# Finite volume method for conservation laws IX

## Second order extension of Godunov scheme

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February 28, 2013

# Basic idea

**Recall the Godunov method**: Finite volume solution is piecewise constant, and defines a Riemann problem at each cell boundary $x_{j+\frac{1}{2}}$. We solve the Riemann problem exactly or approximately, and then advance the solution to next time level. To increase the accuracy of this scheme, we should use a solution representation which is better than the piecewise constant representation.

Let $v_j^n$ denote the cell average value inside cell $j$ at time level $n$. The solution at the next time level $n+1$ is obtained in three steps:

1. **reconstruction**: construct a piecewise polynomial function $\tilde{v}$ from the given cell averages $v_j^n$
2. **evolution**: exact or approximate Riemann solver to advance the solution by a small time-step $\Delta t$
3. **projection**: onto piecewise constant cell-averages $v_j^{n+1}$

**Remark**: We do not impose any continuity requirement on the piecewise polynomial functions. This helps us to represent discontinuous solutions.

# Van Leer method

**Step 1**: Construct a piecewise polynomial of degree one in side each cell

$$\tilde{v}(x, t_n) = v_j^n + \frac{x - x_j}{\Delta x}\delta_j^n, \qquad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \tag{1}$$

where $\delta_j^n \approx \Delta x \frac{\partial v}{\partial x}(x_j, t_n)$. Note that for any value of $\delta_j^n$ we have

$$\tilde{v}(x_j, t_n) = v_j^n = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{v}(x, t_n)\mathrm{d}x$$

We will approximate $\delta_j$ by some finite difference formula.
**Step 2**: We solve (maybe approximately) the following generalized Riemann problem

$$\begin{aligned} &\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}f(w) = 0, \qquad t \in (0, \Delta t) \\ &w(x, 0) = \tilde{v}(x, t_n) \end{aligned}$$

**Step 3**: By averaging $w(x, \Delta t)$ as in Godunov method, we get $v_j^{n+1}$

$$v_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x, \Delta t)\mathrm{d}x$$

# Finite volume form

We integrate the GRP problem over the space-time volume $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times (0, \Delta t)$

$$\int_0^{\Delta t} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left( \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) \right) \mathrm{d}x \mathrm{d}t = 0$$

which leads to

$$(v_j^{n+1} - v_j^n)\Delta x + \int_0^{\Delta t} [f(w(x_{j+\frac{1}{2}}, t)) - f(w(x_{j-\frac{1}{2}}, t))]\mathrm{d}t = 0$$

We can put this in finite volume form

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda(g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n) \\ g_{j+\frac{1}{2}}^n &= \frac{1}{\Delta t} \int_0^{\Delta t} f(w(x_{j+\frac{1}{2}}, t))\mathrm{d}t \end{aligned} \tag{2}$$

# Computation of $\delta_j$

We have several choices for $\delta_j$

$$\Delta v_{j-\frac{1}{2}} = v_j - v_{j-1}, \qquad \hat{\delta}_j = \frac{1}{2}(v_{j+1} - v_{j-1}), \qquad \Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

We want the reconstructed function $\tilde{v}$ to be non-oscillatory. Hence we can choose the smallest slope among the available choices

$$\delta_j = \begin{cases} s \min\{|\Delta v_{j-\frac{1}{2}}|, |\hat{\delta}_j|, |\Delta v_{j+\frac{1}{2}}|\}, & s = \text{sign } \Delta v_{j-\frac{1}{2}} = \text{sign } \Delta v_{j+\frac{1}{2}} = \text{sign } \hat{\delta}_j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

But this is same as

$$\delta_j = \begin{cases} s \min\{|\Delta v_{j-\frac{1}{2}}|, |\Delta v_{j+\frac{1}{2}}|\}, & s = \text{sign } \Delta v_{j-\frac{1}{2}} = \text{sign } \Delta v_{j+\frac{1}{2}} \\ 0, & \text{otherwise} \end{cases}$$

This is called the *minmod* function, so that

$$\delta_j = \text{minmod}(\Delta v_{j-\frac{1}{2}}, \Delta v_{j+\frac{1}{2}})$$

Look at some examples of shock and local extrema.

