Finite volume method for conservation laws IV

Incremental and viscosity form

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February 21, 2013

Definition (Incremental form)

We say that scheme H_Δ can be put in incremental form if there exist two functions of $2k$ variables C, D called incremental coefficients such that if we set

$$C_{j+\frac{1}{2}} = C(v_{j-k+1}, \ldots, v_{j+k}), \qquad D_{j+\frac{1}{2}} = D(v_{j-k+1}, \ldots, v_{j+k})$$

the scheme can be written as

$$v_j^{n+1} = v_j^n + C^n_{j+\frac{1}{2}} \Delta v^n_{j+\frac{1}{2}} - D^n_{j-\frac{1}{2}} \Delta v^n_{j-\frac{1}{2}}, \qquad \Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

Proposition (Incremental form for 3-point scheme)

Any 3-point consistent, conservative scheme with numerical flux g admits a unique incremental form with incremental coefficients given by

$$C_{j+\frac{1}{2}} = \lambda \frac{f_j - g_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}, \qquad D_{j+\frac{1}{2}} = \lambda \frac{f_{j+1} - g_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

Proof: Equating the incremental and conservative forms leads to

$$C(v_j, v_{j+1})(v_{j+1} - v_j) - D(v_{j-1}, v_j)(v_j - v_{j-1}) = -\lambda(g(v_j, v_{j+1}) - g(v_{j-1}, v_j))$$

Setting $v_{j-1} = v_j$ and using the consistency condition $g(v_j, v_j) = f_j$ leads to the desired result for C. Similarly, setting $v_{j+1} = v_j$ leads to the formula for D. We also observe that the coefficients satisfy the following consistency condition

$$D_{j+\frac{1}{2}} - C_{j+\frac{1}{2}} = \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

Conversely, any 3-point incremental scheme can be put in conservation form provided the above relation is satisfied, and the numerical flux is defined as

$$g_{j+\frac{1}{2}} = f_j - \frac{1}{\lambda} C_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}}$$

Definition (Essentially 3-point scheme)

A conservative scheme is called essentially 3-point if its numerical flux satisfies the stronger consistency condition

$$g(v_{-k+1}, \ldots, v_{-1}, v, v, v_2, \ldots, v_k) = f(v)$$

Definition (Viscosity form)

Scheme H_Δ can be put in viscosity form if there exists a function of $2k$ variables Q called the *coefficient of numerical viscosity* such that if we set

$$Q_{j+\frac{1}{2}} = Q(v_{j-k+1}, \ldots, v_{j+k})$$

the scheme can be written as

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(f_{j+1}^n - f_{j-1}^n) + \frac{1}{2}[Q_{j+\frac{1}{2}}^n \Delta v_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n \Delta v_{j-\frac{1}{2}}^n]$$

Given a scheme in the above viscosity form, its numerical flux is given by

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2\lambda} Q_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}}$$

This flux is essentially 3-point. Thus, any scheme in viscosity form has a unique conservation form. The converse is also true for essentially 3-point schemes.

Proposition

Any essentially 3-point scheme with numerical flux g admits a unique viscous form whose viscosity coefficient is given by

$$Q_{j+\frac{1}{2}} = \lambda \frac{f_j + f_{j+1} - 2g_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

Moreover, for a 3-point scheme

$$Q_{j+\frac{1}{2}} = C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}$$

where $C_{j+\frac{1}{2}}, D_{j+\frac{1}{2}}$ are the unique incremental coefficients.

Remark: In general for essentially 3-point schemes in viscosity form, the incremental coefficients are not defined in a unique way. But we can define the incremental coefficients from the numerical flux, e.g.

$$\begin{aligned} C_{j+\frac{1}{2}} &= \lambda \frac{f_j - g_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} = \lambda \frac{f_j - \frac{1}{2}(f_j + f_{j+1}) + \frac{1}{2\lambda} Q_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \\ &= -\frac{\lambda}{2} \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} + \frac{1}{2} Q_{j+\frac{1}{2}} \end{aligned}$$

and similarly

$$D_{j-\frac{1}{2}} = \frac{\lambda}{2} \frac{\Delta f_{j-\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}} + \frac{1}{2} Q_{j-\frac{1}{2}}$$

These coefficients satisfy the condition $Q_{j+\frac{1}{2}} = C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}$. However this is not the only possible choice of the incremental coefficients.

