Finite volume method for conservation laws VI

Approximate Riemann Solvers

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in

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Godunov scheme

The finite volume solution is made of piecewise constant states

$$v_\Delta(x,t) = v_j^n, \quad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad t \in [t_n, t_{n+1})$$

which defines a Riemann problem at each cell face $x = x_{j+\frac{1}{2}}$

$$\frac{\partial w_R}{\partial t} + \frac{\partial}{\partial x} f(w_R) = 0, \quad x \in (x_j, x_{j+1}), \quad t \in [t_n, t_{n+1})$$

$$w_R(x,0) = \begin{cases} v_j^n, & x < x_{j+\frac{1}{2}} \\ v_{j+1}^n, & x > x_{j+\frac{1}{2}} \end{cases}$$

Under the CFL condition

$$\lambda \max |a(u)| \le \frac{1}{2}$$

the solution at next time level is given by projecting the Riemann solution onto piecewise constant states

$$v_j^{n+1} = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} w_R(x/\Delta t; v_{j-1}^n, v_j^n) \mathrm{d}x + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 w_R(x/\Delta t; v_j^n, v_{j+1}^n) \mathrm{d}x$$

Approximate Riemann solver

Let $w(x/t; u_l, u_r)$ be an approximation of the exact entropy solution $w_R(x/t; u_l, u_r)$ of the Riemann problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

We will require that the approximate solution be consistent with the exact one in two respects

Conservation: Integrate over rectangle $(-\frac{\Delta x}{2}, +\frac{\Delta x}{2}) \times (0, \Delta t)$, and provided

$$\lambda |a(u)| \le \frac{1}{2}, \qquad \text{for all } u \text{ between } u_l \text{ and } u_r$$

we get

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} w_R(x/\Delta t; u_l, u_r) \mathrm{d}x = \frac{1}{2}(u_l + u_r) + \lambda(f(u_l) - f(u_r))$$

Thus we require the approximate solution to satisfy

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} w(x/\Delta t; u_l, u_r)\mathrm{d}x = \frac{1}{2}(u_l + u_r) + \lambda(f(u_l) - f(u_r))$$

Entropy condition: Integrating the entropy inequality $U_t + F_x \le 0$ yields

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} U(w_R(x/\Delta t; u_l, u_r))\mathrm{d}x \le \frac{1}{2}(U(u_l) + U(u_r)) + \lambda(F(u_l) - F(u_r))$$

For consistency with the entropy condition, we require approximate solution to satisfy

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} U(w(x/\Delta t; u_l, u_r))\mathrm{d}x \le \frac{1}{2}(U(u_l) + U(u_r)) + \lambda(F(u_l) - F(u_r))$$

Continuity: Finally, we require the solution to be continuous wrt the data

$$w(x/t; u, u) = u$$

Godunov-type scheme: With the help of such an approximate Riemann solution w, we define the Godunov-type scheme as follows

$$v_j^{n+1} = \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} w(x/\Delta t; v_{j-1}^n, v_j^n)\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 w(x/\Delta t; v_j^n, v_{j+1}^n)\mathrm{d}x$$

Theorem

Let w be the approximate Riemann solver which satisfies conservation, consistency with entropy condition for an entropy pair (U, F) and is continuous. Then the Godunov-type scheme can be put in conservation form, is consistent with the conservation law and is consistent with the entropy condition associated with (U, F) under the CFL condition CFL $\le \frac{1}{2}$.

Proof: (1) **Conservation form**: We write down a numerical flux inspired by the formulae in the proof of entropy condition

$$\begin{aligned} g_-(u,v) &= f(v) + \frac{1}{\lambda \Delta x}\int_0^{\frac{\Delta x}{2}} w(x/\Delta t; u, v)\mathrm{d}x - \frac{v}{2\lambda} \\ g_+(u,v) &= f(u) - \frac{1}{\lambda \Delta x}\int_{-\frac{\Delta x}{2}}^{0} w(x/\Delta t; u, v)\mathrm{d}x + \frac{u}{2\lambda} \end{aligned}$$

But by conservation

$$g_-(u,v) - g_+(u,v) = \frac{1}{\lambda \Delta x}\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} w(x/\Delta t; u, v)\mathrm{d}x - \frac{1}{2\lambda}(u+v) + f(v) - f(u) = 0$$

and using continuity condition

$$g_-(u,u) = g_+(u,u) = f(u)$$

Hence the Godunov-type scheme can be written in conservation form

$$v_j^{n+1} = v_j^n - \lambda\{g(v_j^n, v_{j+1}^n) - g(v_{j-1}^n, v_j^n)\}$$

with consistent numerical flux $g = g_- = g_+$.

