Divergence-free discontinuous Galerkin method for MHD and Maxwell's equations

Praveen Chandrashekar
praveen@math.tifrbng.res.in

Center for Applicable Mathematics
Tata Institute of Fundamental Research
Bangalore-560065, India
http://cpraveen.github.io

In collaboration with Dinshaw Balsara, Arijit Hazra, Rakesh Kumar

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Maxwell Equations

Linear hyperbolic system

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}$$

$\boldsymbol{B}$ = magnetic flux density

$\boldsymbol{E}$ = electric field

$\boldsymbol{D}$ = electric flux density

$\boldsymbol{H}$ = magnetic field

$\boldsymbol{J}$ = electric current density

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E} \qquad \mu, \varepsilon \in \mathbb{R}^{3\times 3} \text{ symmetric}$$

$$\begin{aligned} \varepsilon &= \text{permittivity tensor} \\ \mu &= \text{magnetic permeability tensor} \\ \sigma &= \text{conductivity} \end{aligned}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{D} = \rho \quad \text{(electric charge density)}, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Ideal MHD equations

Nonlinear hyperbolic system

Compressible Euler equations with Lorentz force

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (pI + \rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{B} \otimes \boldsymbol{B}) &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot ((E + p)\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}) &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) &= 0
\end{aligned}
$$

Magnetic monopoles do not exist: $\quad \Longrightarrow \quad \nabla \cdot \boldsymbol{B} = 0$

Divergence constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\nabla \cdot \nabla \times}_{=0} \boldsymbol{E} = 0$$

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{B} = 0$$

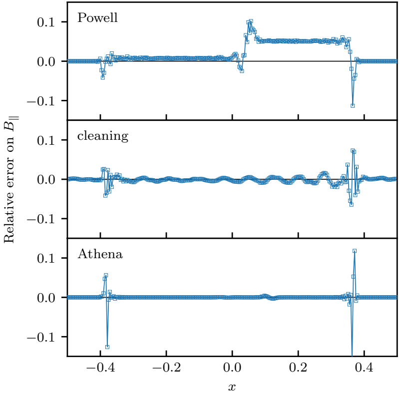
If

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{at } t = 0$$

then

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{for } t > 0$$

Rotated shock tube



Guillet et al., MNRAS 2019

Discrete div-free $\implies$ positivity (Kailiang)

Objectives

- Divergence-free schemes for Maxwell's and compressible MHD
  - Cartesian grids at present
  - Divergence-free reconstruction (BDM)
  - Divergence preserving schemes (RT)
- High order accurate
  - discontinuous-Galerkin
- Upwind-type schemes based on Riemann solvers
- Non-oscillatory schemes for MHD
- Explicit time stepping

Some existing methods

Exactly divergence-free methods

- Yee scheme (Yee (1966))
- Projection methods (Brackbill & Barnes (1980))
- Constrained transport (Evans & Hawley (1989))
- Divergence-free reconstruction (Balsara (2001))
- Globally divergence-free scheme (Li et al. (2011), Fu et al, (2018))

Approximate methods

- Locally divergence-free schemes (Cockburn, Li & Shu (2005))
- Godunov's symmetrized version of MHD (Powell, Gassner et al., C/K)
- Divergence cleaning methods (Dedner et al.)

Approximation of magnetic field

If $\nabla \cdot \boldsymbol{B} = 0$, it is natural to look for approximations in

$$H(div, \Omega) = \{\boldsymbol{B} \in L^2(\Omega) : \mathrm{div}(\boldsymbol{B}) \in L^2(\Omega)\}$$

To approximate functions in $H(div, \Omega)$ on a mesh $\mathcal{T}_h$ with piecewise polynomials, we need

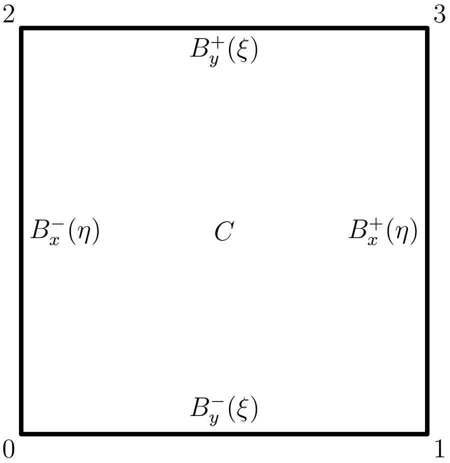
$$\boldsymbol{B} \cdot \boldsymbol{n}$$

continuous across element faces.