## Lemma

Let $\tilde{v}(x, t_n) = v_j^n + \frac{x - x_j}{\Delta x}\delta_j^n$ with $\delta_j$ given by (3). Then

$$TV(\tilde{v}) = TV(v)$$

Proof: We show that the jumps in $\tilde{v}$ have same as sign as jumps in $(v_j)$. Assume that

$$v_{j-1} \le v_j \le v_{j+1}$$

Now

$$\tilde{v}_{j-1}(x_{j-\frac{1}{2}}) = v_{j-1} + \frac{1}{2}\delta_{j-1}, \qquad \tilde{v}_j(x_{j-\frac{1}{2}}) = v_j - \frac{1}{2}\delta_j$$

Then

$$\tilde{v}_j(x_{j-\frac{1}{2}}) - \tilde{v}_{j-1}(x_{j-\frac{1}{2}}) = \frac{1}{2}(\Delta v_{j-\frac{1}{2}} - \delta_{j-1}) + \frac{1}{2}(\Delta v_{j-\frac{1}{2}} - \delta_j)$$

Since

$$\Delta v_{j-\frac{1}{2}} \ge 0 \quad \Longrightarrow \quad 0 \le \delta_{j-1} \le \Delta v_{j-\frac{1}{2}}, \qquad 0 \le \delta_j \le \Delta v_{j-\frac{1}{2}}$$

Hence $\tilde{v}_j(x_{j-\frac{1}{2}}) - \tilde{v}_{j-1}(x_{j-\frac{1}{2}}) \ge 0$. If $v_j$ is a local extremum, then $\delta_j = 0$.

# Linear case

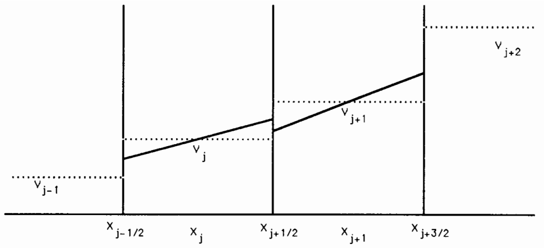
The limiter (3) seems too restrictive in practice, and can be relaxed to the following

$$\delta_j = \begin{cases} s \min\{2|\Delta v_{j-\frac{1}{2}}|, |\hat{\delta}_j|, 2|\Delta v_{j+\frac{1}{2}}|\}, & s = \text{sign } \Delta v_{j-\frac{1}{2}} = \text{sign } \Delta v_{j+\frac{1}{2}} = \text{sign } \hat{\delta}_j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

The above slope ensures that

$$\min(v_{j-1}, v_j, v_{j+1}) \leq \tilde{v}_j(x) \leq \max(v_{j-1}, v_j, v_{j+1}), \qquad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

In this case, we cannot show that $TV(\tilde{v}) = TV(v)$ but for a linear PDE, we can still show the TVD property of the scheme.



# Linear case

Let us consider the linear PDE

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \qquad f(u) = au, \qquad a > 0$$

Then the solution of GRP at $x = x_{j+\frac{1}{2}}$ is given by

$$\begin{aligned} w(x_{j+\frac{1}{2}}, t) &= \tilde{v}_j(x_{j+\frac{1}{2}} - a(t - t_n), t_n) \\ &= v_j^n + \frac{x_{j+\frac{1}{2}} - a(t-t_n) - x_j}{\Delta x} \delta_j^n \\ &= v_j^n + \frac{1}{2}\delta_j^n - \frac{a(t-t_n)}{\Delta x}\delta_j^n \end{aligned}$$

Since this is linear in $t$, the numerical flux is given by

$$g_{j+\frac{1}{2}}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f(w(x_{j+\frac{1}{2}}, t))\mathrm{d}t = f(w(x_{j+\frac{1}{2}}, \frac{1}{2}\Delta t)) = a v_j^n + \frac{1}{2}a(1-\nu)\delta_j^n$$

The scheme can also be written as

$$v_j^{n+1} = v_j^n - \nu \Delta v_{j-\frac{1}{2}}^n - \frac{1}{2}\nu(1-\nu)(\delta_j^n - \delta_{j-1}^n) \tag{5}$$

## Proposition

The scheme (5) where $\delta_j$ is given by (4) is TVD and $L^\infty$ stable under the CFL condition $0 < \nu \le 1$.

