Finite difference method for 2-D heat equation

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Heat equation

- Partial differential equation in $\Omega = (0,1) \times (0,1)$, $\mu > 0$

$$\begin{aligned} u_t &= \mu(u_{xx} + u_{yy}), \quad (x,y) \in \Omega, \quad t > 0 \\ u(x,y,0) &= f(x,y), \quad (x,y) \in \Omega \\ u(x,y,t) &= g(x,y,t), \qquad (x,y) \in \Omega, \quad t > 0 \end{aligned}$$

- Space mesh of $(M_x + 1) \times (M_y + 1)$ points in Ω

$$\Delta x = \frac{1}{M_x}, \qquad x_j = j\Delta x, \qquad j = 0, 1, \ldots, M_x$$

$$\Delta y = \frac{1}{M_y}, \qquad y_k = k\Delta y, \qquad k = 0, 1, \ldots, M_y$$

FTCS scheme

$$D_t^+ u_{j,k}^n = \mu(D_x^+ D_x^- u_{j,k}^n + D_y^+ D_y^- u_{j,k}^n)$$

$$D_t^+ u_{j,k}^n = \frac{u_{j,k}^{n+1} - u_{j,k}^n}{\Delta t}, \quad D_x^+ D_x^- u_{j,k}^n = \frac{u_{j-1,k}^n - 2u_{j,k}^n + u_{j+1,k}^n}{\Delta x^2}$$

$$D_y^+ D_y^- u_{j,k}^n = \frac{u_{j,k-1}^n - 2u_{j,k}^n + u_{j,k+1}^n}{\Delta y^2}$$

Define

$$r_x = \frac{\mu \Delta t}{\Delta x^2}, \qquad r_y = \frac{\mu \Delta t}{\Delta y^2}$$

Define undivided finite differences

$$\delta_x^2 u_{j,k}^n = u_{j-1,k}^n - 2u_{j,k}^n + u_{j+1,k}^n, \qquad \delta_y^2 u_{j,k}^n = u_{j,k-1}^n - 2u_{j,k}^n + u_{j,k+1}^n$$

FTCS scheme

$$u_{j,k}^{n+1} = u_{j,k}^n + r_x \delta_x^2 u_{j,k}^n + r_y \delta_y^2 u_{j,k}^n$$

or

$$\boxed{u_{j,k}^{n+1} = (1 + r_x \delta_x^2 + r_y \delta_y^2) u_{j,k}^n}$$

FTCS scheme: maximum stability

$$u_{j,k}^{n+1} = (1 - 2r_x - 2r_y)u_{j,k}^n + r_x(u_{j-1,k}^n + u_{j+1,k}^n) + r_y(u_{j,k-1}^n + u_{j,k+1}^n)$$

If

$$r_x + r_y \le \frac{1}{2}$$

then

$$\min(u_{j\pm1,k}^n, u_{j,k\pm1}^n, u_{j,k}^n) \le u_{j,k}^{n+1} \le \max(u_{j\pm1,k}^n, u_{j,k\pm1}^n, u_{j,k}^n)$$

so that the scheme is stable in maximum norm. If $\Delta x = \Delta y = h$, then

$$\frac{2\mu\Delta t}{h^2} \le \frac{1}{2} \quad \Longrightarrow \quad \Delta t \le \frac{h^2}{4\mu}$$

This time step restriction is half the value in one dimension.

FTCS scheme: Fourier stability

Take Fourier mode with wave-number $(\xi, \eta) \in (-\pi, +\pi) \times (-\pi, +\pi)$

$$u_{j,k}^n = \hat{u}^n \mathrm{e}^{\mathrm{i}(j\xi + k\eta)}$$

Then

$$\hat{u}^{n+1} = \rho \hat{u}^n$$

where

$$\begin{aligned} \rho(\xi, \eta) &= 1 + 2r_x(\cos\xi - 1) + 2r_y(\cos\eta - 1) \\ &= 1 - 4r_x \sin^2(\xi/2) - 4r_y \sin^2(\eta/2) \end{aligned}$$