DG scheme based on divergence-free reconstruction of 2-D vector fields

Approximation spaces

$\boldsymbol{B} \cdot \boldsymbol{n}$ on the faces must be continuous: approximate $\boldsymbol{B} \cdot \boldsymbol{n}$ by polynomial functions $\mathbb{P}_k$ on the faces.



$$[\Delta x,\, \Delta y] \rightarrow [-\tfrac{1}{2}, +\tfrac{1}{2}] \times [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

$$\xi = \frac{x - x_c}{\Delta x}, \qquad \eta = \frac{y - y_c}{\Delta y}$$

$$B_x^{\pm}(\eta) = \sum_{j=0}^{k} a_j^{\pm} \phi_j(\eta)$$

$$B_y^{\pm}(\xi) = \sum_{j=0}^{k} b_j^{\pm} \phi_j(\xi)$$

Approximation spaces

$\phi_j : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ are mutually orthogonal polynomials; hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} B_x^{\pm}(\eta)\mathsf{d}\eta = a_0^{\pm}, \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} B_y^{\pm}(\xi)\mathsf{d}\xi = b_0^{\pm}$$

Given the data $B_x^{\pm}$, $B_y^{\pm}$ on the faces, we want to recontruct a divergence-free vector field $(B_x(\xi,\eta), B_y(\xi,\eta))$ inside the cell.

$$\text{Necessary condition:} \qquad \int_{\partial C} \boldsymbol{B}\cdot\boldsymbol{n} = \int_C \nabla\cdot\boldsymbol{B} = 0$$

Writing this condition in the reference coordinates yields

$$\Delta y \int_{-\frac{1}{2}}^{\frac{1}{2}} (B_x^+(\eta) - B_x^-(\eta))\mathsf{d}\eta + \Delta x \int_{-\frac{1}{2}}^{\frac{1}{2}} (B_y^+(\xi) - B_y^-(\xi))\mathsf{d}\xi = 0$$

or

$$\boxed{\int_{\partial C} \boldsymbol{B}\cdot\boldsymbol{n} = (a_0^+ - a_0^-)\Delta y + (b_0^+ - b_0^-)\Delta x = 0}$$

Raviart-Thomas (RT) polynomials

Space of tensor product polynomials

$$\mathbb{Q}_{r,s} = \mathrm{span}\{\xi^i \eta^j : 0 \le i \le r, 0 \le j \le s\}$$

For $k \ge 0$, approximate the vector field inside the cells by tensor product polynomials

$$B_x(\xi, \eta) = \sum_{i=0}^{k+1} \sum_{j=0}^{k} a_{ij} \phi_i(\xi) \phi_j(\eta) \in \mathbb{Q}_{k+1,k}$$

$$B_y(\xi, \eta) = \sum_{i=0}^{k} \sum_{j=0}^{k+1} b_{ij} \phi_i(\xi) \phi_j(\eta) \in \mathbb{Q}_{k,k+1}$$

$$\mathrm{RT}(k) = \mathbb{Q}_{k+1,k} \times \mathbb{Q}_{k,k+1}, \qquad \dim\ \mathrm{RT}(k) = 2(k+1)(k+2)$$

$$\mathrm{div}\ \mathrm{RT}(k) \in \mathbb{Q}_{k,k}$$

Brezzi-Douglas-Marini (BDM) polynomials

The BDM polynomials are of the form (Brezzi & Fortin, Section III.3.2)

$$\mathrm{BDM}(k) = (\mathbb{P}_k)^2 + r\nabla \times (x^{k+1}y) + s\nabla \times (xy^{k+1})$$

$$\dim\, \mathrm{BDM}(k) = (k+1)(k+2)+2, \qquad \mathrm{div}\, \mathrm{BDM}(k) \in \mathbb{P}_{k-1}$$

