Finite Volume Method

An Introduction

Praveen. C

CTFD Division
National Aerospace Laboratories
Bangalore 560 037
email: praveen@cfdlab.net

April 7, 2006

Outline

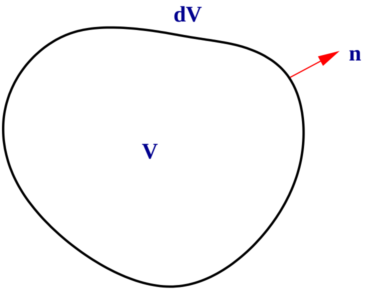
Outline

Divergence theorem

- In Cartesian coordinates, $\vec{q} = (u, v, w)$

$$\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Divergence theorem

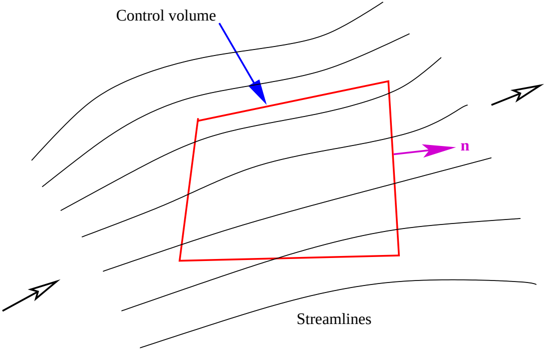


$$\boxed{\int_V \nabla \cdot \vec{q} \, \mathrm{d}V = \oint_{\partial V} \vec{q} \cdot \hat{n} \, \mathrm{d}S}$$

Outline

Conservation laws

- Basic laws of physics are conservation laws: mass, momentum, energy, charge



Rate of change of ϕ in V = -(net flux across ∂V)

$$\frac{\partial}{\partial t}\int_V \phi \, \mathrm{d}V = -\oint_{\partial V} \vec{F}\cdot\hat{n} \, \mathrm{d}S$$

Conservation laws

- If $\vec{F}$ is differentiable, use divergence theorem

$$\frac{\partial}{\partial t}\int_V \phi \, \mathrm{d}V = -\int_V \nabla \cdot \vec{F} \, \mathrm{d}V$$

or

$$\int_V \left[\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{F}\right] \mathrm{d}V = 0, \quad \text{for every control volume } V$$

$$\boxed{\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{F} = 0}$$

- In Cartesian coordinates, $\vec{F} = (f, g, h)$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0}$$

Differential and integral form

- Differential form

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{F} = 0$$

$$\frac{\partial}{\partial t}(\text{conserved quantity}) + \text{divergence(flux)} = 0$$

- Integral form

$$\frac{\partial}{\partial t} \int_V \phi \, dV = - \oint_{\partial V} \vec{F} \cdot \hat{n} \, dS$$

Mass conservation equation

- $\phi = \rho$, $\vec{F} = \rho\vec{q}$

$$\vec{F} \cdot \hat{n} \, \mathrm{d}S = \text{amount of mass flowing across } \mathrm{d}S \text{ per unit time}$$

$$\frac{\partial}{\partial t} \int_V \rho \, \mathrm{d}V + \oint_{\partial V} \rho\vec{q} \cdot \hat{n} \, \mathrm{d}S = 0$$

- Using divergence theorem

$$\frac{\partial}{\partial t} \int_V \rho \, \mathrm{d}V + \int_V \nabla \cdot (\rho\vec{q}) \, \mathrm{d}S = 0$$

or

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{q}) \right] \mathrm{d}V = 0$$

Mass conservation equation

- Differential form

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0}$$

- In Cartesian coordinates, $\vec{q} = (u, v, w)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

- Non-conservative form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

or in vector notation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0}$$

Navier-Stokes equations

$$\begin{aligned}
\text{Mass:} \quad & \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) &&= 0 \\
\text{Momentum:} \quad & \frac{\partial}{\partial t}(\rho \vec{q}) + \nabla \cdot (\rho \vec{q} \vec{q}) + \nabla p &&= \nabla \cdot \tau \\
\text{Energy:} \quad & \frac{\partial E}{\partial t} + \nabla \cdot (E + p)\vec{q} &&= \nabla \cdot (\vec{q} \cdot \tau) + \nabla \cdot \vec{Q}
\end{aligned}$$