(2) **Entropy consistency**: We define numerical entropy flux

$$\begin{aligned} G_-(u,v) &= F(v) + \frac{1}{\lambda \Delta x}\int_0^{\frac{\Delta x}{2}} U(w(x/\Delta t; u, v))\mathrm{d}x - \frac{U(v)}{2\lambda} \\ G_+(u,v) &= F(u) - \frac{1}{\lambda \Delta x}\int_{-\frac{\Delta x}{2}}^{0} U(w(x/\Delta t; u, v))\mathrm{d}x + \frac{U(u)}{2\lambda} \end{aligned}$$

We still have consistency

$$G_-(u,u) = G_+(u,u) = F(u)$$

and from entropy consistency of approximate Riemann solver

$$\begin{aligned}\lambda[G_-(u,v) - G_+(u,v)] =& \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} U(w(x/\Delta t; u,v))\mathrm{d}x \\ &- \frac{1}{2}[U(u) + U(v)] - \lambda[F(u) - F(v)] \le 0\end{aligned}$$

we can only conclude that

$$G_-(u,v) \le G_+(u,v)$$

But this is sufficient to show entropy condition using Jensen's inequality

$$
\begin{aligned}
U(v_j^{n+1}) &\le \frac{1}{2} U\left(\frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} w(x/\Delta t; v_{j-1}^n, v_j^n) \mathrm{d}x \right) + \\
&\quad \frac{1}{2} U\left(\frac{2}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 w(x/\Delta t; v_j^n, v_{j+1}^n) \mathrm{d}x \right) \\
&\le \frac{1}{2} \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} U(w(x/\Delta t; v_{j-1}^n, v_j^n)) \mathrm{d}x + \\
&\quad \frac{1}{2} \frac{2}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 U(w(x/\Delta t; v_j^n, v_{j+1}^n)) \mathrm{d}x \\
&= U(v_j^n) - \lambda\{G_+(v_j, v_{j+1}) - G_-(v_{j-1}, v_j)\} \\
&\le U(v_j^n) - \lambda\{G_+(v_j, v_{j+1}) - G_+(v_{j-1}, v_j)\}
\end{aligned}
$$

so that we obtain entropy consistency with numerical entropy flux

$$
G_{j+\frac{1}{2}} = G_+(v_j, v_{j+1})
$$

Roe scheme and its entropy modification

The Roe scheme is an approximate Riemann solver with

$$w(x/t; u_l, u_r) = \begin{cases} u_l, & x/t < a(u_l, u_r) \\ u_r, & x/t > a(u_l, u_r) \end{cases}$$

with numerical flux

$$g^R(u, v) = \frac{1}{2}(f(u) + f(v)) - \frac{1}{2}|a(u, v)|(v - u)$$

But we have seen that this admits entropy violating shocks. This solution has only shocks and hence there will be problem when the solution is a rarefaction. Harten and Hyman proposed the following approximate Riemann solver

$$w(x/t; u_l, u_r) = \begin{cases} u_l, & x/t < a_l \\ u^*, & a_l < x/t < a_r \\ u_r, & x/t > a_r \end{cases}$$

Here the intermediate state u^* and a_l, a_r are yet to be specified.