Remark: Conversely, given a conservative scheme in incremental form, it cannot necessarily be written in viscosity form. However we can define a pseudo-viscous form as follows. Equating the conservative and incremental forms,

$$g_{j+\frac{1}{2}} + \frac{1}{\lambda} C_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} = g_{j-\frac{1}{2}} + \frac{1}{\lambda} D_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}$$

Let us define a *modified flux* by

$$\tilde{f}_j = g_{j+\frac{1}{2}} + \frac{1}{\lambda} C_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} = g_{j-\frac{1}{2}} + \frac{1}{\lambda} D_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}$$

In the case of a 3-point scheme $\tilde{f}_j = f_j$ so that $\tilde{f}$ can be considered as a perturbation of f. Now let us define a *modified viscosity coefficient* by

$$\tilde{Q}_{j+\frac{1}{2}} = C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}$$

then we get the *modified viscous form*

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(\tilde{f}_{j+1}^n - \tilde{f}_{j-1}^n) + \frac{1}{2}[\tilde{Q}_{j+\frac{1}{2}}^n \Delta v_{j+\frac{1}{2}}^n - \tilde{Q}_{j-\frac{1}{2}}^n \Delta v_{j-\frac{1}{2}}^n]$$

This $\tilde{Q}$ satisfies analogous equations with f replaced by $\tilde{f}$

$$\tilde{Q}_{j+\frac{1}{2}} = \lambda \frac{\tilde{f}_j + \tilde{f}_{j+1} - 2g_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

and the incremental coefficients are related by

$$C_{j+\frac{1}{2}} = -\frac{\lambda}{2}\frac{\Delta \tilde{f}_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} + \frac{1}{2}\tilde{Q}_{j+\frac{1}{2}}, \qquad D_{j-\frac{1}{2}} = \frac{\lambda}{2}\frac{\Delta \tilde{f}_{j-\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}} + \frac{1}{2}\tilde{Q}_{j-\frac{1}{2}}$$

The modified viscosity form will be useful for constructing second order TVD schemes.

Proposition (TVD condition for incremental form)

Assume that the difference scheme H_Δ can be written in incremental form and that the incremental coefficients satisfy the conditions

$$C_{j+\frac{1}{2}} \geq 0, \qquad D_{j+\frac{1}{2}} \geq 0, \qquad C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

Then the scheme is TVD.

Proof: Let us set $w = H_\Delta(v)$. Then

$$w_j = v_j + C_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - C_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}$$

so that

$$\Delta w_{j+\frac{1}{2}} = C_{j+\frac{3}{2}} \Delta v_{j+\frac{3}{2}} + (1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}) \Delta v_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}}$$

By assumption all the coefficients on the right are positive

$$|\Delta w_{j+\frac{1}{2}}| \leq C_{j+\frac{3}{2}} |\Delta v_{j+\frac{3}{2}}| + (1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}) |\Delta v_{j+\frac{1}{2}}| + D_{j-\frac{1}{2}} |\Delta v_{j-\frac{1}{2}}|$$

so that

$$TV(w) = \sum_j |\Delta w_{j+\frac{1}{2}}| \leq \sum_j |\Delta v_{j+\frac{1}{2}}| = TV(v)$$

Corollary (TVD condition for viscosity form)

Assume that the difference scheme H_Δ can be written in viscosity form and that the viscosity coefficients satisfy the conditions

$$\lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

Then the scheme is TVD.

