High order methods for linear conservation laws

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High resolution schemes

- We have seen two notions of stability: Fourier stability and maximum stability (monotone scheme, positive scheme, LED scheme)
- Fourier stability is always necessary; it determines stability to small perturbations.
- A scheme which is not Fourier stable is useless since it will cause blow-up of solution.
- Maximum stability is a stronger notion of stability. It prevents solution from oscillating. It is required when solution has discontinuities and steep gradients.
- Maximum stable first order schemes: upwind, Lax-Friedrichs
- Second order scheme: Lax-Wendroff, Fourier stable under CFL condition, but not stable in maximum norm.
- Basic idea of high order scheme: blend first order and second order scheme with a switching function, called **limiter**. If solution is locally in danger of developing oscillations, switch from second order to first order scheme. It is second order in smooth regions.

Godunov's order barrier theorem

$$v_i^{n+1} = \sum_j b_j v_{i+j}^n, \qquad \{b_j\} \text{ are some constants}$$

Taylor series in space for exact solution u

$$u_{i+j}^n = u_i^n + \sum_{m=1}^{\infty} \frac{(jh)^m}{m!} \frac{\partial^m u}{\partial x^m}$$

Taylor series in time for exact solution u

$$u_i^{n+1} = u_i^n + \sum_{m=1}^{\infty} \frac{(\Delta t)^m}{m!} \frac{\partial^m u}{\partial t^m}$$

Local truncation error: scheme is p'th order accurate if

$$\tau_i^n = \frac{1}{\Delta t} \left(u_i^{n+1} - \sum_j b_j u_{i+j}^n \right) = \mathcal{O}\left(h^p\right)$$

Godunov's order barrier theorem

Use PDE: $u_t = -au_x$, $u_{tt} = a^2 u_{xx}$, etc. Then

$$\begin{aligned} \tau_i^n \Delta t &= (1 - \sum_j b_j) u_i^n - (\sigma + \sum_j j b_j) h u_x \\ &\quad + (\sigma^2 - \sum_j j^2 b_j) \frac{1}{2} h^2 u_{xx} + \mathcal{O}\left(h^3\right) \end{aligned}$$

For first order accuracy, we need

$$\sum_j b_j = 1, \qquad \sum_j j b_j = -\sigma \qquad \Longrightarrow \qquad \tau_i^n = \mathcal{O}\left(h\right)$$

For second order accuracy, we need

$$\sum_j j^2 b_j = \sigma^2 \qquad \Longrightarrow \qquad \tau_i^n = \mathcal{O}\left(h^2\right)$$

Godunov's order barrier theorem

Assume that the scheme is positive, $b_j \geq 0$ and second order accurate. Then, by Cauchy-Schwartz inequality

$$\begin{aligned} \sigma^2 &= \left(\sum_j j b_j\right)^2 = \left(\sum_j j\sqrt{b_j}\sqrt{b_j}\right)^2 \\ &\leq (\sum_j j^2 b_j)(\sum_j b_j) = \sigma^2 \end{aligned}$$

This is possible only if equality holds in Cauchy-Schwartz inequality, i.e., if

$$j\sqrt{b_j} = c\sqrt{b_j}, \qquad \text{for some constant } c$$

This implies that $j = -\sigma$ and requires σ to be an integer in which case we obtain exact solution. But in general, this condition cannot be satisfied.

Any linear, positive scheme for $u_t + a u_x = 0$ is at most first order accurate.

Second order upwind scheme

Assume $a > 0$. The semi-discrete SOU scheme is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}(3v_i - 4v_{i-1} + v_{i-2}) = \frac{2a}{h}(v_{i-1} - v_i) - \frac{a}{2h}(v_{i-2} - v_i)$$

Write as first order upwind scheme + correction

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}(v_i - v_{i-1}) - \frac{a}{h}\left[\frac{1}{2}(v_i - v_{i-1}) - \frac{1}{2}(v_{i-1} - v_{i-2})\right]$$

Define difference ratios to measure local solution smoothness

$$r_{i-1} = \frac{v_i - v_{i-1}}{v_{i-1} - v_{i-2}}, \qquad r_i = \frac{v_{i+1} - v_i}{v_i - v_{i-1}}$$

Introduce switching functions

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}\left[(v_i - v_{i-1}) + \frac{1}{2}\Psi(r_i)(v_i - v_{i-1}) - \frac{1}{2}\Psi(r_{i-1})(v_{i-1} - v_{i-2})\right]$$

Second order upwind scheme

which can be written as

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{a}{h}\left[1 + \frac{1}{2}\Psi(r_i) - \frac{1}{2}\frac{\Psi(r_{i-1})}{r_{i-1}}\right](v_i - v_{i-1})$$

