Finite volume method for conservation laws V

Schemes satisfying entropy condition

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in

February 21, 2013

Entropy condition

Let $U(u)$ be any convex entropy of the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

with associated entropy flux $F(u)$, i.e.,

$$U'(u)f'(u) = F'(u)$$

Then the unique entropy solution of the conservation law satisfies in the sense of distributions

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(u) \leq 0$$

Definition (Entropy consistent numerical scheme)

The difference scheme H is said to be consistent with the entropy condition if there exists a continuous function $G : \mathbb{R}^{2k} \to \mathbb{R}$ which satisfies the following requirements

(1) consistency with the entropy flux F

$$G(v, \ldots, v) = F(v)$$

(2) cell entropy inequality

$$\frac{U(v_j^{n+1}) - U(v_j^n)}{\Delta t} + \frac{G(v_{j-k+1}^n, \ldots, v_{j+k}^n) - G(v_{j-k}^n, \ldots, v_{j+k-1}^n)}{\Delta x} \leq 0$$

The function G is called the *numerical entropy flux*.

If we set

$$U_j^n = U(v_j^n), \qquad G_{j+\frac{1}{2}}^n = G(v_{j-k+1}^n, \ldots, v_{j+k}^n)$$

then we have to verify that

$$U_j^{n+1} \leq U_j^n - \lambda(G_{j+\frac{1}{2}}^n - G_{j-\frac{1}{2}}^n)$$

When the scheme is consistent with any entropy function, we say that it is an "entropy scheme".

Proposition 4.1 (GR1, Chap. 3)

Let H be a 3-point difference scheme with C^2 numerical flux which is consistent with any entropy condition. Then it is at most first order accurate.

Theorem

Assume that the hypothesis of Lax-Wendroff theorem hold. Assume moreover that the scheme is consistent with any entropy condition. Then the limit u is the unique entropy solution of the conservation law.

Theorem

A monotone consistent scheme is consistent with any entropy condition.

Proof: It is enough to check the entropy condition for the Kruzkov's entropy functions: for any $l \in \mathbb{R}$

$$U(u) = |u - l|, \qquad F(u) = \text{sign}(u - l)(f(u) - f(l))$$

Set

$$a \wedge b = \min(a,b), \qquad a \vee b = \max(a,b)$$

and define the numerical entropy flux G as follows

$$G(v_{-k+1},\dots,v_k) = g(v_{-k+1} \vee l,\dots,v_k \vee l) - g(v_{-k+1} \wedge l,\dots,v_k \wedge l)$$

We first prove the following identity

$$|v_j - l| - \lambda(G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) = H(v_{j-k} \vee l,\dots,v_{j+k} \vee l) - H(v_{j-k} \wedge l,\dots,v_{j+k} \wedge l)$$

By definition of $G_{j+\frac{1}{2}}$

$$\begin{aligned} G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}} &= [g(v_{j-k+1} \vee l,\dots,v_{j+k} \vee l) - g(v_{j-k+1} \wedge l,\dots,v_{j+k} \wedge l)] \\ &\quad -[g(v_{j-k} \vee l,\dots,v_{j+k-1} \vee l) - g(v_{j-k} \wedge l,\dots,v_{j+k-1} \wedge l)] \\ &= [g(v_{j-k+1} \vee l,\dots,v_{j+k} \vee l) - g(v_{j-k} \vee l,\dots,v_{j+k-1} \vee l)] \\ &\quad -[g(v_{j-k+1} \wedge l,\dots,v_{j+k} \wedge l) - g(v_{j-k} \wedge l,\dots,v_{j+k-1} \wedge l)] \end{aligned}$$

and using the finite volume scheme, we get

$$\begin{aligned} -\lambda[G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}] &= [H(v_{j-k} \vee l,\dots,v_{j+k} \vee l) - v_j \vee l] \\ &\quad -[H(v_{j-k} \wedge l,\dots,v_{j+k} \wedge l) - v_j \wedge l] \end{aligned}$$

which gives the identity we wanted to show since

$$v_j \vee l - v_j \wedge l = |v_j - l|$$

Now since H is monotone and consistent

$$H(v_{j-k} \vee l, \ldots, v_{j+k} \vee l) \geq H(v_{j-k}, \ldots, v_{j+k}) \vee H(l, \ldots, l) = v_j^{n+1} \vee l$$

In the same way we obtain

$$H(v_{j-k} \wedge l, \ldots, v_{j+k} \wedge l) \leq v_j^{n+1} \wedge l$$

Combining the above results we verify the entropy condition

$$|v_j^{n+1} - l| - |v_j^n - l| + \lambda(G^n_{j+\frac{1}{2}} - G^n_{j-\frac{1}{2}}) \leq 0$$

The consistency condition of G is readily satisfied since

$$\begin{aligned} G(v, \ldots, v) &= g(v \vee l, \ldots, v \vee l) - g(v \wedge l, \ldots, v \wedge l) \\ &= f(v \vee l) - f(v \wedge l) = \operatorname{sign}(v - l)(f(v) - f(l)) \\ &= F(u) \end{aligned}$$

Definition (E-scheme)

A consistent, conservative scheme is called an E-scheme if its numerical flux satisfies

$$\operatorname{sign}(v_{j+1} - v_j)(g_{j+\frac{1}{2}} - f(u)) \leq 0$$

for all u between v_j and v_{j+1}.