Proof: We write the scheme in incremental form with

$$C_{j+\frac{1}{2}} = 0, \qquad D_{j-\frac{1}{2}} = \nu + \nu(1-\nu)\frac{(\delta_j - \delta_{j-1})}{2\Delta v_{j-\frac{1}{2}}}$$

We have to check that $0 \le D_{j-\frac{1}{2}} \le 1$. Consider the case

$$\delta_j > 0 \quad \Longrightarrow \quad \Delta v_{j-\frac{1}{2}} > 0 \quad \Longrightarrow \quad \delta_{j-1} \ge 0$$

Then

$$\delta_{j-1} \le 2\Delta v_{j-\frac{1}{2}} \quad \Longrightarrow \quad \frac{\delta_j - \delta_{j-1}}{2\Delta v_{j-\frac{1}{2}}} \ge \frac{\delta_j}{2\Delta v_{j-\frac{1}{2}}} - 1 > -1$$

which yields the lower bound

$$D_{j-\frac{1}{2}} > \nu - \nu(1-\nu) = \nu^2 > 0$$

Also, using $\delta_{j-1} \ge 0$ and $\delta_j \le 2\Delta v_{j-\frac{1}{2}}$, we get

$$D_{j-\frac{1}{2}} \le \nu + \nu(1-\nu)\frac{\delta_j}{2\Delta v_{j-\frac{1}{2}}} \le 2\nu - \nu^2 = 1 - (1-\nu)^2 < 1$$

## Theorem (General case)

Assume $u_0 \in \mathrm{BV}(\mathbb{R})$ and define

$$v_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) \mathrm{d}x$$

Then the scheme given by (1), (2), (3) is TVD and $L^\infty$ stable provided $\mathrm{CFL} \leq \frac{1}{2}$.

Proof:

$$\begin{aligned} \mathrm{TV}(v^{n+1}) &= \sum_j |\Delta v^{n+1}_{j+\frac{1}{2}}| \\ &= \sum_j \frac{1}{\Delta x} \left| \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (w(x+\Delta x, \Delta t) - w(x, \Delta t)) \mathrm{d}x \right| \\ &\leq \frac{1}{\Delta x} \int_{\mathbb{R}} |w(x+\Delta x, \Delta t) - w(x, \Delta t)| \mathrm{d}x \leq \mathrm{TV}(w(\cdot, \Delta t)) \end{aligned}$$

But the exact solution operator of the conservation law is TVD so that

$$\mathrm{TV}(w(\cdot,\Delta t)) \leq \mathrm{TV}(\tilde{v}(\cdot,t_n)) = \sum_j |\Delta v^n_{j+\frac{1}{2}}|$$

By induction, we obtain the result $\mathrm{TV}(v^n) \leq \mathrm{TV}(v^0) \leq \mathrm{TV}(u_0)$. Stability in maximum norm follows from the stability of the exact evolution operator, and the stability of the reconstruction scheme.

$$|v^{n+1}_j| = \left|\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x,\Delta t)\mathrm{d}t\right| \leq \max_{[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]} |w(x,\Delta t)|$$

Hence

$$\max_j |v^{n+1}_j| \leq \|w(\cdot,\Delta t)\|_{L^\infty(\mathbb{R})} \leq \|\tilde{v}(\cdot,t_n)\|_{L^\infty(\mathbb{R})} = \max_j |v^n_j|$$

# FV scheme for non-linear case

The numerical flux is given by

$$g_{j+\frac{1}{2}}^n = \frac{1}{\Delta t} \int_0^{\Delta t} f(w(x_{j+\frac{1}{2}}, t)) \mathrm{d}t$$

It is not possible to evaluate this integral exactly since the solution of the GRP is complicated. We can retain second order accuracy if we approximate the time integral to second order accuracy. This can be achieved by the mid-point rule of integration since

$$g_{j+\frac{1}{2}}^n = f(w(x_{j+\frac{1}{2}}, \frac{1}{2}\Delta t)) + \mathcal{O}\left(\Delta t\right)^2$$

Hancock proposed a two-step procedure to approximate $w(x_{j+\frac{1}{2}}, \frac{1}{2}\Delta t)$. First obtain the states at the cell face by

$$v_{j+\frac{1}{2},-}^n = \tilde{v}_j(x_{j+\frac{1}{2}}) = v_j^n + \frac{1}{2}\delta_j^n$$
$$v_{j+\frac{1}{2},+}^n = \tilde{v}_{j+1}(x_{j+\frac{1}{2}}) = v_{j+1} - \frac{1}{2}\delta_{j+1}^n$$