In reference coordinates, we take the polynomials to be of the form

$$B_x(\xi,\eta) = \sum_{\substack{i,j=0\\ i+j\le k}}^{k} a_{ij}\phi_i(\xi)\phi_j(\eta) + r\frac{(\xi^{k+1}-M_{k+1})}{\Delta y} + s\frac{(k+1)\xi\eta^k}{\Delta y}$$

$$B_y(\xi,\eta) = \sum_{\substack{i,j=0\\ i+j\le k}}^{k} b_{ij}\phi_i(\xi)\phi_j(\eta) - r\frac{(k+1)\xi^k\eta}{\Delta x} - s\frac{(\eta^{k+1}-M_{k+1})}{\Delta x}$$

Brezzi-Douglas-Fortin-Marini (BDFM) polynomials

The BDFM polynomials are of the form

$$\mathrm{BDFM}(k) = (\mathbb{P}_{k+1})^2 \setminus (0, x^{k+1}) \setminus (y^{k+1}, 0)$$

$$\dim\,\mathrm{BDFM}(k) = (k+2)(k+3) - 2, \qquad \mathrm{div}\,\mathrm{BDFM}(k) \in \mathbb{P}_k$$

In reference cell coordinates, the polynomials can be written as

$$B_x(\xi,\eta) = \sum_{i=0}^{k+1}\sum_{j=0}^{k-i} a_{ij}\phi_i(\xi)\phi_j(\eta), \qquad B_y(\xi,\eta) = \sum_{j=0}^{k+1}\sum_{i=0}^{k-j} b_{ij}\phi_i(\xi)\phi_j(\eta)$$

Remark: Raviart-Thomas space is the largest space considered here, and in fact we have

$$\mathrm{BDM}(k) \subset \mathrm{BDFM}(k) \subset \mathrm{RT}(k)$$

Conditions for the reconstruction

For all the three polynomial spaces

$$B_x(\pm\tfrac{1}{2}, \eta) \in \mathbb{P}_k \qquad \text{and} \qquad B_y(\xi, \pm\tfrac{1}{2}) \in \mathbb{P}_k$$

Match $\boldsymbol{B}(\xi, \eta)$ to $\boldsymbol{B} \cdot \boldsymbol{n}$ on the four faces

$$B_x(\pm\tfrac{1}{2}, \eta) = B_x^{\pm}(\eta) \quad \forall \eta \in [-\tfrac{1}{2}, +\tfrac{1}{2}], \qquad B_y(\xi, \pm\tfrac{1}{2}) = B_y^{\pm}(\xi) \quad \forall \xi \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

In addition, we have to make the vector field divergence-free

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\Delta x} \frac{\partial B_x}{\partial \xi} + \frac{1}{\Delta y} \frac{\partial B_y}{\partial \eta} = 0, \qquad \forall\ \xi, \eta \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

Questions:

- Do we have enough equations ? (Not always)
- Can we solve them ? (Yes)

Approach:

- All three spaces contain $\mathbb{P}_k$, necessary to obtain h^{k+1}'th order accuracy.
- Throw away any basis function outside $\mathbb{P}_k$ without affecting the order of accuracy.

Summary of reconstruction

- All three spaces lead to same solution: $\mathrm{BDM}(\mathsf{k})$
- $k = 0, 1, 2$: reconstruction determined by face solution alone
- $k \geq 3$: Require additional information to solve reconstruction problem

Reconstruction at degree $k = 3$

The face solution is a polynomial of degree three and is of the form

Right/left face

$$B_x^{\pm}(\eta) = a_0^{\pm}\phi_0(\eta) + a_1^{\pm}\phi_1(\eta) + a_2^{\pm}\phi_2(\eta) + a_3^{\pm}\phi_3(\eta)$$

Top/bottom face

$$B_y^{\pm}(\eta) = b_0^{\pm}\phi_0(\xi) + b_1^{\pm}\phi_1(\xi) + b_2^{\pm}\phi_2(\xi) + b_3^{\pm}\phi_3(\xi)$$