For stability we need $|\rho(\xi, \eta)| \leq 1$. Maximum value of ρ

$$\rho = 1 \qquad \text{at} \qquad (\xi, \eta) = (0, 0)$$

and minimum value of ρ

$$\rho = 1 - 4r_x - 4r_y \qquad \text{at} \qquad (\xi, \eta) = (\pi, \pi)$$

Hence we have stability provided

$$r_x + r_y \leq \frac{1}{2}$$

BTCS scheme

$$u_{j,k}^{n+1} = u_{j,k}^{n} + r_x \delta_x^2 u_{j,k}^{n+1} + r_y \delta_y^2 u_{j,k}^{n+1}$$

or

$$\boxed{(1 - r_x \delta_y^2 - r_y \delta_y^2) u_{j,k}^{n+1} = u_{j,k}^{n}}$$

Amplification factor Amplification factor

$$\rho = \frac{1}{1 + 4r_x \sin^2(\xi/2) + 4r_y \sin^2(\eta/2)} \leq 1$$

BTCS scheme is unconditionally stable.

Crank-Nicholson scheme

$$u_{j,k}^{n+1} = u_{j,k}^{n} + r_x \delta_x^2 \frac{u_{j,k}^{n} + u_{j,k}^{n+1}}{2} + r_y \delta_y^2 \frac{u_{j,k}^{n} + u_{j,k}^{n+1}}{2}$$

or

$$\boxed{(1 - \frac{1}{2} r_x \delta_x^2 - \frac{1}{2} r_y \delta_y^2) u_{j,k}^{n+1} = (1 + \frac{1}{2} r_x \delta_x^2 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^{n}}$$

Amplification factor

$$\rho = \frac{1 - 2r_x \sin^2(\xi/2) - 2r_y \sin^2(\eta/2)}{1 + 2r_x \sin^2(\xi/2) + 2r_y \sin^2(\eta/2)} \leq 1$$

Crank-Nicholson scheme is unconditionally stable.

Peaceman-Rachford scheme

Update solution in two steps

$$t^n \to t^n + \frac{1}{2}\Delta t \to t^{n+1}, \qquad u^n \to u^{n+\frac{1}{2}} \to u^{n+1}$$

Step 1: Implicit in x

$$\frac{u_{j,k}^{n+\frac{1}{2}} - u_{j,k}^n}{\frac{1}{2}\Delta t} = \frac{\mu}{\Delta x^2}\delta_x^2 u_{j,k}^{n+\frac{1}{2}} + \frac{\mu}{\Delta y^2}\delta_y^2 u_{j,k}^n$$

or

$$\boxed{(1 - \frac{1}{2}r_x\delta_x^2)u_{j,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2}r_y\delta_y^2)u_{j,k}^n}$$

Step 2: Implicit in y

$$\frac{u_{j,k}^{n+1} - u_{j,k}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = \frac{\mu}{\Delta x^2}\delta_x^2 u_{j,k}^{n+\frac{1}{2}} + \frac{\mu}{\Delta y^2}\delta_y^2 u_{j,k}^{n+1}$$

or

$$\boxed{(1 - \frac{1}{2}r_y\delta_y^2)u_{j,k}^{n+1} = (1 + \frac{1}{2}r_x\delta_x^2)u_{j,k}^{n+\frac{1}{2}}}$$

Peaceman-Rachford scheme: Fourier stability

Step 1:

$$[1 + 2r_x \sin^2(\xi/2)]\hat{u}^{n+\frac{1}{2}} = [1 - 2r_y \sin^2(\eta/2)]\hat{u}^n$$

Step 2:

$$[1 + 2r_y \sin^2(\eta/2))\hat{u}^{n+1} = [1 - 2r_x \sin^2(\xi/2)]\hat{u}^{n+\frac{1}{2}}$$

Combining the two

$$\rho = \frac{\hat{u}^{n+1}}{\hat{u}^n} = \frac{[1 - 2r_x \sin^2(\xi/2)][1 - 2r_y \sin^2(\eta/2)]}{[1 + 2r_x \sin^2(\xi/2)][1 + 2r_y \sin^2(\eta/2)]} \leq 1$$

The scheme is unconditionally stable.