# FV scheme for non-linear case

Then update these values to the mid-point in time by

$$v^{n+\frac{1}{2}}_{j+\frac{1}{2},-} = v^{n}_{j+\frac{1}{2},-} - \frac{\lambda}{2}(f(v_{j+\frac{1}{2},-}) - f(v_{j-\frac{1}{2},+}))$$
$$v^{n+\frac{1}{2}}_{j+\frac{1}{2},+} = v^{n}_{j+\frac{1}{2},+} - \frac{\lambda}{2}(f(v_{j+\frac{3}{2},-}) - f(v_{j+\frac{1}{2},+}))$$

The above intermediate values are an approximation to $w(x_{j+\frac{1}{2}}, \frac{1}{2}\Delta t)$

$$v^{n+\frac{1}{2}}_{j+\frac{1}{2},\pm} \approx w(x_{j+\frac{1}{2}} \pm 0, \frac{1}{2}\Delta t)$$

Then we solve the Riemann problem with piecewise constant data

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0$$
$$w(x,0) = \begin{cases} v^{n+\frac{1}{2}}_{j+\frac{1}{2},-}, & x < x_{j+\frac{1}{2}} \\ v^{n+\frac{1}{2}}_{j+\frac{1}{2},+}, & x > x_{j+\frac{1}{2}} \end{cases}$$

# FV scheme for non-linear case

Let $w_R$ denote the solution of the Riemann problem. Then the flux is evaluated as

$$g^n_{j+\frac{1}{2}} = f(w_R(0; v^{n+\frac{1}{2}}_{j+\frac{1}{2},-}, v^{n+\frac{1}{2}}_{j+\frac{1}{2},+}))$$

or some using some approximate Riemann solver.

**Remark**: The above scheme when applied to the linear advection equation leads to the scheme (5).

**Remark**: In the methods we have considered, the time and space discretization is combined to be second order accurate. This is not easy to extend to higher orders of accuracy. It is more useful to adopt the ***Method of Lines*** approach, which decouples the space and time discretizations. The space discretization is made high order accurate by some reconstruction procedure. For time discretization, we use high order accurate Runge-Kutta schemes.

# Method of lines

Integrate over space only

$$\frac{\mathrm{d}v_j}{\mathrm{d}t} + \frac{g_{j+\frac{1}{2}}(t) - g_{j-\frac{1}{2}}(t)}{\Delta x} = 0$$

where

$$g_{j+\frac{1}{2}}(t) = g(v^-_{j+\frac{1}{2}}(t), v^+_{j+\frac{1}{2}}(t))$$

The two states are obtained by some piecewise polynomial reconstruction $\tilde{v}$,

$$v^-_{j+\frac{1}{2}}(t) = \tilde{v}_j(x_{j+\frac{1}{2}}, t), \qquad v^+_{j+\frac{1}{2}}(t) = \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, t)$$

The space discretization is second order accurate. We need atleast a second order discretization in time so that the overall scheme is second order accurate. Let us write the system of ODE as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = L(v)$$

Let us also assume that the first order in time discretization is *stable*, i.e.,

$$\Delta t \le \Delta t_1 \quad \implies \quad \|v + \Delta t L(v)\| \le \|v\|$$

# First order time integration scheme

First order scheme: Forward Euler

$$v^{n+1} = v^n + \Delta t L(v^n)$$

Let us apply this scheme to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$$

which yields

$$v^{n+1} = [1 + \lambda \Delta t] v^n$$

while the exact solution is

$$u^{n+1} = \mathrm{e}^{\lambda \Delta t} v^n = [1 + \lambda \Delta t + \mathcal{O}\left(\Delta t\right)^2] v^n$$

The numerical scheme agrees with the exact solution upto $\mathcal{O}\left(\Delta t\right)$.

# Second order time integration scheme

Second order scheme (2-stage)

$$
\begin{aligned}
v^{(0)} &= v^n \\
v^{(1)} &= v^{(0)} + \Delta t L(v^{(0)}) \\
v^{(2)} &= \frac{1}{2} v^{(0)} + \frac{1}{2}[v^{(1)} + \Delta t L(v^{(1)})] \\
v^{n+1} &= v^{(2)}
\end{aligned}
$$

Applying this scheme to the ODE $\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$ yields

$$
v^{n+1} = [1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2] v^n
$$

which agrees with the exact solution upto $\mathcal{O}\left(\Delta t\right)^2$. Since the scheme is a convex combination, we obtain stability for $\Delta t \leq \Delta t_1$

$$
\left\| v^{n+1} \right\| \leq \frac{1}{2} \left\| v^{(0)} \right\| + \frac{1}{2} \left\| v^{(1)} + \Delta t L(v^{(1)}) \right\| \leq \frac{1}{2} \left\| v^n \right\| + \frac{1}{2} \left\| v^{(1)} \right\| \leq \left\| v^n \right\|
$$

Such time integration schemes are known as *Strong Stability Preserving RK* schemes.