Reconstruction using $\mathrm{BDFM}(3)$

The vector field has the form

$$\begin{aligned}B_x(\xi,\eta) = {} & a_{00} + a_{10}\xi + a_{01}\eta + a_{20}(\xi^2 - \tfrac{1}{12}) + a_{11}\xi\eta + a_{02}(\eta^2 - \tfrac{1}{12}) + \\ & a_{30}(\xi^3 - \tfrac{3}{20}\xi) + a_{21}(\xi^2 - \tfrac{1}{12})\eta + a_{12}\xi(\eta^2 - \tfrac{1}{12}) + a_{03}(\eta^3 - \tfrac{3}{20}\eta) + \\ & a_{40}(\xi^4 - \tfrac{3}{14}\xi^2 + \tfrac{3}{560}) + a_{13}\xi(\eta^3 - \tfrac{3}{20}\eta) + a_{31}(\xi^3 - \tfrac{3}{20}\xi)\eta + \\ & a_{22}(\xi^2 - \tfrac{1}{12})(\eta^2 - \tfrac{1}{12}) \in \mathbb{P}_4 \setminus \{y^4\}\end{aligned}$$

$$\begin{aligned}B_y(\xi,\eta) = {} & b_{00} + b_{10}\xi + b_{01}\eta + b_{20}(\xi^2 - \tfrac{1}{12}) + b_{11}\xi\eta + b_{02}(\eta^2 - \tfrac{1}{12}) + \\ & b_{30}(\xi^3 - \tfrac{3}{20}\xi) + b_{21}(\xi^2 - \tfrac{1}{12})\eta + b_{12}\xi(\eta^2 - \tfrac{1}{12}) + b_{03}(\eta^3 - \tfrac{3}{20}\eta) + \\ & b_{22}(\xi^2 - \tfrac{1}{12})(\eta^2 - \tfrac{1}{12}) + b_{13}\xi(\eta^3 - \tfrac{3}{20}\eta) + b_{31}(\xi^3 - \tfrac{3}{20}\xi)\eta + \\ & b_{04}(\eta^4 - \tfrac{3}{14}\eta^2 + \tfrac{3}{560}) \in \mathbb{P}_4 \setminus \{x^4\}\end{aligned}$$

which has a total of 28 coefficients . The divergence-free condition gives 9 equations and matching the face solution gives 16 equations for a total of 26 equations .

Reconstruction using $\mathrm{BDFM}(3)$

We can first solve for some of the coefficients completely in terms of the face solution

$$
\begin{aligned}
a_{00} &= \frac{1}{2}(a_0^- + a_0^+) + \frac{1}{12}(b_1^+ - b_1^-)\frac{\Delta x}{\Delta y} \\
a_{10} &= a_0^+ - a_0^- + \frac{1}{30}(b_2^+ - b_2^-)\frac{\Delta x}{\Delta y} \\
a_{20} &= -\frac{1}{2}(b_1^+ - b_1^-)\frac{\Delta x}{\Delta y} + \frac{3}{140}(b_3^+ - b_3^-)\frac{\Delta x}{\Delta y} \\
a_{30} &= -\frac{1}{3}(b_2^+ - b_2^-)\frac{\Delta x}{\Delta y} \\
a_{03} &= \frac{1}{2}(a_3^- + a_3^+) \\
a_{12} &= a_2^+ - a_2^- \\
a_{40} &= -\frac{1}{4}(b_3^+ - b_3^-)\frac{\Delta x}{\Delta y} \\
a_{13} &= a_3^+ - a_3^-
\end{aligned}
$$

$$
\begin{aligned}
b_{00} &= \frac{1}{2}(b_0^- + b_0^+) + \frac{1}{12}(a_1^+ - a_1^-)\frac{\Delta y}{\Delta x} \\
b_{01} &= b_0^+ - b_0^- + \frac{1}{30}(a_2^+ - a_2^-)\frac{\Delta y}{\Delta x} \\
b_{02} &= -\frac{1}{2}(a_1^+ - a_1^-)\frac{\Delta y}{\Delta x} + \frac{3}{140}(a_3^+ - a_3^-)\frac{\Delta y}{\Delta x} \\
b_{30} &= \frac{1}{2}(b_3^- + b_3^+) \\
b_{03} &= -\frac{1}{3}(a_2^+ - a_2^-)\frac{\Delta y}{\Delta x} \\
b_{21} &= b_2^+ - b_2^- \\
b_{31} &= b_3^+ - b_3^- \\
b_{04} &= -\frac{1}{4}(a_3^+ - a_3^-)\frac{\Delta y}{\Delta x}
\end{aligned}
$$