Remark: Note that an E-scheme is essentially 3-point. Indeed letting $v_{j+1} \to v_j$ with first $v_{j+1} \geq v_j$ and then with $v_{j+1} \leq v_j$ shows that g is essentially 3-point.

Remark: A 3-point monotone scheme is an E-scheme. Since $g(u, v)$ is non-decreasing in u and non-increasing in v, we obtain

$$\begin{aligned} g(u,v) \leq g(u,w) \leq g(w,w) = f(w) \quad &\text{if} \quad u \leq w \leq v \\ g(u,v) \geq g(w,v) \geq g(w,w) = f(w) \quad &\text{if} \quad u \geq w \geq v \end{aligned}$$

and therefore

$$\operatorname{sign}(v-u)(g(u,v) - f(w)) \leq 0, \qquad \text{for all } w \text{ between } u \text{ and } v$$

In particular, the Godunov scheme is an E-scheme under CFl ≤ 1.

Lemma

Assume that CFL ≤ 1. Then E-fluxes are characterized by

$$\begin{cases} g_{j+\frac{1}{2}} \leq g^G_{j+\frac{1}{2}} & \text{if } v_j < v_{j+1} \\ g_{j+\frac{1}{2}} \geq g^G_{j+\frac{1}{2}} & \text{if } v_j > v_{j+1} \end{cases}$$

where g^G stands for Godunov numerical flux.

Proof: Under CFL ≤ 1, the Godunov flux is given by

$$g^G_{j+\frac{1}{2}} = \begin{cases} \min_{u\in[v_j,v_{j+1}]} f(u) & \text{if } v_j < v_{j+1} \\ \max_{u\in[v_{j+1},v_j]} f(u) & \text{if } v_j \geq v_{j+1} \end{cases}$$

Assume $v_j < v_{j+1}$. Then E-flux satisfies

$$g_{j+\frac{1}{2}} \leq f(u), \qquad v_j \leq u \leq v_{j+1}$$

Assume $v_j > v_{j+1}$. Then E-flux satisfies

$$g_{j+\frac{1}{2}} \geq f(u), \qquad v_{j+1} \leq u \leq v_j \quad \square$$

Lemma

Assume that CFL ≤ 1. E-schemes are characterized by

$$0 \leq Q^G_{j+\frac{1}{2}} \leq Q_{j+\frac{1}{2}}, \qquad \forall j \in \mathbb{Z}$$

Proof: Assume $v_j < v_{j+1}$. Then for E-scheme $g_{j+\frac{1}{2}} \leq g^G_{j+\frac{1}{2}}$

$$\frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}Q_{j+\frac{1}{2}}(v_{j+1} - v_j) \leq \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}Q^G_{j+\frac{1}{2}}(v_{j+1} - v_j) \quad \square$$

Proposition

An E-scheme with differentiable numerical flux is at most first order accurate.

Theorem (Viscous form and entropy condition)

Assume that the CFL condition

$$\lambda \max |a(u)| \leq \frac{1}{2}$$

holds. An E-scheme whose coefficient of numerical viscosity satisfies

$$Q^G_{j+\frac{1}{2}} \leq Q_{j+\frac{1}{2}} \leq \frac{1}{2}$$

is consistent with any entropy condition.

The proof requires two lemmas which we first prove. The basic idea is to write any E-scheme as a convex combination of the Godunov scheme and a modified Lax-Friedrichs scheme, both of which satisfy entropy condition.

Lemma (Godunov scheme)

Assume that CFL $\leq \frac{1}{2}$. The Godunov scheme can be written in the following way

$$H^G(v_{j-1}, v_j, v_{j+1}) = \frac{1}{2}(v^{G-}_{j-\frac{1}{2}} + v^{G+}_{j+\frac{1}{2}})$$

where

$$\begin{aligned} v^{G-}_{j-\frac{1}{2}} &= \frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_{j-1}, v_j)\mathrm{d}x = v_j - 2\lambda(f_j - g^G_{j-\frac{1}{2}}) \\ v^{G+}_{j+\frac{1}{2}} &= \frac{2}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} w_R(x/\Delta t; v_j, v_{j+1})\mathrm{d}x = v_j + 2\lambda(f_j - g^G_{j+\frac{1}{2}}) \end{aligned}$$