Convection and diffusion

$$\begin{aligned}
\text{Mass:} \quad & \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) && = 0 \\
\text{Momentum:} \quad & \frac{\partial}{\partial t}(\rho \vec{q}) + \nabla \cdot (\rho \vec{q}\vec{q}) + \nabla p && = \nabla \cdot \tau \\
\text{Energy:} \quad & \frac{\partial E}{\partial t} + \nabla \cdot (E + p)\vec{q} && = \nabla \cdot (\vec{q} \cdot \tau) + \nabla \cdot \vec{Q}
\end{aligned}$$

Outline

- Convection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

- a, μ are constants, $\mu > 0$
- Conservation law form

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} &= 0 \\ f &= f^c + f^d \\ f^c &= au \\ f^d &= -\mu\frac{\partial u}{\partial x} \end{aligned}$$

Convection equation

- Scalar convection equation

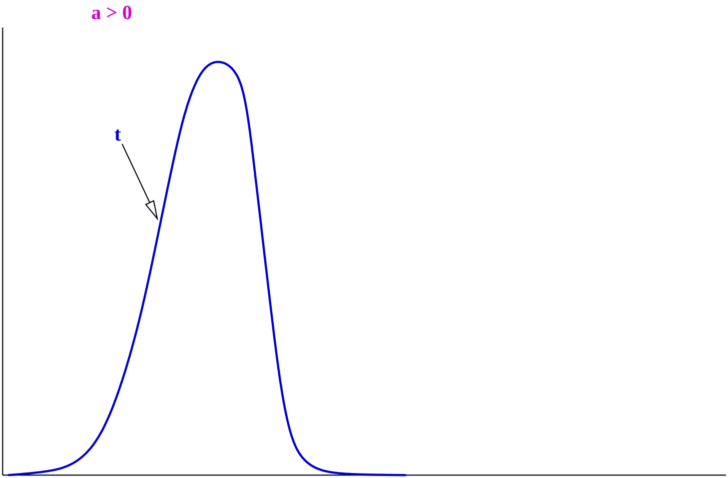
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- Exact solution

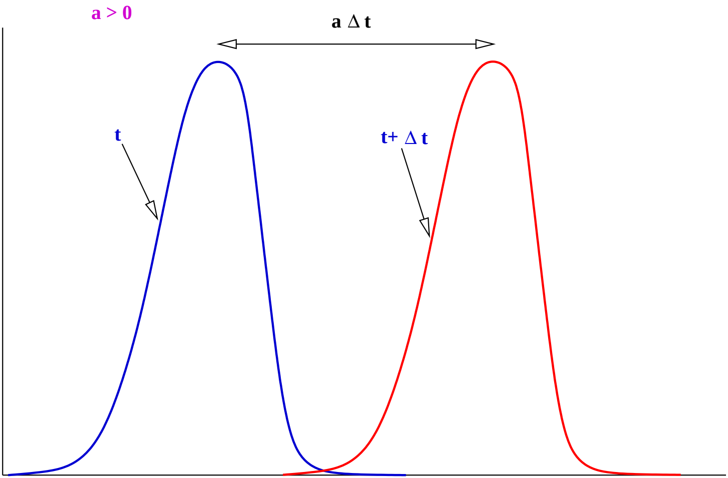
$$u(x, t + \Delta t) = u(x - a\Delta t, t)$$

- Wave-like solution: convected at speed a
- Wave moves in a particular direction
- Solution can be discontinuous
- Hyperbolic equation

Scalar convection equation



Scalar convection equation



Convection equation

- Scalar convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- Exact solution

$$u(x,t+\Delta t) = u(x - a\Delta t, t)$$

- Wave-like solution: convected at speed a
- Wave moves in a particular direction
- Solution can be discontinuous
- Hyperbolic equation

Diffusion equation

- Scalar diffusion equation

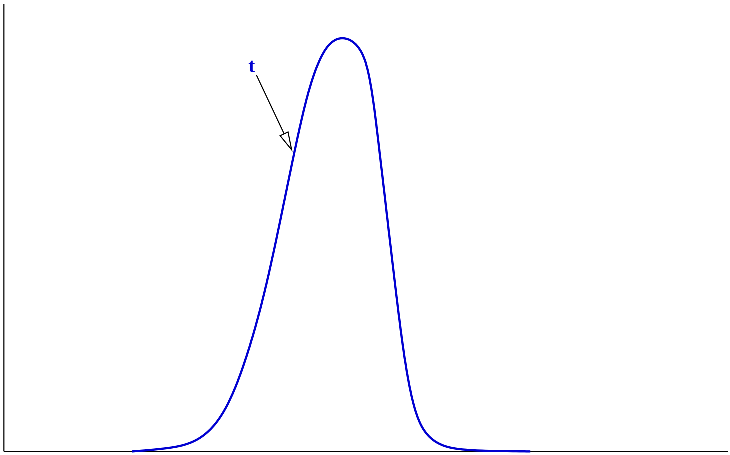
$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

