{j=-k}^{+k} j \frac{\partial H}{\partial v_j}(u,\ldots,u) = \sum_{j=-k}^{+k} j \sqrt{\frac{\partial H}{\partial v_j}(u,\ldots,u)} \sqrt{\frac{\partial H}{\partial v_j}(u,\ldots,u)}$$

and by Cauchy-Schwarz inequality

$$(\lambda f'(u))^2 \leq \sum_{j=-k}^{+k} j^2 \frac{\partial H}{\partial v_j}(u,\ldots,u) \sum_{j=-k}^{+k} \frac{\partial H}{\partial v_j}(u,\ldots,u)$$

By consistency

$$\sum_{j=-k}^{+k} \frac{\partial H}{\partial v_j}(u,\ldots,u) = 1$$

so that

$$(\lambda f'(u))^2 \leq \sum_{j=-k}^{+k} j^2 \frac{\partial H}{\partial v_j}(u,\ldots,u)$$

which implies that $\beta(u,\lambda) \geq 0$. Moreover $\beta(u,\lambda) = 0$ iff equality holds in Cauchy-Schwarz inequality, which is the case if for some constant C

$$j\sqrt{\frac{\partial H}{\partial v_j}} = C\sqrt{\frac{\partial H}{\partial v_j}}$$

This implies that $\frac{\partial H}{\partial v_j} = 0$ except for one subscript j_0 for which $j_0 = C$. Then

$$1 = \sum_{j=-k}^{+k} \frac{\partial H}{\partial v_j}(u,\ldots,u) = \frac{\partial H}{\partial v_{j_0}}$$

and

$$-\lambda f'(u) = \sum_{j=-k}^{+k} j\frac{\partial H}{\partial v_j}(u,\ldots,u) = j_0\frac{\partial H}{\partial v_{j_0}} = j_0$$

This is the case if $f(u)$ is linear, $f'(u) = a$ and $j_0 = -a\lambda = -a\frac{\Delta t}{\Delta x}$ so that

$$v_j^{n+1} = H(v_{j-k}^n, \ldots, v_{j+k}^n) = v_{j+j_0}^n$$

Note that $j_0 \le -1$ if $a < 0$ and $j_0 \ge +1$ if $a > 0$ which gives exact solution for linear equation as $|a|\lambda = 1$ and the backward characteristic from $(j, n+1)$ exactly hits the grid point $(j + j_0, n)$. Obviously, the local truncation error is zero

$$\begin{aligned} & u(x, t + \Delta t) - H(u(x - k\Delta x, t), \dots, u(x + k\Delta x, t)) \\ = & \, u(x, t + \Delta t) - u(x + j_0 \Delta x, t) \\ = & \, u(x, t + \Delta t) - u(x - a\Delta t, t) = 0 \end{aligned}$$

Some discrete norms:
Given a sequence $(v_j)_{j\in\mathbb{Z}}$ define

$$\begin{aligned} \|v\|_{L^1(\Delta)} &= \Delta x \sum_{j\in\mathbb{Z}} |v_j| \\ \|v\|_{L^\infty(\Delta)} &= \max_{j\in\mathbb{Z}} |v_j| \\ \mathrm{TV}(v) &= \sum_{j\in\mathbb{Z}} |v_{j+1} - v_j| \end{aligned}$$

Definition (TVD scheme)

The finite difference scheme H_Δ is said to be total variation diminishing (TVD) if

$$\mathrm{TV}(H_\Delta(v)) \le \mathrm{TV}(v)$$

Definition (L^∞ stability)

The finite difference scheme H_Δ is said to be L^∞-stable if $\exists C > 0$ independent of n and Δt such that

$$\|v^n\|_{L^\infty(\Delta)} \le C, \qquad \forall n \ge 0$$

Lemma 5.3 (Crandall-Tartar, GR1, Chap. 2)