# SSP RK schemes

Third order scheme (3-stage)

$$
\begin{aligned}
v^{(0)} &= v^n \\
v^{(1)} &= v^{(0)} + \Delta t L(v^{(0)}) \\
v^{(2)} &= \frac{3}{4} v^{(0)} + \frac{1}{4}[v^{(1)} + \Delta t L(v^{(1)})] \\
v^{(3)} &= \frac{1}{3} v^{(0)} + \frac{2}{3}[v^{(2)} + \Delta t L(v^{(2)})] \\
v^{n+1} &= v^{(3)}
\end{aligned}
$$

Applying this scheme to the ODE $\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$ yields

$$
v^{n+1} = [1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \frac{1}{6}(\lambda\Delta t)^3] v^n
$$

# SSP RK schemes

A general $m$-stage RK scheme is of the form

$$\begin{aligned} v^{(0)} &= v^n \\ v^{(i)} &= \sum_{k=0}^{i-1} \left[ \alpha_{ik} v^{(k)} + \Delta t \beta_{ik} L(v^{(k)}) \right], \quad i = 1, \ldots, m \\ v^{n+1} &= v^{(m)} \end{aligned}$$

By consistency (take $L(v) \equiv 0$), we must have

$$\sum_{k=0}^{i-1} \alpha_{ik} = 1, \qquad i = 1, \ldots, m$$

# SSP RK schemes

**Lemma (Stability of SSP RK scheme)**

$$\text{If} \qquad \Delta t \le \Delta t_1 \quad \Longrightarrow \quad \|v + \Delta t L(v)\| \le \|v\|$$

then the $m$-stage RK scheme is stable under CFL condition

$$\Delta t \le c \Delta t_1, \qquad c = \min_{i,k} \frac{\alpha_{ik}}{\beta_{ik}}$$

provided that $\alpha_{ik} \ge 0$, $\beta_{ik} \ge 0$.

Proof: We write the RK scheme as a convex combination

$$v^{(i)} = \sum_{k=0}^{i-1} \alpha_{ik} \left[ v^{(k)} + \Delta t \frac{\beta_{ik}}{\alpha_{ik}} L(v^{(k)}) \right], \quad i = 1, \ldots, m$$

and

$$\left\| v^{(k)} + \Delta t \frac{\beta_{ik}}{\alpha_{ik}} L(v^{(k)}) \right\| \le \left\| v^{(k)} \right\| \qquad \text{if} \qquad \Delta t \frac{\beta_{ik}}{\alpha_{ik}} \le \Delta t_1$$

# MUSCL scheme

From Taylor formula

$$v(x) = v(x_j) + (x - x_j)v_x(x_j) + \frac{1}{2}(x - x_j)^2 v_{xx}(x_j) + \mathcal{O}\left(\Delta x^3\right)$$

But $v(x_j) \neq v_j$ while we want conservation, so ignoring terms $\mathcal{O}\left(\Delta x\right)^3$ and above

$$\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(x)\mathrm{d}x = v_j \quad \Longrightarrow \quad v(x_j) = v_j - \frac{\Delta x^2}{24} v_{xx}(x_j)$$

Hence

$$v(x) = v_j + (x - x_j)v_x(x_j) + \frac{1}{2}\left[(x - x_j)^2 - \frac{\Delta x^2}{12}\right] v_{xx}(x_j) + \mathcal{O}\left(\Delta x^3\right)$$

Degree two polynomial in cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$\tilde{v}_j(x) = v_j + (x - x_j)\frac{v_{j+1} - v_{j-1}}{2\Delta x} + \frac{3\kappa}{2}\left[(x - x_j)^2 - \frac{\Delta x^2}{12}\right]\frac{v_{j-1} - 2v_j + v_{j+1}}{\Delta x^2}$$