Peaceman-Rachford scheme: Consistency

From Step 1

$$(1 - \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

Multiply on both sides

$$(1 + \frac{1}{2} r_x \delta_x^2)(1 - \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2} r_x \delta_x^2)(1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

Operators on left commute

$$(1 - \frac{1}{2} r_x \delta_x^2) \underbrace{(1 + \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}}}_{\text{use Step 2}} = (1 + \frac{1}{2} r_x \delta_x^2)(1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

Using Step 2

$$\boxed{(1 - \frac{1}{2} r_x \delta_x^2)(1 - \frac{1}{2} r_y \delta_y^2) u_{j,k}^{n+1} = (1 + \frac{1}{2} r_x \delta_x^2)(1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n}$$

Peaceman-Rachford scheme: Consistency

Expand difference operators on both sides

$$\frac{u_{j,k}^{n+1} - u_{j,k}}{\Delta t} = \frac{\mu}{\Delta x^2}\delta_x^2 \frac{u_{j,k}^{n} + u_{j,k}^{n+1}}{2} + \frac{\mu}{\Delta y^2}\delta_y^2 \frac{u_{j,k}^{n} + u_{j,k}^{n+1}}{2}$$
$$- \frac{\mu^2 \Delta t}{4} \frac{\delta_x^2}{\Delta x^2} \frac{\delta_y^2}{\Delta y^2} (u_{j,k}^{n+1} - u_{j,k}^{n})$$

This is Crank-Nicholson scheme with an extra term. This term is $\mathcal{O}\left(\Delta t^2\right)$ since

$$\frac{\delta_x^2}{\Delta x^2} \frac{\delta_y^2}{\Delta y^2} (u_{j,k}^{n+1} - u_{j,k}^{n}) = \Delta t \left(\frac{\partial^5 u}{\partial t \partial^2 x \partial^2 y} \right)_{j,k}^{n} + \mathcal{O}\left(\Delta t \Delta x^2 + \Delta t \Delta y^2 + \Delta t^2\right)$$

Hence the Peaceman-Rachford scheme is second order accurate in space and time.

D'Yakonov Scheme

Peaceman-Rachford scheme

$$(1 - \frac{1}{2}r_x\delta_x^2)\underbrace{(1 - \frac{1}{2}r_y\delta_y^2)u_{j,k}^{n+1}}_{u_{j,k}^*} = (1 + \frac{1}{2}r_x\delta_x^2)(1 + \frac{1}{2}r_y\delta_y^2)u_{j,k}^n$$

Step 1: Solve for u^*

$$(1 - \frac{1}{2}r_x\delta_x^2)u_{j,k}^* = (1 + \frac{1}{2}r_x\delta_x^2)(1 + \frac{1}{2}r_y\delta_y^2)u_{j,k}^n$$

Step 2: Solve for u^{n+1}

$$(1 - \frac{1}{2}r_y\delta_y^2)u_{j,k}^{n+1} = u_{j,k}^*$$

Approximate factorization

Peaceman-Rachford scheme is close to Crank-Nicholson scheme

$$(1 - \frac{1}{2}r_x\delta_x^2 - \frac{1}{2}r_y\delta_y^2)u_{j,k}^{n+1} = (1 + \frac{1}{2}r_x\delta_x^2 + \frac{1}{2}r_y\delta_y^2)u_{j,k}^n$$

Factorise operator on left hand side

$$(1 - \frac{1}{2}r_x\delta_x^2 - \frac{1}{2}r_y\delta_y^2) = (1 - \frac{1}{2}r_x\delta_x^2)(1 - \frac{1}{2}r_y\delta_y^2) - \frac{r_x r_y}{4}\delta_x^2\delta_y^2$$