- Solution

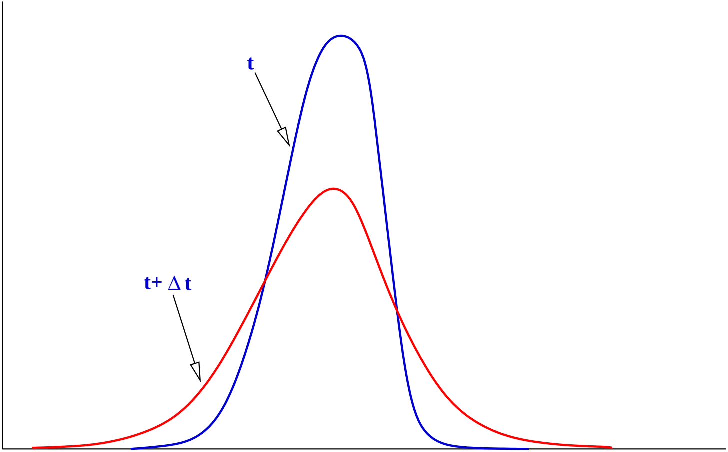
$$u(x,t) \sim e^{-\mu k^2 t} f(x;k)$$

- Solution is smooth and decays with time
- Elliptic equation

Diffusion equation: Initial condition



Diffusion equation



Diffusion equation

- Scalar diffusion equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

- Solution

$$u(x,t) \sim e^{-\mu k^2 t} f(x;k)$$

- Solution is smooth and decays with time
- Elliptic equation

Outline

FVM in 1-D: Conservation law

- 1-D conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad a < x < b, \quad t > 0$$

- Initial condition

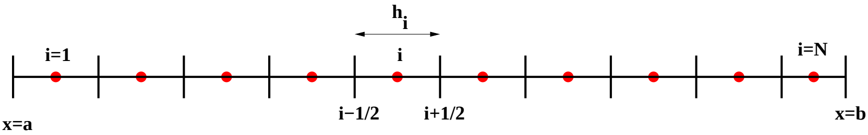
$$u(x, 0) = g(x)$$

- Boundary conditions at $x = a$ and/or $x = b$

FVM in 1-D: Grid

- Divide computational domain into N cells

$$a = x_{1/2}, x_{1+1/2}, x_{2+1/2}, \ldots, x_{N-1/2}, x_{N+1/2} = b$$



- Cell number i

$$C_i = [x_{i-1/2}, x_{i+1/2}]$$

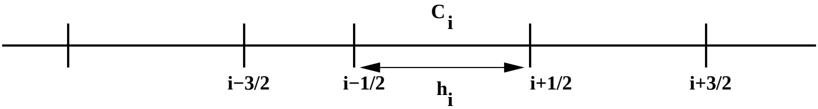
- Cell size (width)

$$h_i = x_{i+1/2} - x_{i-1/2}$$

FVM in 1-D

- Conservation law applied to cell C_i

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \mathrm{d}x = 0$$



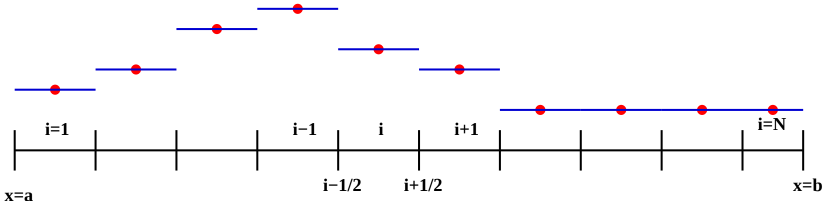
- Cell-average value

$$u_i(t) = \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t) \, \mathrm{d}x$$

- Integral form

$$h_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + f(x_{i+1/2}, t) - f(x_{i-1/2}, t) = 0$$

FVM in 1-D



- Cell-average value is dicontinuous at the interface
- What is the flux ?
- Numerical flux function

$$f(x_{i+1/2}, t) \approx F_{i+1/2} = F(u_i, u_{i+1})$$

- Finite volume update equation (ODE)