Proof: Defining the incremental coefficients from the viscosity coefficient

$$C_{j+\frac{1}{2}} = -\frac{\lambda}{2} \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} + \frac{1}{2} Q_{j+\frac{1}{2}}, \qquad D_{j+\frac{1}{2}} = \frac{\lambda}{2} \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} + \frac{1}{2} Q_{j+\frac{1}{2}}$$

and noting that $Q_{j+\frac{1}{2}} = C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}$, we obtain the desired condition on Q by using the previous proposition.

Theorem (3-point TVD scheme)

A 3-point scheme which is written in an incremental form is TVD iff it is monotonicity preserving. And a necessary and sufficient condition for the scheme to be TVD is given by

$$C_{j+\frac{1}{2}} \geq 0, \qquad D_{j+\frac{1}{2}} \geq 0, \qquad C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

Proof; We have already shown that any TVD scheme is monotonicity preserving. It is thus enough to prove that for a 3-point monotonicity preserving (MP) scheme, the conditions on incremental coefficients are satisfied, i.e., MP $\Longrightarrow$ TVD. Consider the particular sequence with

$$v_j = \begin{cases} v_l & j \leq J \\ v_r & j > J \end{cases}$$

for which

$$\begin{aligned} \Delta v_{j+\frac{1}{2}} &= 0, \qquad j \leq J-1 \quad \text{or} \quad j \geq J+1 \\ \Delta v_{J+\frac{1}{2}} &= v_r - v_l \end{aligned}$$

With $w = H_\Delta(v)$ we have

$$\begin{aligned} \Delta w_{J+\frac{3}{2}} &= D_{J+\frac{1}{2}} \Delta v_{J+\frac{1}{2}} \\ \Delta w_{J+\frac{1}{2}} &= (1 - C_{J+\frac{1}{2}} - D_{J+\frac{1}{2}}) \Delta v_{J+\frac{1}{2}} \\ \Delta w_{J-\frac{1}{2}} &= C_{J+\frac{1}{2}} \Delta v_{J+\frac{1}{2}} \end{aligned}$$

By monotonicity preserving, the sign of $\Delta w_{J+\frac{3}{2}}$ must be same as sign of $\Delta v_{J+\frac{1}{2}}$. Hence

$$D_{J+\frac{1}{2}} = D(v_J, v_{J+1}) = D(v_l, v_r) \geq 0$$

for arbitrary $v_l, v_r \in \mathbb{R}$. The same is true of the other coefficients which leads to the necessary conditions on the incremental coefficients.

Remark: For a 3-point conservative scheme, the incremental and viscosity forms are uniquely defined. Then the conditions for TVD a scheme are also necessary, which we state as a corollary.

Corollary

A 3-point scheme with Lipschitz continuous numerical flux is TVD if and only if

$$C_{j+\frac{1}{2}} \geq 0, \qquad D_{j+\frac{1}{2}} \geq 0, \qquad C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

or equivalently

$$\lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

holds.

Theorem (TVD and L^∞-stability)

Let H_Δ be an essentially 3-point scheme which is written in a viscous form. Assume that the coefficient of numerical viscosity satisfies

$$\lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq \frac{1}{2}, \qquad \forall j \in \mathbb{Z}$$

Then the scheme is TVD and L^∞-stable.

Proof: We already know that the scheme is TVD. We will show the local maximum principle

$$\min_{j-1,j,j+1} v_l \leq (H_\Delta(v))_j \leq \max_{j-1,j,j+1} v_l$$

which implies L^∞ stability. Define the incremental coefficient

$$C_{j+\frac{1}{2}} = -\frac{\lambda}{2}\frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} + \frac{1}{2}Q_{j+\frac{1}{2}} \leq \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$$

and similarly $D_{j+\frac{1}{2}} \leq \frac{1}{2}$. Then

$$(H_\Delta(v))_j = D_{j-\frac{1}{2}}v_{j-1} + (1 - C_{j+\frac{1}{2}} - D_{j-\frac{1}{2}})v_j + C_{j+\frac{1}{2}}v_{j+1}$$

is a convex combination of the values v_{j-1}, v_j, v_{j+1} which leads to the local maximum principle.