Moreover if (U, F) is any entropy pair, we have

$$U(v^{G\pm}_{j\pm\frac{1}{2}}) \leq U(v_j) \pm 2\lambda[F(v_j) - G^G_{j\pm\frac{1}{2}}]$$

where

$$G^G(u, v) = F(w_R(0; u, v))$$

Proof: The formulae for $v^{G\pm}_{j\pm\frac{1}{2}}$ follow from the derivation of the Godunov scheme. Now since U is a convex function, we obtain by Jensen's inequality

$$U\left(\frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_{j-1}, v_j)\mathrm{d}x\right) \leq \frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} U(w_R(x/\Delta t; v_{j-1}, v_j))\mathrm{d}x$$

Since $w_R(x/\Delta t; v_{j-1}, v_j)$ is by definition the entropy solution of the Riemann problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}f(u) = 0, \quad t \in [0, \Delta t]$$
$$u(x,0) = \begin{cases} v_{j-1} & x < 0 \\ v_j & x > 0 \end{cases}$$

it satisfies an entropy inequality

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0$$

Integrating this last inequality on the domain $(x_{j-\frac{1}{2}}, x_j) \times (0, \Delta t)$ we get

$$\begin{aligned} U(v^{G-}_{j-\frac{1}{2}}) &\leq \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} U(w_R(x/\Delta t; v_{j-1}, v_j)) \mathrm{d}x \\ &\leq U(v_j) - 2\lambda [F(v_j) - G^G_{j-\frac{1}{2}}] \end{aligned}$$

In the same way we have

$$\begin{aligned} U(v^{G+}_{j+\frac{1}{2}}) &\leq \frac{2}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 U(w_R(x/\Delta t; v_j, v_{j+1})) \mathrm{d}x \\ &\leq U(v_j) + 2\lambda [F(v_j) - G^G_{j+\frac{1}{2}}] \end{aligned}$$

Remark: This also shows that Godunov scheme satisfies the entropy condition associated with (U, F) since

$$\begin{aligned} U(H^G(v_{j-1}, v_j, v_{j+1})) &\leq \frac{1}{2} [U(v^{G-}_{j-\frac{1}{2}}) + U(v^{G+}_{j+\frac{1}{2}})] \\ &\leq U(v_j) - \lambda [G^G_{j+\frac{1}{2}} - G^G_{j-\frac{1}{2}}] \end{aligned}$$

Lemma (Lax-Friedrichs modified)

Consider the 3-point scheme

$$H^M(v_{j-1}, v_j, v_{j+1}) = \frac{1}{4}(v_{j-1} + 2v_j + v_{j+1}) - \frac{1}{2}\lambda(f_{j+1} - f_{j-1})$$

Then under the condition CFL $\leq \frac{1}{2}$, we can write

$$H^M(v_{j-1}, v_j, v_{j+1}) = \frac{1}{2}(v^{M-}_{j-\frac{1}{2}} + v^{M+}_{j+\frac{1}{2}})$$

where

$$\begin{aligned} v^{M-}_{j-\frac{1}{2}} &= \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_{j-1}, v_j)\mathrm{d}x = \frac{1}{2}(v_j + v_{j-1}) - \lambda(f_j - f_{j-1}) \\ v^{M+}_{j+\frac{1}{2}} &= \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_j, v_{j+1})\mathrm{d}x = \frac{1}{2}(v_j + v_{j+1}) - \lambda(f_{j+1} - f_j) \end{aligned}$$

Moreover if (U, F) is any entropy pair, we have

$$U(v^{M\pm}_{j\pm\frac{1}{2}}) \leq U(v_j) \pm 2\lambda[F(v_j) - G^M_{j\pm\frac{1}{2}}]$$

where

$$G^M(u, v) = \frac{1}{2}(F(u) + F(v)) - \frac{1}{4\lambda}(U(v) - U(u))$$

Proof: Under the condition CFL $\le \frac{1}{2}$, we obtain the desired results by integrating the conservation law over the domains $(x_{j-1}, x_j) \times (t_n, t_{n+1})$ and $(x_j, x_{j+1}) \times (t_n, t_{n+1})$. We get

$$\begin{aligned} 0 &= \int_{t_n}^{t_{n+1}} \int_{x_{j-1}}^{x_j} \left(\frac{\partial w_R}{\partial t} + \frac{\partial}{\partial x} f(w_R) \right) \mathrm{d}x \mathrm{d}t \\ &= \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_{j-1}, v_j) \mathrm{d}x - \frac{\Delta x}{2}(v_j + v_{j-1}) + \\ &\quad \int_0^{\Delta t} [f(w_R(\Delta x/(2t); v_{j-1}, v_j)) - f(w_R(-\Delta x/(2t); v_{j-1}, v_j))] \mathrm{d}t \end{aligned}$$

But the waves from $x_{j-\frac{1}{2}}$ do not reach x_{j-1}, x_j so that

$$w_R(\Delta x/(2t); v_{j-1}, v_j) = v_j, \qquad w_R(-\Delta x/(2t); v_{j-1}, v_j) = v_{j-1}$$

This proves the formulae for $v_{j\pm\frac{1}{2}}^{M\pm}$.