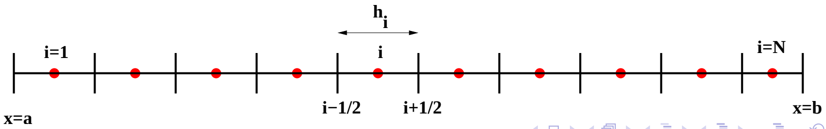
$$\boxed{h_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + F_{i+1/2} - F_{i-1/2} = 0, \quad i = 1, \ldots, N}$$

- Telescopic collapse of fluxes

$$\sum_{i=1}^{N} \left(h_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + F_{i+1/2} - F_{i-1/2} \right) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} h_i u_i + F_{N+1/2} - F_{1/2} = 0$$

Rate of change of total u = -(Net flux across the domain)



Time integration

- At time $t = 0$ set the initial condition

$$u_i^o := u_i(0) = \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} g(x)\, \mathrm{d}x$$

- Choose a time step Δt (stability and accuracy) or M
- Break time $(0, T)$ into M intervals

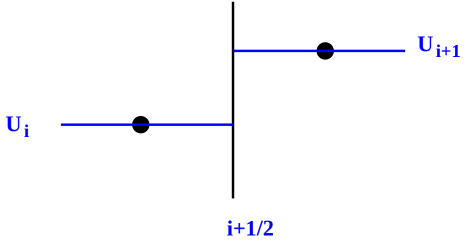
$$\Delta t = \frac{T}{M}, \quad t^n = n\Delta t, \quad n = 0, 1, 2, \ldots, M$$

- For $n = 0, 1, 2, \ldots, M$, compute $u_i^n = u_i(t^n) \approx u(x_i, t^n)$

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta t}{h_i}(F_{i+1/2}^n - F_{i-1/2}^n)}$$

Numerical Flux Function

- Take average of the left and right cells



$$F_{i+1/2} = \frac{1}{2}[f(u_i) + f(u_{i+1})]$$

- Central difference scheme

$$\frac{du_i}{dt} + \frac{f_{i+1} - f_{i-1}}{h_i} = 0$$

- Suitable for elliptic equations (diffusion phenomena)
- Unstable for hyperbolic equations (convection phenomena)

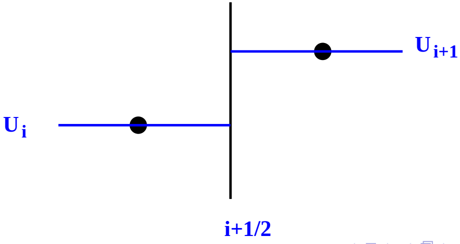
Convection equation: Centered flux

- Convection equation, $a = \text{constant}$

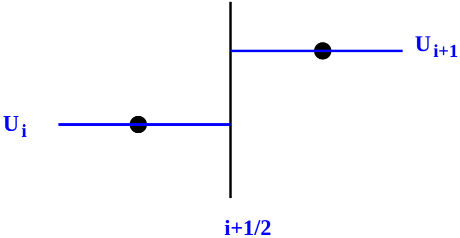
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad f(u) = au$$

- Centered flux, $F_{i+1/2} = (au_i + au_{i+1})/2$

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + a\frac{u_{i+1} - u_{i-1}}{h_i} = 0$$



- Upwind flux



$$F_{i+1/2} = \begin{cases} a u_i, & \text{if } a > 0 \\ a u_{i+1}, & \text{if } a < 0 \end{cases}$$

or

$$F_{i+1/2} = \frac{a + |a|}{2} u_i + \frac{a - |a|}{2} u_{i+1}$$

- Upwind scheme

$$\begin{aligned} \frac{du_i}{dt} + a\frac{u_i - u_{i-1}}{h_i} &= 0, \quad \text{if} \quad a > 0 \\ \frac{du_i}{dt} + a\frac{u_{i+1} - u_i}{h_i} &= 0, \quad \text{if} \quad a < 0 \end{aligned}$$

or

$$\frac{du_i}{dt} + \frac{a + |a|}{2}\frac{u_i - u_{i-1}}{h_i} + \frac{a - |a|}{2}\frac{u_{i+1} - u_i}{h_i} = 0$$

Diffusion equation

- Diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad f = -\mu \frac{\partial u}{\partial x}$$

- No directional dependence; use centered formula

$$F_{i+1/2} = -\mu \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

- On uniform grid

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \mu \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

Outline

Choice of numerical parameters

- What scheme to use ?
- What should be the grid size h ?
- What should be the time step Δt ?