Let (Ω, μ) be a measure space and $\mathcal{C}$ be a subset of $L^1(\Omega)$ such that

$$f, g \in \mathcal{C} \quad \Longrightarrow \quad \max(f, g) \in \mathcal{C}$$

Let $T : \mathcal{C} \to L^1(\Omega)$ such that

$$\int_\Omega T(f) \mathrm{d}\mu = \int_\Omega f \mathrm{d}\mu$$

Then the following three properties are equivalent:

(1) T is order preserving, i.e.,

$$f, g \in \mathcal{C} \quad \text{and} \quad f \le g \quad a.e. \quad \Longrightarrow \quad T(f) \le T(g) \quad a.e.$$

(2) $\int_\Omega (T(f) - T(g))_+ \mathrm{d}\mu \le \int_\Omega (f - g)_+ \mathrm{d}\mu, \quad \forall f, g \in \mathcal{C}$

(3) $\int_\Omega |T(f) - T(g)| \mathrm{d}\mu \le \int_\Omega |f - g| \mathrm{d}\mu, \quad \forall f, g \in \mathcal{C}$

Proof: (1) $\Longrightarrow$ (2). Let $f, g \in \mathcal{C}$. We have the identity

$$\max(f,g) = g + (f-g)_+ \geq g$$

Since $\max(f,g) \in \mathcal{C}$ and T is order preserving

$$T(\max(f,g)) \geq T(g) \qquad \Longrightarrow \qquad T(\max(f,g)) - T(g) \geq 0$$

and similarly

$$T(\max(f,g)) \geq T(f) \qquad \Longrightarrow \qquad T(\max(f,g)) - T(g) \geq T(f) - T(g)$$

Subtract $T(g)$ from both sides of above inequality to get

$$T(\max(f,g)) - T(g) \geq \max(T(f) - T(g), 0) = (T(f) - T(g))_+$$

Integrating previous inequality over Ω and using conservation property

$$\begin{aligned} \int_\Omega (T(f) - T(g))_+ \mathrm{d}\mu &\leq \int_\Omega (T(\max(f,g)) - T(g)) \mathrm{d}\mu \\ &= \int_\Omega (\max(f,g) - g) \mathrm{d}\mu \\ &= \int_\Omega (f-g)_+ \mathrm{d}\mu \end{aligned}$$

(2) $\Longrightarrow$ (3). Using the identity

$$|f-g| = (f-g)_+ + (g-f)_+$$

we have

$$\begin{aligned}
\int_\Omega |T(f)-T(g)|\mathrm{d}\mu &= \int_\Omega (T(f)-T(g))_+\mathrm{d}\mu + \int_\Omega (T(g)-T(f))_+\mathrm{d}\mu \\
&\leq \int_\Omega (f-g)_+\mathrm{d}\mu + \int_\Omega (g-f)_+\mathrm{d}\mu \\
&= \int_\Omega |f-g|\mathrm{d}\mu
\end{aligned}$$

(3) $\Longrightarrow$ (1). See GR1, Chap. 2, Lemma 5.2

Theorem (Monotone = TVD and L^∞ stability)

Let H_Δ be a monotone difference scheme which can be put in conservation form. Then it is TVD and L^∞-stable

$$\|v^n\|_{L^\infty(\Delta)} \leq \left\|v^0\right\|_{L^\infty(\Delta)}$$

Moreover for any sequences u and v, we have

$$\|H_\Delta(u) - H_\Delta(v)\|_{L^1(\Delta)} \leq \|u - v\|_{L^1(\Delta)}$$

(Monotone $\Longrightarrow$ L^1 contraction $\Longrightarrow$ TVD)

Proof: **(1) L^∞-stability**: We will show that H_Δ satisfies maximum principle

$$\min_{j-k\leq l\leq j+k} v_l \leq (H_\Delta(v))_j \leq \max_{j-k\leq l\leq j+k} v_l$$

which implies L^∞-stability. Let w be the constant sequence

$$w_j = \max_{j-k\leq l\leq j+k} v_l = c, \qquad j-k \leq j \leq j+k, \qquad w_j = \max_{l\in\mathbb{Z}} v_l \qquad \text{otherwise}$$

Then due to consistency

$$(H_\Delta(w))_j = c$$

and due to monotone property of the scheme, since $v \le w$, we get

$$H_\Delta(v) \le H_\Delta(w)$$

which gives the right side of the inequality. The left side can be obtained by a similar argument.