Proposition (Construction of 3-point TVD scheme)

Consider a 3-point difference scheme in viscous form whose numerical viscosity coefficient satisfies

$$Q(u, v) = Q(\lambda a(u, v))$$

where

$$a(u, v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \neq v \\ a(u) = f'(u) & \text{if } u = v \end{cases}$$

and $Q(\cdot)$ is some given continuous function. Then if Q satisfies

$$|x| \leq Q(x) \leq 1, \qquad 0 \leq |x| \leq \mu \leq 1$$

the finite difference scheme is TVD under the CFL condition

$$\lambda \max_{j,n} |a(v_j^n, v_{j+1}^n)| \leq \mu$$

Example: Lax-Friedrichs scheme
The numerical flux is given by

$$g(u,v) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2\lambda}(v - u)$$

for which the viscosity coefficient is

$$Q^{LF}_{j+\frac{1}{2}} = 1$$

This corresponds to the upper bound in the TVD condition, i.e., taking $Q(x) = 1$. Hence the scheme is TVD provided the CFL condition

$$\lambda \max_j \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| \leq 1$$

is satisfied.

Example: Murman-Roe scheme
The numerical flux is given by

$$g^R(u,v) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}|a(u,v)|(v-u)$$

and its numerical viscosity coefficient is

$$Q^R_{j+\frac{1}{2}} = \lambda|a(v_j, v_{j+1})| = \lambda\left|\frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\right|$$

This corresponds to the lower bound in the TVD condition, i.e., taking $Q(x) = |x|$. The scheme is TVD provided the CFL condition

$$\lambda \max_j \left|\frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\right| \leq 1$$

is satisfied. However, we know that this scheme admits entropy violating shocks.

Example: Engquist-Osher scheme

The numerical flux is given by

$$g^{EO}(u,v) = \frac{1}{2}(f(u)+f(v)) - \frac{1}{2}\int_u^v |a(\xi)|\mathrm{d}\xi$$

and its numerical viscosity coefficient is given by

$$Q^{EO}_{j+\frac{1}{2}} = \frac{\lambda}{\Delta v_{j+\frac{1}{2}}}\int_{v_j}^{v_{j+1}} |a(\xi)|\mathrm{d}\xi$$

In the strictly convex case, the numerical flux is given by

$$g^{EO}(u,v) = f^+(u) + f^-(v) - f(\bar{u})$$

where $\bar{u}$ is the only sonic point ($a(\bar{u}) = f'(\bar{u}) = 0$) and

$$f^+(u) = f(\max(u,\bar{u})), \qquad f^-(u) = f(\min(u,\bar{u}))$$

When $v_j \leq \bar{u} \leq v_{j+1}$ then

$$Q^{EO}_{j+\frac{1}{2}} = \lambda\frac{f_j + f_{j+1} - 2f(\bar{u})}{\Delta v_{j+\frac{1}{2}}}$$

while in domains where f is monotone, it reduces to the standard upwind scheme, as Roe's scheme. Thus for $v_j, v_{j+1} < \bar{u}$ or $v_j, v_{j+1} > \bar{u}$ we get

$$Q^{EO}_{j+\frac{1}{2}} = \lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| = Q^{R}_{j+\frac{1}{2}}$$

It deviates from Roe scheme by introducing non-zero dissipation near a sonic point. Moreover, it is monotone provided the CFL condition is satisfied.

Remark: Note that in the case $v_j \leq \bar{u} \leq v_{j+1}$ Roe scheme gives entropy violating shocks if $f_j = f_{j+1}$ and $Q^{R}_{j+\frac{1}{2}} = 0$. In case of EO scheme, $Q^{EO}_{j+\frac{1}{2}} > 0$ and $Q^{EO}_{j+\frac{1}{2}} = 0$ only if $v_j = v_{j+1} = \bar{u}$.