This scheme is positive provided

$$\frac{\Psi(r_{i-1})}{r_{i-1}} - \Psi(r_i) \le 2$$

We have lot of freedom in choosing the function Ψ. Let us restrict Ψ to be a positive function

$$\Psi(r) \ge 0, \qquad r \ge 0$$

When $\Psi = 0$, the scheme becomes first order accurate. If there is shock around some grid point, i.e., $r_i < 0$, we want the scheme to become first order accurate. Hence

$$\Psi(r) = 0, \qquad r \le 0$$

Second order upwind scheme

These conditions imply that

$$0 \leq \Psi(r) \leq 2r$$

Symmetry property: Backward and forward differences are treated in the same manner

$$\frac{\Psi(r)}{r} = \Psi\left(\frac{1}{r}\right)$$

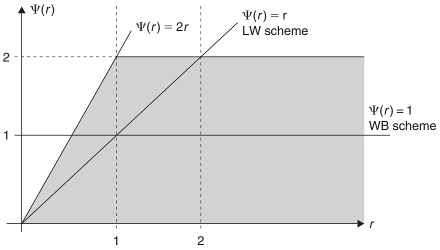
This leads to the condition

$$\Psi(r) \leq 2$$

Combining all the above conditions, we get

$$0 \leq \Psi(r) \leq \min(2, 2r)$$

Second order upwind scheme

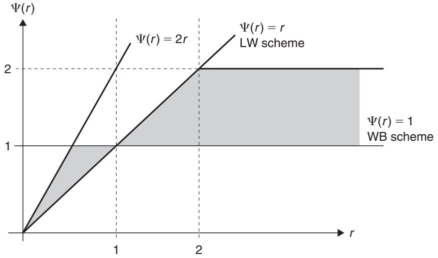


If v is a linear function of x, then the scheme should give exact solution. In this case $r_i = 1$ and we should have

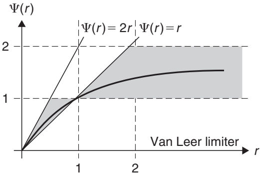
$$\Psi(1) = 1$$

This condition is required to achieve second order accuracy in smooth regions.

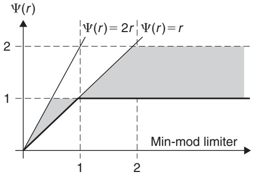
Second order upwind scheme



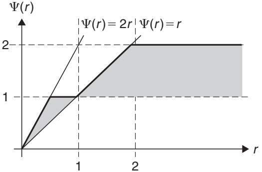
Limiter functions



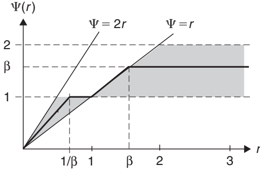
(a) Van Leer's limiter $\Psi=(r+|r|)/(1+r)$



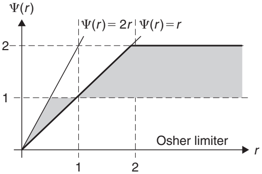
(b) Min-mod Limiter $\Psi(r)=\text{min-mod}\ (r,1)$



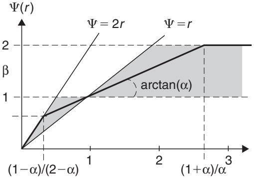
(c) Roe's "Superbee" Limiter
$\Psi=\text{Max}\ [0, \min(2r,1), \min(r,2)]$



(d) General β Limiters
$\Psi=\text{Max}\ [0, \min(\beta r,1), \min(r,\beta)]$



(e) Chakravarthy and Osher Limiter
$\Psi(r)=\text{Max}[0, \min(r,\beta)]$



(f) General α Limiter
$\Psi=\text{Max}[0, \min(2r,\alpha r+1-\alpha,2)]$

Lax-Wendroff scheme

$$v_i^{n+1} = \frac{1}{2}\sigma(1+\sigma)v_{i-1}^n + (1-\sigma^2)v_i^n - \frac{1}{2}\sigma(1-\sigma)v_{i+1}^n$$

If $a > 0$ so that $\sigma > 0$, then coefficient of v_{i+1}^n is negative and the scheme is not monotone.

Define ratios

$$R_i = \frac{v_i - v_{i-1}}{v_{i+1} - v_i} = \frac{\text{backward difference}}{\text{forward difference}}$$

If the solution v_{i-1}, v_i, v_{i+1} is smooth then $R_i = \mathcal{O}(1)$. In fact if solution is linear in x, then $R_i = 1$.

Modification of Lax-Wendroff scheme

Step 1: Write high order scheme as low order positive scheme + correction term

$$
\begin{aligned}
u_i^{n+1} \;=\;& u_i^n - \sigma(u_i^n - u_{i-1}^n) \\
& -\frac{\sigma}{2}(1-\sigma)(u_{i+1}^n - u_i^n) + \frac{\sigma}{2}(1-\sigma)(u_i^n - u_{i-1}^n)
\end{aligned}
$$

Step 2: Introduce switching function in correction terms

$$
\begin{aligned}
u_i^{n+1} \;=\;& u_i^n - \sigma(u_i^n - u_{i-1}^n) \\
& -\frac{\sigma}{2}(1-\sigma)\Psi(R_i)(u_{i+1}^n - u_i^n) + \frac{\sigma}{2}(1-\sigma)\Psi(R_{i-1})(u_i^n - u_{i-1}^n)
\end{aligned}
$$

or, re-arranging

$$
u_i^{n+1} = u_i^n - \sigma\left\{1 + \frac{1}{2}(1-\sigma)\left[\frac{\Psi(R_i)}{R_i} - \Psi(R_{i-1})\right]\right\}(u_i^n - u_{i-1}^n)
$$