(2) L^1 contraction: From L^∞-stability

$$(H_\Delta(v))_j \le \max_{j-k\le l\le j+k} |v_l| \le \sum_{j-k\le l\le j+k} |v_l|$$

so that

$$\|H_\Delta(v)\|_{L^1(\Delta)} = \Delta x \sum_{j\in\mathbb{Z}} |H_\Delta(v)_j| \le (2k+1)\Delta x \sum_{j\in\mathbb{Z}} |v_j|$$

and hence $H_\Delta : L^1(\Delta) \to L^1(\Delta)$. Since H_Δ is conservative and monotone, we can apply Crandall-Tartar Lemma to obtain L^1 contraction property.

(3) TVD property: Define the shifted sequence $w = (w_j)_{j\in\mathbb{Z}}$

$$w_j = v_{j+1}$$

Now the total variation can be written as

$$\mathrm{TV}(H_\Delta(v)) = \sum_{j\in\mathbb{Z}} |H_\Delta(v)_{j+1} - H_\Delta(v)_j| = \frac{1}{\Delta x} \|H_\Delta(w) - H_\Delta(v)\|_{L^1(\Delta)}$$

Then by L^1 contraction property of H_Δ

$$\mathrm{TV}(H_\Delta(v)) \leq \frac{1}{\Delta x} \|w - v\|_{L^1(\Delta)} = \mathrm{TV}(v)$$

Corollary

If the difference scheme H_Δ can be put in conservation form and is monotone, then v_Δ satisfies the following estimates $\forall t \geq 0$

(1) $\|v_\Delta(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|v_\Delta(\cdot, 0)\|_{L^\infty(\mathbb{R})}$

(2) $\|v_\Delta(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|v_\Delta(\cdot, 0)\|_{L^1(\mathbb{R})}$

(3) $\mathrm{TV}(v_\Delta(\cdot, t)) \leq \mathrm{TV}(v_\Delta(\cdot, 0))$

Property (2) can be obtained from L^1 contraction property.

Definition (Monotonicity preserving scheme)

A difference scheme H_Δ is called *monotonicity preserving* if it maps a monotone sequence into a monotone sequence.

Proposition (TVD and monotonicity preservation)

A difference scheme H_Δ which is TVD is monotonicity preserving.

Proof: Consider a monotone sequence of the form

$$v_j = \begin{cases} v_- & \forall j \leq J_- \\ \text{monotone} & J_- \leq j \leq J_+ \\ v_+ & \forall j \geq J_+ \end{cases}$$

for which $\mathrm{TV}(v) = |v_+ - v_-|$. If $w = H_\Delta(v)$ were not monotone, it would have atleast one local minimum and one local maximum. Denote by v_m and v_M the values of the first two successive local extrema. Then

$$\mathrm{TV}(w) \geq |v_+ - v_-| + |v_M - v_m| > \mathrm{TV}(v)$$

which contradicts the assumption that the scheme is TVD.

Proposition (Linear MP schemes)

In the linear constant coefficient case, scheme H_Δ is monotonicity preserving iff the coefficients c_l, $-k \le c_l \le +k$ are non-negative.

Proof: Since $v_j^{n+1} = \sum_l c_l v_{j+l}^n$, we have

$$v_{j+1}^{n+1} - v_j^{n+1} = \sum_{l=-k}^{+k} c_l (v_{j+l+1}^n - v_{j+l}^n)$$

and a sufficient condition for monotonicity preservation is $c_l \ge 0$. This condition is also necessary. For suppose that $c_L < 0$ for some $-k \le L \le +k$ and consider initial condition

$$v_j^0 = \begin{cases} 1, & j \le 0 \\ 0, & j \ge 1 \end{cases}$$

Then

$$v^1_{-L+1} - v^1_{-L} = \sum_{l=-k}^{+k} c_l (v^0_{-L+l+1} - v^0_{-L+l}) = c_L (v_1^0 - v_0^0) > 0$$

which shows that the scheme is not monotonicity preserving.