Reconstruction using $\mathrm{BDFM}(3)$

The remaining coefficients satisfy the following equations

$$\begin{aligned} a_{01} + \frac{1}{6} a_{21} &= \frac{1}{2}(a_1^- + a_1^+) \\ a_{02} + \frac{1}{6} a_{22} &= \frac{1}{2}(a_2^- + a_2^+) \\ b_{10} + \frac{1}{6} b_{12} &= \frac{1}{2}(b_1^- + b_1^+) \\ b_{20} + \frac{1}{6} b_{22} &= \frac{1}{2}(b_2^- + b_2^+) \end{aligned}$$

$$\begin{aligned} 15 a_{11} \Delta y - b_{22} \Delta x &= 15(a_1^+ - a_1^-)\Delta y \\ 15 b_{11} \Delta x - a_{22} \Delta y &= 15(b_1^+ - b_1^-)\Delta x \\ b_{12} \Delta x + a_{21} \Delta y &= 0 \\ 3 a_{31} \Delta y + 2 b_{22} \Delta x &= 0 \\ 3 b_{13} \Delta x + 2 a_{22} \Delta y &= 0 \end{aligned}$$

More unknowns than equations; set the fourth order terms to zero since they are not required to get fourth order accuracy,

$$a_{31} = a_{22} = b_{13} = b_{22} = 0$$

Then we get

$$a_{02} = \frac{1}{2}(a_2^- + a_2^+), \quad b_{20} = \frac{1}{2}(b_2^- + b_2^+), \quad a_{11} = a_1^+ - a_1^-, \quad b_{11} = b_1^+ - b_1^-$$

Reconstruction using $\mathrm{BDFM}(3)$

and the remaining unknowns satisfy

$$
\begin{aligned}
a_{01} + \frac{1}{6} a_{21} &= \frac{1}{2}(a_1^- + a_1^+) =: r_1 \\
b_{10} + \frac{1}{6} b_{12} &= \frac{1}{2}(b_1^- + b_1^+) =: r_2 \\
b_{12}\Delta x + a_{21}\Delta y &= 0
\end{aligned}
$$

We have four unknowns but only three equations. All unknowns at degree 3 and below. Let us assume that we know the value of ω such that

$$
b_{10} - a_{01} = \omega
$$

ω provides information about the curl of the vector field in the cell

$$
\omega = \frac{\partial B_y}{\partial \xi}(0,0) - \frac{\partial B_x}{\partial \eta}(0,0)
$$

Reconstruction using $\mathrm{BDFM}(3)$

Then we can solve for remaining coefficients to obtain

$$
\begin{aligned}
a_{01} &= \frac{1}{1+\frac{\Delta y}{\Delta x}}\left[r_1\frac{\Delta y}{\Delta x}+r_2-\omega\right] \\
b_{10} &= \omega + a_{01} \\
a_{21} &= 6(r_1 - a_{01}) \\
b_{12} &= 6(r_2 - b_{10})
\end{aligned}
$$

$B_x(\eta)$, $B_y(\xi)$ **on faces,** ω **in cell**; ω is a dof which we will evolve in time using the induction/Maxwell's equation.

4'th order scheme of Balsara

- $\boldsymbol{B}$ inside cells

$$B_x \in (\mathbb{P}_4 \setminus \{y^4\}) + \mathrm{span}\{x^4 y, x^2 y^3\}$$

$$B_y \in (\mathbb{P}_4 \setminus \{x^4\}) + \mathrm{span}\{x y^4, x^3 y^2\}$$

- a_{21} and b_{12} determined using WENO
- $a_{41} = b_{14}$ and $a_{23} = b_{32}$ (*energy minimzation*)

Reconstruction at degree $k = 4$

- On the faces: $B_x^\pm(\eta) \in \mathbb{P}_4$, $B_y^\pm(\xi) \in \mathbb{P}_4$
- Need three extra information

$$b_{10} - a_{01} = \omega_1, \qquad b_{20} - a_{11} = \omega_2, \qquad b_{11} - a_{02} = \omega_3$$

correspond to curl and its gradient

$$\omega = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}, \qquad \frac{\partial \omega}{\partial x}, \qquad \frac{\partial \omega}{\partial y}$$