Modification of Lax-Wendroff scheme

Scheme is positive if

$$\Psi(R_{i-1}) - \frac{\Psi(R_i)}{R_i} \le \frac{2}{1-\sigma}$$

Assuming symmetry property for limiter, this condition is satisfied by choosing

$$0 \le \Psi(R) \le \min\left(\frac{2R}{\sigma}, \frac{2}{1-\sigma}\right)$$

Since by CFL condition $0 \le \sigma \le 1$, we can take the more restrictive condition

$$0 \le \Psi(R) \le \min(2, 2R)$$

This is the same condition as we obtained for the SOU scheme and the allowed region for Ψ is as before.

For better accuracy, one should use the condition involving the CFL number σ, see Hirsch, page 387.

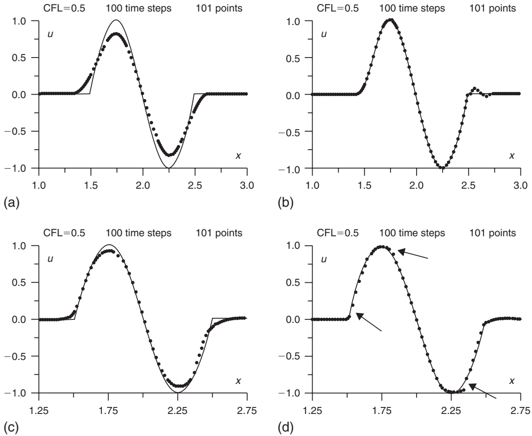


Figure 8.3.5 *Effects of limiters on the linear convection of a sinusoidal wave (a) first order upwind scheme (b) second order upwind scheme (c) second order upwind scheme with min-mod limiter (d) second order upwind scheme with superbee limiter.*

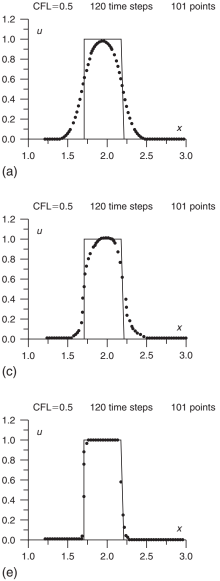


Figure 8.3.6 *Effects of limiters on the linear convection of a square wave after 120 time steps: (a) first order upwind scheme, (b) second order upwind scheme, (c) second order upwind scheme with min-mod limiter, (d) second order upwind scheme with Van Leer limiter and (e) second order upwind scheme with superbee limiter.*

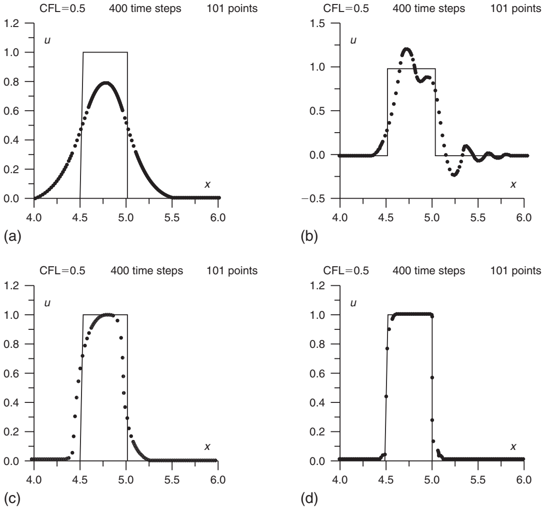


Figure 8.3.7 *Effects of limiters on the linear convection of a square wave after 400 time steps: (a) first order upwind scheme, (b) second order upwind scheme, (c) second order upwind scheme with Van Leer limiter and (d) second order upwind scheme with superbee limiter.*

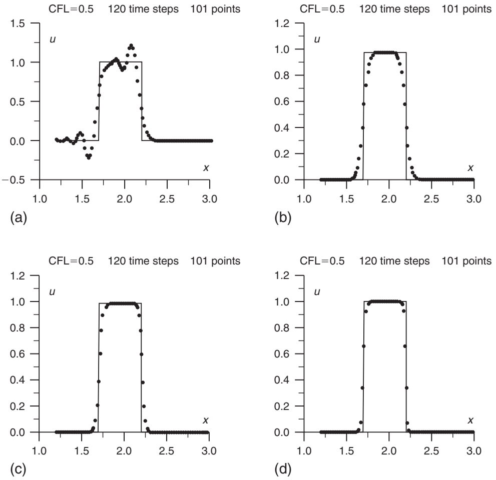


Figure 8.3.8 *Effects of limiters on the linear convection of a square wave after 120 time steps: (a) standard LW scheme, (b) second order high-resolution LW scheme with min-mod limiter, (c) second order high-resolution LW scheme with Van Leer limiter and (d) second order high-resolution LW scheme with superbee limiter.*