- For a linear scheme, if all coefficients are positive, scheme is also monotone, and thus only first order accurate.
  TVD $\Leftrightarrow$ Monotonicity preserving $\Leftrightarrow c_l \geq 0 \Leftrightarrow$ monotone $\Rightarrow$ order one
  For linear scheme, all concepts are equivalent.
- We will see that for a 3-point TVD scheme (not necessarily linear)
  TVD $\Longleftrightarrow$ Monotonicity preserving $\Longrightarrow$ order one
- Remember that for any TVD scheme
  TVD $\Longrightarrow$ Monotonicity preserving
- A general 3-point linear scheme

$$\begin{aligned} v_j^{n+1} &= v_j^n - \frac{a\lambda}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{q}{2}(v_{j-1}^n - 2v_j^n + v_{j+1}^n) \\ &= \frac{1}{2}(a\lambda + q)v_{j-1}^n + (1-q)v_j^n + (-a\lambda + q)v_{j+1}^n \end{aligned}$$

The scheme is monotone provided

$$|a|\lambda \leq q \leq 1$$

while the scheme is L^2-stable provided

$$(a\lambda)^2 \le q \le 1$$

For Lax-Wendroff scheme, $q^{LW} = (a\lambda)^2$ but this is not TVD/monotone. For upwind scheme, $q^U = |a|\lambda$ and it is TVD/monotone.

- Upwind scheme has least numerical dissipation among all 3-point, TVD schemes.
- However, *there are no TVD, linear, second order accurate schemes.* We will see that even for linear problems, we have to construct non-linear schemes to achieve TVD property together with second order accuracy.

$$v_j^{n+1} = \sum_{l=-k}^{+k} c_l(a, \lambda, v^n) v_{j+l}^n$$

We next give results for TVD schemes which are analogous to those for monotone schemes. However, we have to explicitly add the condition of L^∞ stability, while monotone schemes are automatically L^∞-stable.

Lemma (TVD time estimate)

Let H_Δ be a TVD scheme which can be put in conservation form with a Lipschitz continuous numerical flux g. Assume moreover that the scheme is L^∞-stable. Then $\exists C > 0$ such that $\forall n \geq m \geq 0$

$$\|H^n_\Delta(v) - H^m_\Delta(v)\|_{L^1(\Delta)} \leq C(n-m)\Delta t \ \mathrm{TV}(v)$$

<u>Proof</u>: Due to conservation form

$$(H_\Delta(v))_j - v_j = -\lambda(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}})$$

$$g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}} = g(v_{j-k+1}, \ldots, v_{j+k}) - g(v_{j-k}, \ldots, v_{j+k-1})$$

and since g satisfies a local Lipschitz condition, $\exists c = c(g, \|v\|_{L^\infty(\Delta)})$

$$|(H_\Delta(v))_j - v_j| \leq \lambda c\left(|v_{j-k+1} - v_{j-k}| + \ldots + |v_{j+k} - v_{j+k-1}|\right)$$

hence

$$\|H_\Delta(v) - v\|_{L^1(\Delta)} \leq 2kc\Delta t \ \mathrm{TV}(v)$$

In the same way, since $H_\Delta^{s+1}(v) = H_\Delta(H_\Delta^s(v))$

$$\left\| H_\Delta^{s+1}(v) - H_\Delta^s(v) \right\|_{L^1(\Delta)} \leq 2kc\Delta t \ \mathrm{TV}(H_\Delta^s(v)) \leq 2kc\Delta t \ \mathrm{TV}(v)$$

Now for $n > m$ we can write

$$H_\Delta^n(v) - H_\Delta^m(v) = \sum_{s=m}^{n-1} (H_\Delta^{s+1}(v) - H_\Delta^s(v))$$

which yields the desired result.