Example: Lax-Wendroff scheme
The numerical flux is given by

$$g^{LW}(u,v) = \frac{1}{2}(f(u)+f(v)) - \frac{1}{2}\lambda a(u,v)(f(v)-f(u))$$

Hence its numerical viscosity coefficient is

$$Q^{LW}_{j+\frac{1}{2}} = (\lambda a_{j+\frac{1}{2}})^2 = \lambda^2 \left(\frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right)^2$$

which *does not satisfy the TVD condition.* This also implies that it does not preserve monotonicity. To illustrate this, consider the linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition

$$v^0_j = \begin{cases} 0 & \text{for } j \le 0 \\ 1 & \text{for } j \ge 1 \end{cases}$$

which is increasing function. After one time step, the solution is given by

$$\begin{aligned} v_j^1 &= 0, \quad \text{for } j \le -1 \\ v_0^1 &= \frac{1}{2}(\lambda + \lambda^2) \\ v_1^1 &= 1 + \frac{\lambda}{2} - \frac{\lambda^2}{2} \\ v_j^1 &= 1, \quad \text{for } j \ge 2 \end{aligned}$$

Thus for $\lambda < 1$,

$$v_0^1 = \frac{1}{2}(\lambda + \lambda^2) < \frac{1}{2}(1 + \lambda) < 1$$

$$v_1^1 = 1 + \frac{\lambda}{2} - \frac{\lambda^2}{2} > 1 + \frac{\lambda}{2}(1 - \lambda) > 1$$

$$\ldots v_{-1}^1 < v_0^1 < 1 < v_1^1 > v_2^1 \ldots$$

and thus v^1 is not an increasing function. Thus the Lax-Wendroff scheme tends to generate oscillations in the neighbourhood of discontinuities and non-smooth regions.

Theorem

Any essentially 3-point conservative difference scheme which satisfies the TVD requirement

$$\lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}} \right| \leq Q_{j+\frac{1}{2}} \leq 1, \qquad \forall j \in \mathbb{Z}$$

is at most first order accurate.

Proof: The numerical flux of an essentially 3-point scheme can be written as

$$g(v_{-k+1}, \ldots, v_k) = \frac{1}{2}(f(v_0) + f(v_1)) - \frac{1}{2\lambda} Q(v_{-k+1}, \ldots, v_k)(v_1 - v_0)$$

Comparing it to the second order accurate Lax-Wendroff flux we get

$$g - g^{LW} = \frac{1}{2\lambda}(\lambda^2 a^2(v_0, v_1) - Q)(v_1 - v_0)$$

Since the scheme is TVD, Q satisfies

$$\lambda |a(v_0, v_1)| \leq Q \leq 1$$

so that

$$
\begin{aligned}
|g - g^{LW}| &= \frac{1}{2\lambda}(Q - \lambda^2 a^2(v_0, v_1))|v_1 - v_0| \\
&\geq \frac{1}{2}|a(v_0, v_1)|\ (1 - \lambda|a(v_0, v_1)|)\ |v_1 - v_0| \\
&= \mathcal{O}\left(\Delta x\right) \quad \text{for smooth solutions}
\end{aligned}
$$

But the numerical flux of a second order accurate scheme has to satisfy for any smooth soution u

$$
|g(u(x+(-k+1)\Delta x, t), \ldots, u(x+k\Delta x, t) - g^{LW}(u(x, t), u(x+\Delta x, t))| = \mathcal{O}\left(\Delta x^2\right)
$$

However for the TVD scheme, this difference is only $\mathcal{O}\left(\Delta x\right)$, unless $\lambda|a(v_0, v_1)| = 1$ everywhere, which corresponds to the trivial case of a pure translation.

Corollary

A 3-point TVD conservative difference scheme is at most first order accurate.