We cannot neglect the last term since it is $\mathcal{O}\left(\Delta t^2\right)$ and the scheme would only be first order accurate

$$\frac{r_x r_y}{4}\delta_x^2\delta_y^2 u_{j,k}^{n+1} = \frac{\Delta t^2}{4}\left(\frac{\partial^4 u}{\partial^2 x \partial^2 y}\right)_{j,k}^{n+1} + \text{higher order terms}$$

We factorise the right hand side also

$$(1-\frac{1}{2}r_x\delta_x^2)(1-\frac{1}{2}r_y\delta_y^2)u_{j,k}^{n+1} = (1+\frac{1}{2}r_x\delta_x^2)(1+\frac{1}{2}r_y\delta_y^2)u_{j,k}^n + \underbrace{\frac{r_x r_y}{4}\delta_x^2\delta_y^2(u_{j,k}^{n+1} - u_{j,k}^n)}_{\mathcal{O}(\Delta t^3)}$$

Peaceman-Rachford Scheme: Dirichlet BC

Step 1 is

$$(1 - \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

What boundary condition to use for $u_{j,k}^{n+\frac{1}{2}}$ at $j = 0,\ M_x$?

Option I: Since $u_{j,k}^{n+\frac{1}{2}} \approx u(x_j, y_k, t^n + \frac{1}{2}\Delta t)$

$$u_{j,k}^{n+\frac{1}{2}} = g(x_j, y_k, t^n + \frac{1}{2}\Delta t), \qquad j = 0, M_x$$

Option II: Using Step 1 and 2

$$u_{j,k}^{n+\frac{1}{2}} = \frac{1}{2}(1 - \frac{1}{2} r_y \delta_y^2) u_{j,k}^{n+1} + \frac{1}{2}(1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

At $j = 0, M_x$

$$u_{j,k}^{n+\frac{1}{2}} = \frac{1}{2}(1 - \frac{1}{2} r_y \delta_y^2) g(x_j, y_k, t^n + \frac{1}{2}\Delta t) + \frac{1}{2}(1 + \frac{1}{2} r_y \delta_y^2) g(x_j, y_k, t^n)$$

If BC do not depend on t, then $u_{j,k}^{n+\frac{1}{2}} = g(x_j, y_k)$

D'Yakonov Scheme: Dirichlet BC

What boundary condition to use for $u^*_{j,k}$ at $j = 0, M_x$? Unlike $u^{n+\frac{1}{2}}$, we do not know what u^* represents. Instead use the numerical scheme itself; from Step 2

$$u^*_{j,k} = (1 - \frac{1}{2} r_y \delta_y^2) u^{n+1}_{j,k}$$

Boundary condition for $u^*_{j,k}$

$$u^*_{j,k} = (1 - \frac{1}{2} r_y \delta_y^2) g(x_j, y_k, t^{n+1}), \qquad j = 0, M_x$$

E.g., at $j = 0$

$$u^*_{0,k} = -\frac{r_y}{2} g^{n+1}_{j,k-1} + (1 + r_y) g^{n+1}_{j,k} - \frac{r_y}{2} g^{n+1}_{j,k+1}$$

Peaceman-Rachford Scheme: Implementation

Step 1: For $j = 1, \ldots, M_x - 1$ and $k = 1, \ldots, M_y - 1$

$$(1 - \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n$$

For $k = 1, \ldots, M_y - 1$, solve tridiagonal matrix

$$-\frac{r_x}{2} u_{j-1,k}^{n+\frac{1}{2}} + (1 + r_x) u_{j,k}^{n+\frac{1}{2}} - \frac{r_x}{2} u_{j+1,k}^{n+\frac{1}{2}} = (1 + \frac{1}{2} r_y \delta_y^2) u_{j,k}^n, \quad j = 1, \ldots, M_x - 1$$

Step 2: For $j = 1, \ldots, M_x - 1$ and $k = 1, \ldots, M_y - 1$

$$(1 - \frac{1}{2} r_y \delta_y^2) u_{j,k}^{n+1} = (1 + \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}}$$

For $j = 1, \ldots, M_x - 1$, solve tridiagonal matrix

$$-\frac{r_y}{2} u_{j,k-1}^{n+1} + (1 + r_y) u_{j,k}^{n+1} - \frac{r_y}{2} u_{j,k+1}^{n+1} = (1 + \frac{1}{2} r_x \delta_x^2) u_{j,k}^{n+\frac{1}{2}}, \quad k = 1, \ldots, M_y - 1$$

Remark: Total work scales as $\mathcal{O}(M_x M_y)$ and hence is optimal.