Stability: Convection equation

- Take $a > 0$
- CFL number

$$\lambda = \frac{a\Delta t}{h}$$

- Centered flux

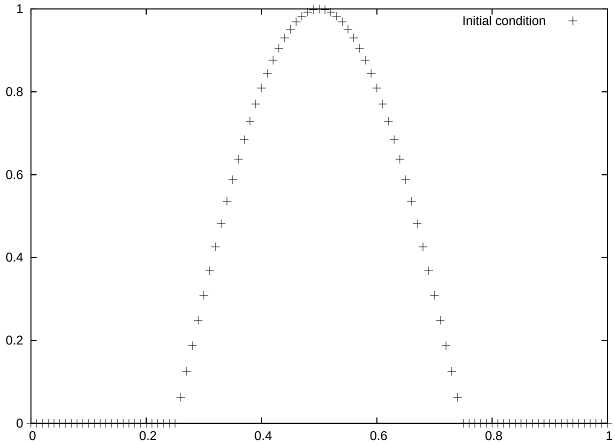
$$u_i^{n+1} = u_i^n - \frac{\lambda}{2}u_{i+1}^n + \frac{\lambda}{2}u_{i-1}^n$$

- Upwind flux

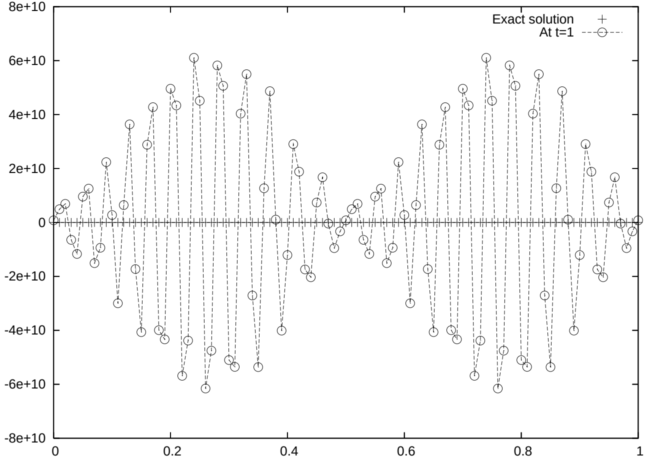
$$u_i^{n+1} = (1-\lambda)u_i^n + \lambda u_{i-1}^n$$

- Solution at $t = 1$

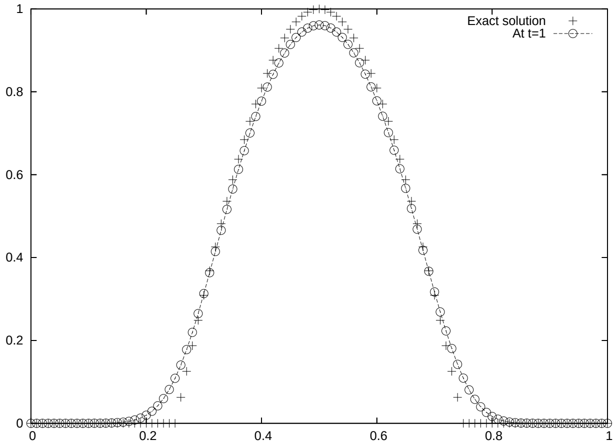
Convection equation: Initial condition



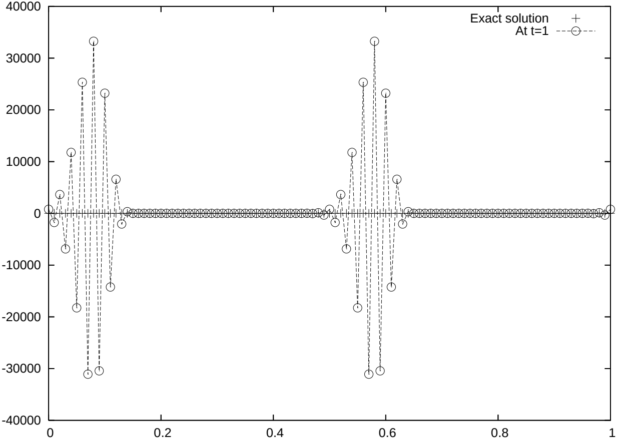
Convection equation: Centered scheme, $\lambda = 0.8$