For a rarefaction wave, we define a_l and a_r to be approximations of the limit slopes of the rarefaction fan, whereas for a shock wave we set $a_l = a_r = a(u_l, u_r)$. Consistency with conservation condition yields

$$(a_r - a_l)u^* = a_r u_r - a_l u_l - (f(u_r) - f(u_l))$$

so that once a_l, a_r are specified, then u^* is uniquely determined by the above condition. We can also write

$$(a_r - a_l)u^* = a_r u_r - a_l u_l - a(u_l, u_r)(u_r - u_l)$$

or

$$(a_r - a_l)u^* = u_l(a(u_l, u_r) - a_l) + u_r(a_r - a(u_l, u_r))$$

In order to satisfy entropy condition, let us choose

$$a_l = a(u_l, u_r) - \delta, \qquad a_r = a(u_l, u_r) + \delta$$

where

$$\delta = \sup_{u \in (u_l, u_r)} \max\{0, a(u_l, u_r) - a(u_l, u), a(u, u_r) - a(u_l, u_r)\}$$

This choice of a_l, a_r yields the intermediate state

$$u^* = \frac{1}{2}(u_l + u_r)$$

Example: Burgers equation, $a(u_l, u_r) = \frac{1}{2}(u_l + u_r)$

$$\delta = \frac{1}{2}|u_l - u_r|$$

Note that $\delta \ge 0$ and $a_l \le a_r$. Also, we have that $\delta = 0$ if and only if

$$a(u, u_r) \le a(u_l, u_r) \le a(u_l, u), \qquad \text{for all } u \text{ between } u_l \text{ and } u_r$$

Also, $a_l = a_r = a(u_l, u_r)$ and there is no intermediate state; the solution is a shock and the above condition is the Oleneik entropy condition.

f is convex: We have $\delta = 0$ iff $u_l \ge u_r$ so that the solution is a shock. On the other hand if $\delta > 0$, i.e., if $u_l < u_r$, then the exact solution is a rarefaction fan limited by the characteristics with slopes $f'(u_l) = a(u_l, u_l)$ and $f'(u_r) = a(u_r, u_r)$. Thus the approximate Riemann solver satisfies

$$w(x/t; u_l, u_r) = u_l = w_R(x/t; u_l, u_r), \qquad x/t < a_l \le f'(u_l)$$
$$w(x/t; u_l, u_r) = u_r = w_R(x/t; u_l, u_r), \qquad x/t > a_r \ge f'(u_r)$$

f is non-convex: Suppose that $u_l < u_r$. Then for any $u \in [u_l, u_r]$

$$a_l = a(u_l, u_r) - \delta \le a(u_l, u)$$
$$a_r = a(u_l, u_r) + \delta \ge a(u, u_r)$$

The fan in the (x, t) plane bordered by the straight lines with slopes $s_1 = a(u_l, u_1)$ and $s_p = a(u_p, u_r)$ for some u_1, u_p lying between u_l and u_r. Hence this fan is contained in the sector bounded by the straight lines with slopes a_l and a_r. This implies that $w(x/t; u_l, u_r)$ is equal to w_R outside the interval $(a_l \Delta t, a_r \Delta t)$ just as in the convex case.

Under the condition CFL $\le \frac{1}{2}$

$$-\frac{1}{2} \le \lambda a_l \le \lambda a_r \le \frac{1}{2}$$

so that the interval $(a_l \Delta t, a_r \Delta t)$ is contained in $(-\frac{\Delta x}{2}, +\frac{\Delta x}{2})$. By conservation

$$\int_{-\frac{\Delta x}{2}}^{+\frac{\Delta x}{2}} w(x/\Delta t; u_l, u_r) \mathrm{d}x = \int_{a_l \Delta t}^{a_r \Delta t} w_R(x/\Delta t; u_l, u_r) \mathrm{d}x$$

and hence

$$u^* \cdot (a_r - a_l)\Delta t = \int_{a_l \Delta t}^{a_r \Delta t} w(x/\Delta t; u_l, u_r) \mathrm{d}x = \int_{a_l \Delta t}^{a_r \Delta t} w_R(x/\Delta t; u_l, u_r) \mathrm{d}x$$

By Jensen's inequality

$$U(u^*) \le \frac{1}{(a_r - a_l)\Delta t} \int_{a_l \Delta t}^{a_r \Delta t} U(w_R(x/\Delta t; u_l, u_r)) \mathrm{d}x$$