Douglas-Rachford Scheme

This is an approximate factorised version of the BTCS scheme.

$$(1 - r_x \delta_y^2 - r_y \delta_y^2) u^{n+1} = u^n$$

Factorise LHS operator

$$(1 - r_x \delta_y^2 - r_y \delta_y^2) = (1 - r_x \delta_x^2)(1 - r_y \delta_y^2) - r_x r_y \delta_x^2 \delta_y^2$$

We can throw away the last term since we will get first order time accuracy. But we can make a better approximation as follows.

$$(1 - r_x \delta_x^2)(1 - r_y \delta_y^2) u^{n+1} = u^n + r_x r_y \delta_x^2 \delta_y^2 u^{n+1}$$

$$(1 - r_x \delta_x^2)(1 - r_y \delta_y^2) u^{n+1} = (1 + r_x r_y \delta_x^2 \delta_y^2) u^n + \underbrace{r_x r_y \delta_x^2 \delta_y^2 (u^{n+1} - u^n)}_{\mathcal{O}(\Delta t^3)}$$

Douglas-Rachford scheme is given by

$$\boxed{(1 - r_x \delta_x^2)(1 - r_y \delta_y^2) u^{n+1} = (1 + r_x r_y \delta_x^2 \delta_y^2) u^n}$$

Douglas-Rachford Scheme: Two step scheme

Introduce an intermediate state u^*

$$(1 - r_x\delta_x^2)\underbrace{(1 - r_y\delta_y^2)u^{n+1}}_{u^*} = (1 + r_x r_y \delta_x^2\delta_y^2)u^n$$

Two-step process

$$\begin{aligned}(1 - r_x\delta_x^2)u^* &= (1 + r_y\delta_y^2)u^n \\ (1 - r_y\delta_y^2)u^{n+1} &= u^* - r_y\delta_y^2 u^n\end{aligned}$$

We avoid fourth differences $\delta_x^2\delta_y^2$. The equivalence is seen as follows.

$$\begin{aligned}(1 - r_x\delta_x^2)(1 - r_y\delta_y^2)u^{n+1} &= (1 - r_x\delta_x^2)u^* - (1 - r_x\delta_x^2)r_y\delta_y^2 u^n \\ &= (1 + r_y\delta_y^2)u^n - (1 - r_x\delta_x^2)r_y\delta_y^2 u^n \\ &= (1 + r_x r_y \delta_x^2\delta_y^2)u^n\end{aligned}$$

Douglas-Rachford Scheme: Fourier stability

Amplification factor

$$\rho(\xi, \eta) = \frac{1 + 16 r_x r_y \sin^2(\xi/2) \sin^2(\eta/2)}{[1 + 4 r_x \sin^2(\xi/2)][1 + 4 r_y \sin^2(\eta/2)]} \leq 1$$

The scheme is unconditionally stable.

Douglas-Rachford Scheme: Dirichlet BC

We need BC for u^* at $j = 0, M_x$. Using Step 2 of the scheme

$$u^* = (1 - r_y \delta_y^2) u^{n+1} - r_y \delta_y^2 u^n$$

This gives the boundary conditions

$$u^*_{j,k} = (1 - r_y \delta_y^2) g^{n+1}_{j,k} - r_y \delta_y^2 g^n_{j,k}, \qquad j = 0, M_x$$

Source terms

Heat equation with a forcing term

$$u_t = \mu(u_{xx} + u_{yy}) + F(x, y, t)$$

Crank-Nicholson scheme, second order in time and space

$$(1 - \frac{1}{2}r_x\delta_x^2 - \frac{1}{2}r_y\delta_y^2)u^{n+1} = (1 + \frac{1}{2}r_x\delta_x^2 + \frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}(F^n + F^{n+1})$$