Theorem (Time estimates for $v_\Delta(x,t)$)

Let H_Δ be a TVD scheme which can be put in conservation form with a locally Liptschitz continuous numerical flux g. Assume moreover that the scheme is L^∞-stable. Then $\exists C > 0$ which depends only on g and $\|v\|_{L^\infty(\Delta)}$ such that for any $T > 0$

$$
\begin{aligned}
\|v_\Delta(\cdot,t)\|_{L^1(\Delta)} &\leq \|v_\Delta(\cdot,0)\|_{L^1(\Delta)} + CT\ \mathrm{TV}(v_\Delta(\cdot,0)), \quad 0 \leq t \leq T \\
\|v_\Delta(\cdot,t) - v_\Delta(\cdot,s)\|_{L^1(\Delta)} &\leq C(|t-s| + \Delta t)\mathrm{TV}(v_\Delta(\cdot,0)), \qquad 0 \leq s,t \leq T \\
\mathrm{TV}(v_\Delta(\cdot,t)) &\leq \mathrm{TV}(v_\Delta(v,\cdot,0)), \qquad 0 \leq t \leq T
\end{aligned}
$$

Proof: First property is a consequence of previous lemma, while last one is just the TVD property. We only need to check the second inequality. Let t_m, t_n be such that $t_m \leq s \leq t_{m+1}$ and $t_n \leq t \leq t_{n+1}$. Then

$$|t_n - t_m| \leq |t-s| + \Delta t$$

and

$$v_\Delta(\cdot,t) - v_\Delta(\cdot,s) = v_\Delta(\cdot,t_n) - v_\Delta(\cdot,t_m)$$

Then apply previous lemma to get the result.

Notion of convergence

The global error $v_\Delta - u$ is not well defined when the weak solution u is not unique. Hence we measure the error as the distance of v_Δ from the set of all weak solutions

$$\mathcal{W} = \{w : w \text{ is a weak solution of conservation law}\}$$

We use the 1-norm to measure the distance, which is defined as

$$\|v\|_{1,T} = \int_0^T \|v\|_1 \, \mathrm{d}t = \int_0^T \int_{-\infty}^{+\infty} |v(x,t)| \mathrm{d}x \mathrm{d}t$$

The global error is defined as

$$\operatorname{dist}(v_\Delta, \mathcal{W}) = \inf_{w \in \mathcal{W}} \|v_\Delta - w\|_{1,T}$$

But the space

$$L_{1,T} = \{v : \|v\|_{1,T} < \infty\}$$

is not compact and hence we cannot show convergence.

Example: The space

$$L_1 = \{v : \mathbb{R} \to \mathbb{R}, \quad \|v\|_1 < \infty\}$$

is not compact since the sequence

$$v_j(x) = \begin{cases} 1 & j < x < j+1 \\ 0 & \text{otherwise} \end{cases}$$

has no convergent sub-sequence. Note that the supports of v_j are disjoint.

Example: The space

$$\{v \in L_1 : \quad \|v\|_1 < R, \quad supp(v) \subset [-M, +M]\}$$

is also not compact since the sequence

$$v_j(x) = \begin{cases} \sin(j\pi x) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

has no convergent sub-sequence. These functions become increasingly oscillatory as $j \to \infty$.

In order to obtain a compact set in L_1, we will put a bound on the total variation. Then the set of functions

$$\{v \in L_1 : \quad \mathrm{TV}(v) \leq R, \quad supp(v) \in [-M, +M]\}$$

is compact so that any sequence of functions with uniformly bounded total variation and support contains convergent sub-sequences.