# MUSCL scheme

where we have introduced a parameter $\kappa$. If $\kappa = \frac{1}{3}$ then we obtain third order accuracy in the reconstruction. Using this approximation we can get the states at the cell faces

$$v^+_{j-\frac{1}{2}} = v_j - \frac{1}{4}[(1+\kappa)\Delta v_{j-\frac{1}{2}} + (1-\kappa)\Delta v_{j+\frac{1}{2}}]$$
$$v^-_{j+\frac{1}{2}} = v_j + \frac{1}{4}[(1-\kappa)\Delta v_{j-\frac{1}{2}} + (1+\kappa)\Delta v_{j+\frac{1}{2}}]$$

which can be used to compute the flux

$$g_{j+\frac{1}{2}} = g(v^-_{j+\frac{1}{2}}, v^+_{j+\frac{1}{2}})$$

In order to make the scheme TVD we limit the reconstructed states $v^\pm_{j+\frac{1}{2}}$. We first write

$$v^+_{j-\frac{1}{2}} = v_j - \frac{1}{4}[(1+\kappa)\frac{\Delta v_{j-\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\Delta v_{j+\frac{1}{2}} + (1-\kappa)\frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}\Delta v_{j-\frac{1}{2}}]$$
$$v^-_{j+\frac{1}{2}} = v_j + \frac{1}{4}[(1-\kappa)\frac{\Delta v_{j-\frac{1}{2}}}{\Delta v_{j+\frac{1}{2}}}\Delta v_{j+\frac{1}{2}} + (1+\kappa)\frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}\Delta v_{j-\frac{1}{2}}]$$

# MUSCL scheme

Let us introduce the parameter

$$R_j = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

to measure the local smoothness of the function. Then we introduce a limiter function $\psi$ into the reconstruction scheme

$$\begin{aligned} v^{+}_{j-\frac{1}{2}} &= v_j - \frac{1}{4}[(1+\kappa)\psi(1/R_j)\Delta v_{j+\frac{1}{2}} + (1-\kappa)\psi(R_j)\Delta v_{j-\frac{1}{2}}] \\ v^{-}_{j+\frac{1}{2}} &= v_j + \frac{1}{4}[(1-\kappa)\psi(1/R_j)\Delta v_{j+\frac{1}{2}} + (1+\kappa)\psi(R_j)\Delta v_{j-\frac{1}{2}}] \end{aligned} \tag{6}$$

In smooth regions we expect $R_j \approx 1$ and we should have $\psi(R_j) \approx R_j$. In particular, we need $\psi(1) = 1$ in order to obtain second order accuracy in smooth regions.

## Theorem

The finite volume scheme with monotone Lipschitz continuous numerical flux

$$v_j^{n+1} = v_j^n - \lambda(g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n), \qquad g_{j+\frac{1}{2}} = g(v_{j+\frac{1}{2}}^-, v_{j+\frac{1}{2}}^+)$$

where the states $v_{j+\frac{1}{2}}^{\pm}$ are obtained by the $\kappa$ parameter MUSCL scheme (6) is TVD if $\psi$ satisfies

$$0 \le \psi(R) \le \frac{3-\kappa}{1-\kappa} - (1+\alpha)\frac{1+\kappa}{1-\kappa}, \qquad 0 \le \frac{\psi(R)}{R} \le 2+\alpha$$

where $\alpha \in [-2, 2(1-\kappa)/(1+\kappa)]$ under the CFL time step restriction

$$\lambda\frac{(2-(2+\alpha)\kappa)}{1-\kappa}C_j \le 1, \qquad C_j = \max_{u,v}\left|\frac{\partial g}{\partial u}(u, v_{j+\frac{1}{2}}^+) - \frac{\partial g}{\partial v}(v_{j-\frac{1}{2}}^-, v)\right|$$

where the maximum is taken over all $u$ between $v_{j-\frac{1}{2}}^-$, $v_{j+\frac{1}{2}}^-$ and all $v$ between $v_{j-\frac{1}{2}}^+$, $v_{j+\frac{1}{2}}^+$.