Convection equation: Upwind scheme, $\lambda = 0.8$



Convection equation: Upwind scheme, $\lambda = 1.1$



Stability: Convection equation

- Centered flux: Unstable
- Upwind flux: conditionally stable

$$\lambda \le 1 \quad \text{or} \quad \Delta t \le \frac{h}{|a|}$$

Stability: Diffusion equation

- Central scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$$

- Stability parameter

$$\nu = \frac{\mu \Delta t}{h^2}$$

- Update equation

$$u_i^{n+1} = \nu u_{i+1}^n + (1 - 2\nu) u_i^n + \nu u_{i-1}^n$$

- Stability condition

$$\nu \le \frac{1}{2} \quad \text{or} \quad \Delta t \le \frac{h^2}{2\mu}$$

Stability: Convection-diffusion equation

- Convection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad f = au - \mu \frac{\partial u}{\partial x}$$

- Upwind scheme for convective term;
  central scheme for diffusive term

$$F_{i+1/2} = \frac{a+|a|}{2} u_i + \frac{a-|a|}{2} u_{i+1} - \mu \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

- Update equation

$$\begin{aligned} u_i^{n+1} = & \left(\frac{a^+ \Delta t}{h} + \frac{\mu \Delta t}{h^2} \right) u_{i-1}^n \\ + & \left(1 - \frac{|a| \Delta t}{h} - \frac{\mu \Delta t}{h^2} \right) u_i^n \\ + & \left(- \frac{a^- \Delta t}{h} + \frac{\mu \Delta t}{h^2} \right) u_{i+1}^n \end{aligned}$$

Outline

Non-linear conservation law

- Inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Finite difference scheme

$$\frac{du_i}{dt} + \frac{u_i + |u_i|}{2}\frac{u_i - u_{i-1}}{h} + \frac{u_i - |u_i|}{2}\frac{u_{i+1} - u_i}{h} = 0$$

- Gives wrong results
- Not conservative; cannot be put in FV form

$$h\frac{du_i}{dt} + F_{i+1/2} - F_{i-1/2} = 0$$

Non-linear conservation law

- Conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad f(u) = \frac{u^2}{2}$$

- Upwind finite volume scheme

$$h\frac{\mathrm{d}u_i}{\mathrm{d}t} + F_{i+1/2} - F_{i-1/2} = 0$$

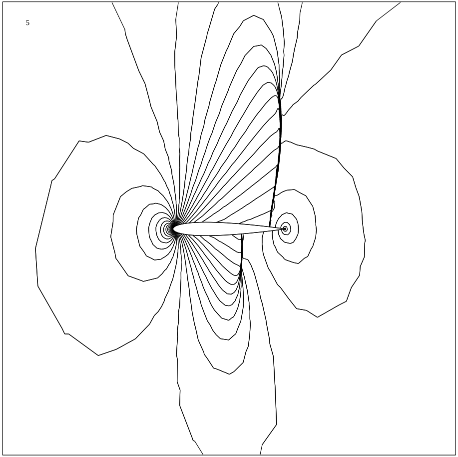
with

$$F_{i+1/2} = \begin{cases} \frac{1}{2}u_i^2 & \text{if } u_{i+1/2} \geq 0 \\ \frac{1}{2}u_{i+1}^2 & \text{if } u_{i+1/2} < 0 \end{cases}$$

- Approximate interface value, for example

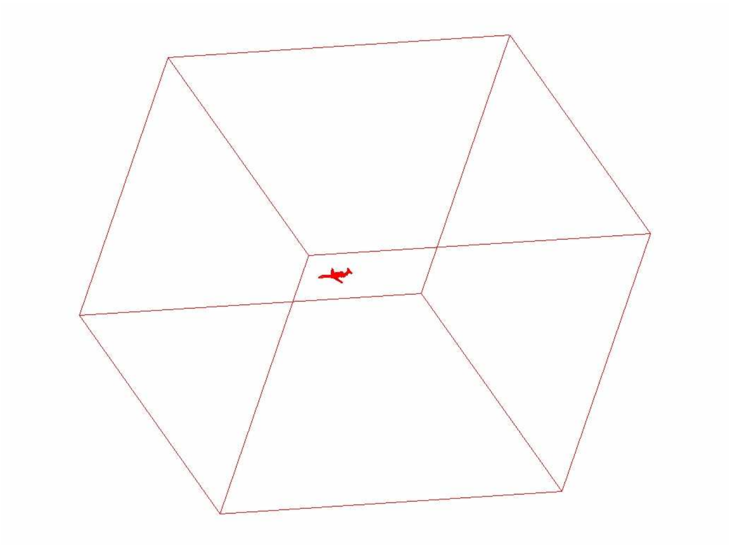
$$u_{i+1/2} = \frac{1}{2}(u_i + u_{i+1})$$