The approximations v_Δ are functions of x and t, and the total variation for such a function is defined as

$$\begin{aligned} \mathrm{TV}_T(u) \quad = \quad & \mathrm{limsup}_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{+\infty} |u(x+\epsilon, t) - u(x,t)| \mathrm{d}x\mathrm{d}t + \\ & \mathrm{limsup}_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{-\infty}^{+\infty} |u(x, t+\epsilon) - u(x,t)| \mathrm{d}x\mathrm{d}t \end{aligned}$$

Then the set

$$\mathcal{K} = \{u \in L_{1,T} : \mathrm{TV}_T(u) \leq R, \quad supp(v(\cdot, t)) \subset [-M, +M] \quad \forall t \in [0, T]\}$$

is compact in $L_{1,T}$. For the numerical solution v_Δ, the total variation is

$$\mathrm{TV}_T(v_\Delta) = \sum_{n=0}^{T/\Delta t} \left\{ \Delta t |v_{j+1}^n - v_j^n| + \Delta x |v_j^{n+1} - v_j^n| \right\}$$

This can be written in terms of the total variation in x and the L_1 norm

$$\mathrm{TV}_T(v_\Delta) = \sum_{n=0}^{T/\Delta t} \left\{ \Delta t \mathrm{TV}(v^n) + \left\| v^{n+1} - v^n \right\|_{L^1(\Delta)} \right\}$$

But for a TVD scheme, we have shown that $\left\| v^{n+1} - v^n \right\|_{L^1(\Delta)}$ is bounded by its total variation in space. Hence the space-time total variation is bounded by the initial total variation in x

$$\mathrm{TV}_T(v_\Delta) \leq CT \ \mathrm{TV}(v_\Delta(\cdot, 0))$$

Theorem (TVD, L^∞-stable schemes are convergent)

Let H_Δ be a TVD and L^∞ stable scheme which Lipschitz continuous flux. Let $u_0 \in \mathrm{BV}(\mathbb{R})$ and define v^0 by cell averaging. Then there exists a sequence $\Delta_k x \to 0$ such that if we set $\Delta_k t = \lambda \Delta_k x$, with λ kept constant, the subsequence $(v_{\Delta_k})_k$ converges to $\mathcal{W}$ for any $T > 0$.

Proof: We first give some uniform bounds on $v_\Delta(\cdot, 0)$.

$$\begin{aligned}
\mathrm{TV}(v_\Delta(\cdot,0)) &= \sum_j |v_{j+1}^0 - v_j^0| = \sum_j \left| \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u_0(x+\Delta x) - u_0(x)) \mathrm{d}x \right| \\
&\leq \frac{1}{\Delta x} \int_{\mathbb{R}} |u_0(x+\Delta x) - u_0(x)| \mathrm{d}x \leq \mathrm{TV}(u_0)
\end{aligned}$$

If $u_0 \in L_1(\mathbb{R})$ then

$$\|v_\Delta(\cdot,0)\|_{L_1(\mathbb{R})} = \Delta x \sum_j |v_j^0| = \sum_j \left| \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) \mathrm{d}x \right| \leq \|u_0\|_{L_1(\mathbb{R})}$$

Hence the sequence of solutions $(v_\Delta)_\Delta$ belongs to the same compact set $\mathcal{K} \subset L_{1,T}$. We now claim that $\text{dist}(v_\Delta, \mathcal{W}) \to 0$. Assume the contrary. Then there is a sequence $(v_{\Delta_k})_k$ such that $\Delta_k x \to 0$ as $k \to \infty$ while

$$\text{dist}(v_{\Delta_k}, \mathcal{W}) > \epsilon, \qquad \forall k$$

Since $v_{\Delta_k} \in \mathcal{K}$ for all k, there is a convergent subsequence converging to some $u \in \mathcal{K}$. Hence

$$\|v_{\Delta_k} - u\|_{1,T} < \epsilon, \qquad \text{for } k \text{ sufficiently large}$$

But since v_{Δ_k} is generated by a conservative and consistent method, it follows from Lax-Wendroff Theorem that the limit u is a weak solution of the conservation law, i.e., $u \in \mathcal{W}$. But this contradicts our initial assumption that $\text{dist}(v_{\Delta_k}, \mathcal{W}) > \epsilon$ and we conclude that $\text{dist}(v_{\Delta_k}, \mathcal{W}) \to 0$.

Remark: Note that we do not have uniqueness. Different sequences $(\Delta_k x)_k$ could converge to different weak solutions $u \in \mathcal{W}$. To show that all the sub-sequences have a unique limit, we need to impose the entropy condition.