We can do a factorization as before

$$\begin{aligned}(1 - \frac{1}{2}r_x\delta_x^2)(1 - \frac{1}{2}r_y\delta_y^2)u^{n+1} &= (1 + \frac{1}{2}r_x\delta_x^2)(1 + \frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}(F^n + F^{n+1}) \\ &\quad \underbrace{- \frac{r_x r_y}{4}\delta_x^2\delta_y^2(u^{n+1} - u^n)}_{\mathcal{O}(\Delta t^3)}\end{aligned}$$

Ignoring last term, the resulting scheme is second order accurate

$$(1 - \frac{1}{2}r_x\delta_x^2)(1 - \frac{1}{2}r_y\delta_y^2)u^{n+1} = (1 + \frac{1}{2}r_x\delta_x^2)(1 + \frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}(F^n + F^{n+1})$$

Source terms: Peaceman-Rachford Scheme

Use the viewpoint of two step process: $t^n \to t^n + \frac{1}{2}\Delta t \to t^{n+1}$

$$
\begin{aligned}
(1 - \frac{1}{2} r_x \delta_x^2) u^{n+\frac{1}{2}} &= (1 + \frac{1}{2} r_y \delta_y^2) u^n + \frac{\Delta t}{2} F^n \\
(1 - \frac{1}{2} r_y \delta_y^2) u^{n+1} &= (1 + \frac{1}{2} r_x \delta_x^2) u^{n+\frac{1}{2}} + \frac{\Delta t}{2} F^{n+1}
\end{aligned}
$$

To check the consistency of the scheme, eliminate $u^{n+\frac{1}{2}}$

$$
\begin{aligned}
(1 - \frac{1}{2} r_x \delta_x^2)(1 - \frac{1}{2} r_y \delta_y^2) u^{n+1} &= (1 - \frac{1}{2} r_x \delta_x^2)(1 + \frac{1}{2} r_x \delta_x^2) u^{n+\frac{1}{2}} \\
&\quad + \frac{\Delta t}{2}(1 - \frac{1}{2} r_x \delta_x^2) F^{n+1} \\
&= (1 + \frac{1}{2} r_x \delta_x^2)(1 + \frac{1}{2} r_y \delta_y^2) u^n + \frac{\Delta t}{2}(F^n + F^{n+1}) \\
&\quad \underbrace{- \frac{r_x \Delta t}{2} \delta_x^2 (F^{n+1} - F^n)}_{\mathcal{O}(\Delta t^3)}
\end{aligned}
$$

Hence the two-step scheme is second order accurate in time and space.

Source terms: D'Yakonov Scheme

Start with approximately factored Crank-Nicholson scheme

$$(1-\frac{1}{2}r_x\delta_x^2)\underbrace{(1-\frac{1}{2}r_y\delta_y^2)u^{n+1}}_{u^*} = (1+\frac{1}{2}r_x\delta_x^2)(1+\frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}(F^n + F^{n+1})$$

Two-step D'Yakonov scheme

$$\begin{aligned}(1-\frac{1}{2}r_x\delta_x^2)u^* &= (1+\frac{1}{2}r_x\delta_x^2)(1+\frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}(F^n + F^{n+1}) \\ (1-\frac{1}{2}r_y\delta_y^2)u^{n+1} &= u^*\end{aligned}$$

Remark: We cannot spread out the source term into the two steps like this

$$\begin{aligned}(1-\frac{1}{2}r_x\delta_x^2)u^* &= (1+\frac{1}{2}r_x\delta_x^2)(1+\frac{1}{2}r_y\delta_y^2)u^n + \frac{\Delta t}{2}F^n \\ (1-\frac{1}{2}r_y\delta_y^2)u^{n+1} &= u^* + \frac{\Delta t}{2}F^{n+1}\end{aligned}$$

This scheme is only first order accurate in time.