Enough to perform divergence-free reconstruction[1]

[1] See https://arxiv.org/abs/1809.03816

Discontinuous Galerkin method

Let us first consider B_x which is stored on the vertical faces $e_x^{\mp}$.

$$\Delta y \frac{\partial B_x}{\partial t} + \frac{\partial E}{\partial \eta} = 0$$

Multiply by test function $\phi_i(\eta)$, $i = 0, 1, \ldots, k$, and integrate over the vertical face (Balsara & Käppeli (2017))

$$\Delta y \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial B_x^+}{\partial t} \phi_i \mathsf{d}\eta - \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{x+} \frac{\partial \phi_i}{\partial \eta} \mathsf{d}\eta + (\tilde{E}\phi_i)_3 - (\tilde{E}\phi_i)_1 = 0$$

$$\Delta y \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial B_x^-}{\partial t} \phi_i \mathsf{d}\eta - \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{x-} \frac{\partial \phi_i}{\partial \eta} \mathsf{d}\eta + (\tilde{E}\phi_i)_2 - (\tilde{E}\phi_i)_0 = 0$$

$$\hat{E}^{x\pm} = \text{Numerical flux on face from 1-D Riemann solver}$$

$$\tilde{E} = \text{Numerical flux at vertex from 2-D Riemann solver}$$

Discontinuous Galerkin method

Similarly, the B_y component stored on the horizontal faces,

$$\Delta x \frac{\partial B_y}{\partial t} - \frac{\partial E}{\partial \xi} = 0$$

multiply this by a test function $\phi_i(\xi)$, $i = 0, 1, \ldots, k$, and integrate over the horizontal edge

$$\Delta x \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial B_y^+}{\partial t} \phi_i \mathsf{d}\xi + \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{y+} \frac{\partial \phi_i}{\partial \xi} \mathsf{d}\xi - (\tilde{E}\phi_i)_3 + (\tilde{E}\phi_i)_2 = 0$$

$$\Delta x \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial B_y^-}{\partial t} \phi_i \mathsf{d}\xi + \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{y-} \frac{\partial \phi_i}{\partial \xi} \mathsf{d}\xi - (\tilde{E}\phi_i)_1 + (\tilde{E}\phi_i)_0 = 0$$

Fourth order scheme

$$\omega = b_{10} - a_{01} = 12 \int \int (B_y \xi - B_x \eta) \mathsf{d}\xi \mathsf{d}\eta$$

Evolution equation for ω using the induction equation

$$\frac{1}{12}\frac{\mathsf{d}\omega}{\mathsf{d}t} = \int \int \left(\frac{\partial B_y}{\partial t}\xi - \frac{\partial B_x}{\partial t}\eta \right) \mathsf{d}\xi \mathsf{d}\eta = \int \int \left(\frac{1}{\Delta x}\frac{\partial E}{\partial \xi}\xi + \frac{1}{\Delta y}\frac{\partial E}{\partial \eta}\eta \right) \mathsf{d}\xi \mathsf{d}\eta$$

Integrate by parts, use numerical flux on faces from a 1-D Riemann solver

$$\begin{aligned} \frac{1}{12}\frac{\mathsf{d}\omega}{\mathsf{d}t} \quad = \quad & \frac{1}{\Delta x}\left[\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{x-} \mathsf{d}\eta + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{x+} \mathsf{d}\eta - \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} E \mathsf{d}\xi \mathsf{d}\eta \right] \\ & + \frac{1}{\Delta y}\left[\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{y-} \mathsf{d}\xi + \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{E}^{y+} \mathsf{d}\xi - \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} E \mathsf{d}\xi \mathsf{d}\eta \right] \end{aligned}$$