Since

$$w(x/\Delta t; u_l, u_r) = u^*, \qquad \text{for } a_l \le x/t \le a_r$$

we obtain

$$\begin{aligned} \int_{a_l \Delta t}^{a_r \Delta t} U(w(x/\Delta t; u_l, u_r)) \mathrm{d}x &= U(u^*)(a_r - a_l)\Delta t \\ &\le \int_{a_l \Delta t}^{a_r \Delta t} U(w_R(x/\Delta t; u_l, u_r)) \mathrm{d}x \end{aligned}$$

But since w and w_R agree outside the interval $(a_l \Delta t, a_r \Delta t)$, we get

$$\int_{-\frac{1}{2}\Delta x}^{+\frac{1}{2}\Delta x} U(w(x/\Delta t; u_l, u_r))\mathrm{d}x \le \int_{-\frac{1}{2}\Delta x}^{+\frac{1}{2}\Delta x} U(w_R(x/\Delta t; u_l, u_r))\mathrm{d}x$$

which implies that w satisfies the entropy consistency condition.

Numerical flux: The numerical flux can be evaluated using the formulas for g_+ or g_-. An easy computation shows that the scheme satisfies the *upwind property*

$$g(u_l, u_r) = \begin{cases} f(u_l), & a_l \ge 0, \quad \text{i.e.,} \quad a(u_l, u_r) \ge +\delta \\ f(u_r), & a_r \le 0, \quad \text{i.e.,} \quad a(u_l, u_r) \le -\delta \end{cases}$$
$$= \frac{1}{2}(f(u_l) + f(u_r)) - \frac{1}{2}|a(u_l, u_r)|(u_r - u_l), \quad |a(u_l, u_r)| \ge \delta$$

Hence it coincides with the original Roe scheme in the domains where $|a(u_l, u_r)| \ge \delta$. In the case $|a(u_l, u_r)| < \delta$, the scheme is modified to have non-zero dissipation as we can see below. Using the expression for g_+

$$\begin{aligned} g(u_l, u_r) &= f(u_l) - \frac{1}{\Delta t}\int_{-\frac{1}{2}\Delta x}^{0} w(x/\Delta t; u_l, u_r)\mathrm{d}x + \frac{u_l}{2\lambda} \\ &= f(u_l) - \frac{1}{\Delta t}\int_{-\frac{1}{2}\Delta x}^{a_l \Delta t} u_l \mathrm{d}x - \frac{1}{\Delta t}\int_{a_l \Delta t}^{0} u^* \mathrm{d}x + \frac{u_l}{2\lambda} \\ &= f(u_l) + \frac{1}{2}a_l(u_r - u_l), \qquad \text{since} \quad u^* = \frac{1}{2}(u_l + u_r) \\ &= f(u_l) + \frac{1}{2}(a(u_l, u_r) - \delta)(u_r - u_l) \end{aligned}$$

This yields the flux

$$g(u_l, u_r) = \frac{1}{2}[f(u_l) + f(u_r)] - \frac{1}{2}\delta(u_r - u_l), \quad \text{if} \quad |a(u_l, u_r)| < \delta$$

Viscosity form: Let us define

$$\epsilon = \lambda\delta$$

The numerical viscosity of the modified scheme is given by

$$Q_{j+\frac{1}{2}} = \begin{cases} \lambda|a(v_j, v_{j+1})|, & \text{if} \quad \lambda|a(v_j, v_{j+1})| \geq \epsilon \\ \epsilon, & \text{if} \quad \lambda|a(v_j, v_{j+1})| < \epsilon \end{cases}$$

We can write this as $Q_{j+\frac{1}{2}} = Q(\lambda a(v_j, v_{j+1}))$ with

$$Q(x) = \begin{cases} |x|, & |x| \geq \epsilon \\ \epsilon, & |x| < \epsilon \end{cases}$$

and hence due to a proposition seen before, the scheme is also TVD.

Remark: We have shown that the modified Roe scheme satisfies the entropy condition. It is also possible to show that it is an E-scheme, which in turn implies the entropy condition.