Transonic flow over NACA0012



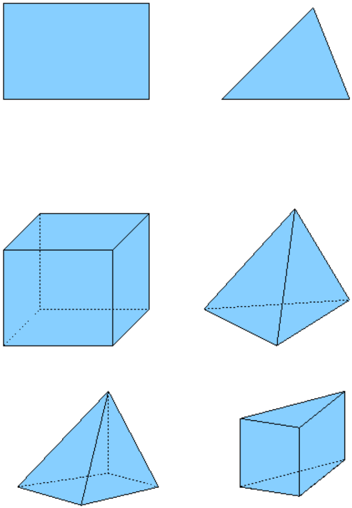
Outline

Computational domain

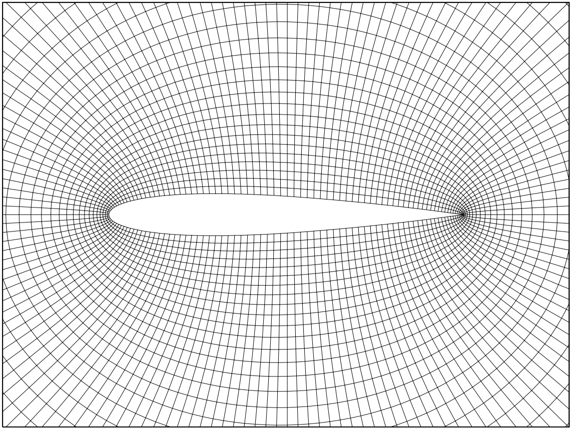


(Josy)