Since w_R is the entropy solution, we integrate the entropy inequality over $(x_{j-1}, x_j) \times (t_n, t_{n+1})$ and using Jensen's inequality

$$
\begin{aligned}
U(v^{M-}_{j-\frac{1}{2}}) &\leq \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} U(w_R(x/\Delta t; v_{j-1}, v_j))\mathrm{d}x \\
&\leq \frac{1}{2}(U(v_j) + U(v_{j-1})) - \lambda(F(v_j) - F(v_{j-1})) \\
&= U(v_j) - 2\lambda(F(v_j) - G^M_{j-\frac{1}{2}})
\end{aligned}
$$

Remark: The modified Lax-Friedrichs scheme is consistent with any entropy condition with numerical flux G^M since

$$
\begin{aligned}
U(H^M(v_{j-1}, v_j, v_{j+1})) &\leq \frac{1}{2}(U(v^{M-}_{j-\frac{1}{2}}) + U(v^{M+}_{j+\frac{1}{2}})) \\
&\leq U(v_j) - \lambda[G^M_{j+\frac{1}{2}} - G^M_{j-\frac{1}{2}}]
\end{aligned}
$$

Note that the numerical viscosity of this scheme is

$$
Q^M_{j+\frac{1}{2}} = \frac{1}{2}
$$

Proof of Theorem (Viscous form and entropy condition): Let us write the scheme in viscous form

$$v_j^{n+1} = v_j^n - \frac{\lambda}{2}(f_{j+1}^n - f_{j-1}^n) + \frac{1}{2}(Q_{j+\frac{1}{2}}^n \Delta v_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n \Delta v_{j-\frac{1}{2}}^n)$$

and in averaged form

$$v_j^{n+1} = \frac{1}{2}(v_{j-\frac{1}{2}}^- + v_{j+\frac{1}{2}}^+)$$

where

$$\begin{aligned} v_{j-\frac{1}{2}}^- &= v_j - \lambda(f_j - f_{j-1}) - Q_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}} \\ v_{j+\frac{1}{2}}^+ &= v_j - \lambda(f_{j+1} - f_j) + Q_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} \end{aligned}$$

We can write the Godunov and modified Lax-Friedrich schemes in the same form with superscript G and M. Now since $Q^M = \frac{1}{2}$ and

$$Q^G \leq Q \leq Q^M = \frac{1}{2}$$

we can write, with some $0 \leq \theta_{j+\frac{1}{2}} \leq 1$

$$Q_{j+\frac{1}{2}} = \theta_{j+\frac{1}{2}} Q^G_{j+\frac{1}{2}} + (1-\theta_{j+\frac{1}{2}}) Q^M_{j+\frac{1}{2}}, \quad \forall j \in \mathbb{Z}$$

It follows that

$$v^{\pm}_{j\pm\frac{1}{2}} = \theta_{j\pm\frac{1}{2}} v^{G\pm}_{j\pm\frac{1}{2}} + (1-\theta_{j\pm\frac{1}{2}}) v^{M\pm}_{j\pm\frac{1}{2}}$$

If (U, F) is any entropy pair, then due to convexity of U

$$\begin{aligned} U(v_j^{n+1}) &\le \frac{1}{2}U(v^-_{j-\frac{1}{2}}) + \frac{1}{2}U(v^+_{j+\frac{1}{2}}) \\ &\le \frac{1}{2}\theta_{j-\frac{1}{2}} U(v^{G-}_{j-\frac{1}{2}}) + \frac{1}{2}(1-\theta_{j-\frac{1}{2}}) U(v^{M-}_{j-\frac{1}{2}}) + \\ &\quad \frac{1}{2}\theta_{j+\frac{1}{2}} U(v^{G+}_{j+\frac{1}{2}}) + \frac{1}{2}(1-\theta_{j+\frac{1}{2}}) U(v^{M+}_{j+\frac{1}{2}}) \\ &\le U(v_j) - \lambda(G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}}) \end{aligned}$$

where

$$G_{j+\frac{1}{2}} = \theta_{j+\frac{1}{2}} G^G_{j+\frac{1}{2}} + (1-\theta_{j+\frac{1}{2}}) G^M_{j+\frac{1}{2}}$$

is a consistent entropy flux associated with the E-scheme under consideration.

Remark: Under the conditions of the above theorem $Q^G \le Q \le \frac{1}{2}$, the E-scheme is also TVD and L^∞ stable.