Proof:

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda[g(v_{j+\frac{1}{2}}^-, v_{j+\frac{1}{2}}^+) - g(v_{j-\frac{1}{2}}^-, v_{j-\frac{1}{2}}^+)] \\ &= v_j^n - \lambda[g(v_{j+\frac{1}{2}}^-, v_{j+\frac{1}{2}}^+) - g(v_{j-\frac{1}{2}}^-, v_{j+\frac{1}{2}}^+) \\ &\qquad + g(v_{j-\frac{1}{2}}^-, v_{j+\frac{1}{2}}^+) - g(v_{j-\frac{1}{2}}^-, v_{j-\frac{1}{2}}^+)] \end{aligned}$$

For some $\tilde{u}$ between $v_{j-\frac{1}{2}}^-$, $v_{j+\frac{1}{2}}^-$ and for some $\tilde{v}$ between $v_{j-\frac{1}{2}}^+$, $v_{j+\frac{1}{2}}^+$, we can perform a mean-value linearization of the numerical flux and write

$$v_j^{n+1} = v_j^n - \lambda \underbrace{\frac{\partial g}{\partial u}(\tilde{u}, v_{j+\frac{1}{2}}^+)}_{\geq 0}(v_{j+\frac{1}{2}}^- - v_{j-\frac{1}{2}}^-) - \lambda \underbrace{\frac{\partial g}{\partial v}(v_{j-\frac{1}{2}}^-, \tilde{v})}_{\leq 0}(v_{j+\frac{1}{2}}^+ - v_{j-\frac{1}{2}}^+)$$

We have

$$v_{j+\frac{1}{2}}^- - v_{j-\frac{1}{2}}^- = \left[1 + \frac{1+\kappa}{4}\left(\psi(R_j) - \frac{\psi(R_{j-1})}{R_{j-1}}\right) + \frac{1-\kappa}{4}\left(\psi\left(\frac{1}{R_j}\right)R_j - \psi\left(\frac{1}{R_{j-1}}\right)\right)\right] \times \Delta v_{j-\frac{1}{2}}$$

and

$$v^+_{j+\frac{1}{2}} - v^+_{j-\frac{1}{2}} = \left[1 - \frac{1+\kappa}{4}\left(\psi\left(\frac{1}{R_{j+1}}\right)R_{j+1} - \psi\left(\frac{1}{R_j}\right)\right) - \frac{1-\kappa}{4}\left(\psi(R_{j+1}) - \frac{\psi(R_j)}{R_j}\right)\right] \times \Delta v_{j+\frac{1}{2}}$$

Under the conditions on $\psi$ given in the theorem, the terms inside the square brackets in the last two equations can be shown to be positive.

$$v_j^{n+1} = v_j^n + C^n_{j+\frac{1}{2}}\Delta v^n_{j+\frac{1}{2}} - D^n_{j-\frac{1}{2}}\Delta v^n_{j-\frac{1}{2}}, \quad C_{j+\frac{1}{2}} \geq 0, \quad D_{j-\frac{1}{2}} \geq 0$$

Using the TVD condition of the incremental form, this implies TVD property of the MUSCL scheme. The condition $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$ gives the CFL condition.

# Minmod limiter

The limiter function

$$\psi_{MM}(R) = \max(0, \min(R, \beta)), \qquad \beta \in [1, (3-\kappa)/(1-\kappa)]$$

satisfies the conditions of the theorem. Let us choose $\kappa = -1$ and $\beta = 1$. Then

$$v^-_{j+\frac{1}{2}} = v_j + \frac{1}{2}\psi_{MM}(1/R_j)\Delta v_{j+\frac{1}{2}}, \qquad R_j = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

If $0 < \Delta v_{j-\frac{1}{2}} < \Delta v_{j+\frac{1}{2}}$, then $\frac{1}{R_j} < 1$ and $\psi_{MM}(\frac{1}{R_j}) = \frac{1}{R_j}$ so that

$$v^-_{j+\frac{1}{2}} = v_j + \frac{1}{2}(1/R_j)\Delta v_{j+\frac{1}{2}} = v_j + \frac{1}{2}\Delta v_{j-\frac{1}{2}}$$

If $\Delta v_{j-\frac{1}{2}} > \Delta v_{j+\frac{1}{2}} > 0$, then $\frac{1}{R_j} > 1$ and $\psi_{MM}(\frac{1}{R_j}) = 1$ so that

$$v^-_{j+\frac{1}{2}} = v_j + \frac{1}{2}\Delta v_{j+\frac{1}{2}}$$

The scheme picks the smaller of the two slopes $\Delta v_{j-\frac{1}{2}}$, $\Delta v_{j+\frac{1}{2}}$ and is identical to the minmod scheme which we have seen before.