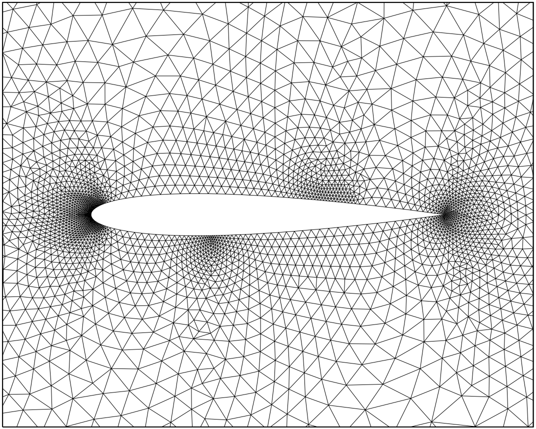
Grids and finite volumes



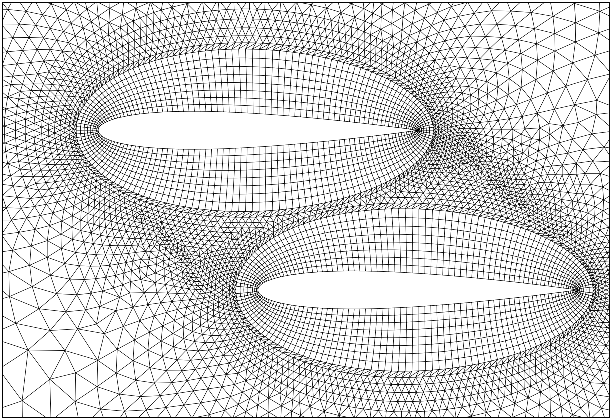
Structured grid



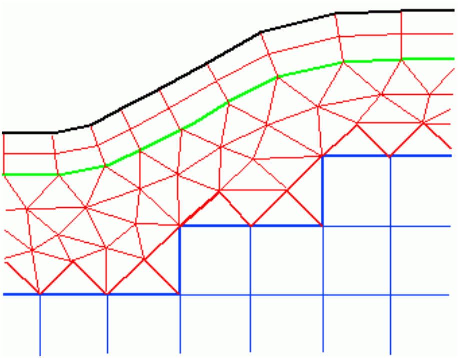
Unstructured grid



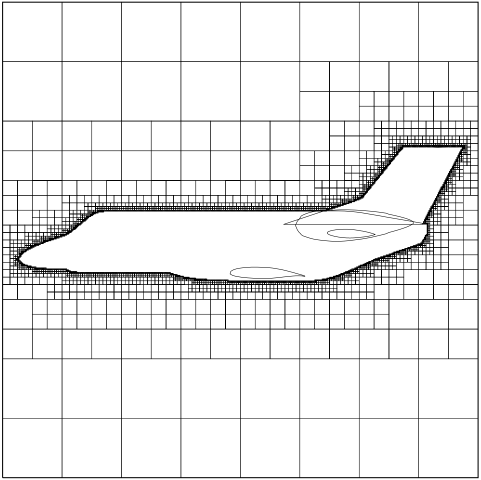
Unstructured hybrid grid



Unstructured hybrid grid

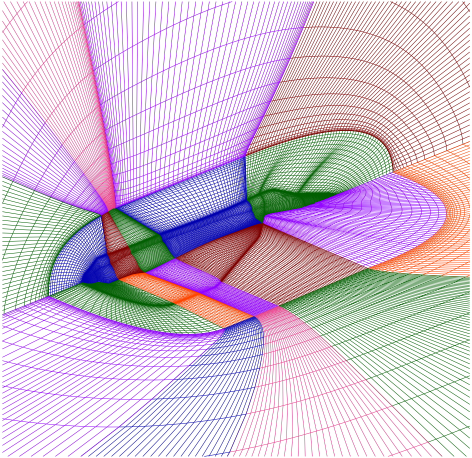


Cartesian grid



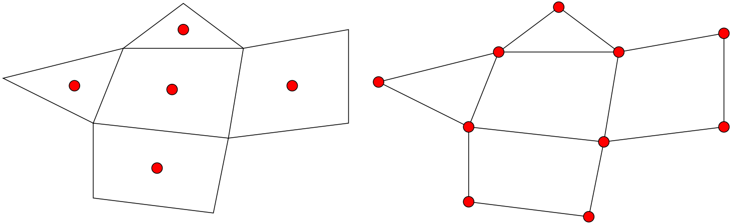
Saras (Josy)

Block-structured grid



Fighter aircraft (Nair)

Cell-centered and vertex-centered



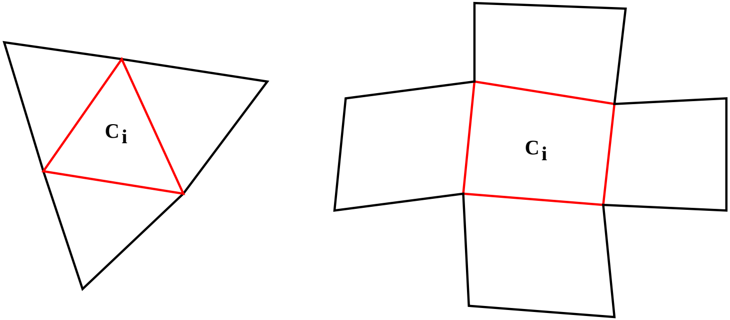
Outline

- Conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

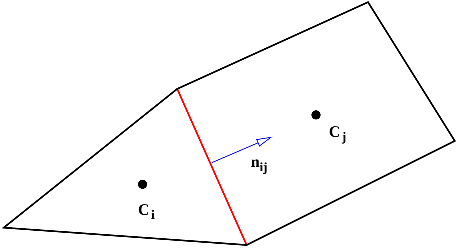
- Integral form

$$A_i \frac{du_i}{dt} + \oint_{\partial C_i} (f n_x + g n_y)\, dS = 0$$



FVM in 2-D

- Approximate flux across the cell face



$$F_{ij} \Delta S_{ij} \approx \int (f n_x + g n_y) \, \mathrm{d}S$$

- Finite volume approximation

$$A_i \frac{\mathrm{d}u_i}{\mathrm{d}t} + \sum_{j \in N(i)} F_{ij} \Delta S_{ij} = 0$$

Outline

Summary

- Integral form of conservation law
- Satisfies conservation of mass, momentum, energy, etc.
- Can capture discontinuities like shocks
- Can be applied on any type of grid
- Useful for complex geometry

Outline

For further reading I

- Blazek J., *Computational Fluid Dynamics: Principles and Applications*, Elsevier, 2004.
- Hirsch Ch., *Numerical Computation of Internal and External Flows*, Vol. 1 and 2, Wiley.
- LeVeque R. J., *Finite Volume Methods for Hyperbolic Equations*, Cambridge University Press, 2002.
- Wesseling P., *Principles of Computational Fluid Dynamics*, Springer, 2001.
- Ferziger J. H. and Peric M., *Computational methods for fluid dynamics*, 3'rd edition, Springer, 2003.
- `http://www.cfd-online.com/Wiki`