Divergence-free property

From the DG equations

$$\Delta y \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial t}(B_x^+ - B_x^-)\mathsf{d}\eta + \Delta x \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial t}(B_y^+ - B_y^-)\mathsf{d}\xi + \tilde{E}_3 - \tilde{E}_1 - \tilde{E}_2 + \tilde{E}_0 - \tilde{E}_3 + \tilde{E}_2 + \tilde{E}_1 - \tilde{E}_0 = 0$$

which simplifies to

$$\frac{\mathsf{d}}{\mathsf{d}t}\int_{\partial C} \boldsymbol{B}\cdot\boldsymbol{n} = \frac{\mathsf{d}}{\mathsf{d}t}[(a_0^+ - a_0^-)\Delta y + (b_0^+ - b_0^-)\Delta x] = 0$$

$$\Downarrow$$

$$\int_{\partial C} \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ at } t = 0 \quad \overset{DG}{\Longrightarrow} \quad \int_{\partial C} \boldsymbol{B}\cdot\boldsymbol{n} = 0 \text{ for } t > 0$$

Conservation property

Is the 2-D conservation law satisfied for $\boldsymbol{B}$?

$$\frac{1}{\Delta x \Delta y}\int_C B_x \mathsf{d}x\mathsf{d}y = a_{00} = \frac{1}{2}(a_0^- + a_0^+) + \frac{1}{12}(b_1^+ - b_1^-)\frac{\Delta x}{\Delta y}$$

The mean value a_{00} in the cell evolves as

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}t}a_{00} =& -\frac{\tilde{E}_2 - \tilde{E}_0}{2\Delta y} - \frac{\tilde{E}_3 - \tilde{E}_1}{2\Delta y} - \frac{1}{\Delta x \Delta y}\int_{top}\hat{E}^{y+}\mathsf{d}x + \frac{\tilde{E}_3 + \tilde{E}_2}{2\Delta y} + \frac{1}{\Delta x \Delta y}\int_{bot}\hat{E}^{y-}\mathsf{d}x - \frac{\tilde{E}_1 + \tilde{E}_0}{2\Delta y}\\ =& -\frac{1}{\Delta x \Delta y}\int_{top}\hat{E}^{y+}\mathsf{d}x + \frac{1}{\Delta x \Delta y}\int_{bot}\hat{E}^{y-}\mathsf{d}x\end{aligned}$$

which is just the 2-D conservation law for B_x.

Numerical results: 2-D Maxwell equation

$$
\begin{aligned}
\frac{\partial D_x}{\partial t} - \frac{\partial H_z}{\partial y} &= 0 \qquad (1)\\
\frac{\partial D_y}{\partial t} + \frac{\partial H_z}{\partial x} &= 0 \qquad (2)\\
\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0 \qquad (3)
\end{aligned}
$$

where

$$
(E_x, E_y) = \frac{1}{\varepsilon(x,y)}(D_x, D_y), \qquad H_z = \frac{1}{\mu(x,y)} B_z
$$

Divergence-free constraint: $\nabla \cdot \boldsymbol{D} = 0$

Flux reconstruction on faces for D_x, D_y, and DG scheme on cells for B_z

Riemann solvers to estimate numerical fluxes at faces/vertices

Plane wave propagation in vacuum

$N_x \times N_y$	$\|\boldsymbol{D}^h - \boldsymbol{D}\|_{L^1}$	Ord	$\|\boldsymbol{D}^h - \boldsymbol{D}\|_{L^2}$	Ord	$\|B_z^h - B_z\|_{L^1}$	Ord	$\|B_z^h - B_z\|_{L^2}$
16×16	3.0557e-07	—	3.5095e-07	—	1.9458e-04	—	2.5101e-04
32×32	1.1040e-08	4.79	1.3428e-08	4.71	1.1275e-05	4.11	1.4590e-05
64×64	5.0469e-10	4.45	6.1548e-10	4.45	6.7924e-07	4.05	8.9228e-07
128×128	2.6834e-11	4.23	3.3945e-11	4.18	4.1802e-08	4.02	5.5449e-08

Table: Plane wave test, degree$=3$: convergence of error

$N_x \times N_y$	$\|\boldsymbol{D}^h - \boldsymbol{D}\|_{L^1}$	Ord	$\|\boldsymbol{D}^h - \boldsymbol{D}\|_{L^2}$	Ord	$\|B_z^h - B_z\|_{L^1}$	Ord	$\|B_z^h - B_z\|_{L^2}$
8×8	2.4982e-07	—	2.8435e-07	—	2.5514e-04	—	3.2612e-04
16×16	6.3357e-09	5.30	7.2655e-09	5.29	6.4340e-06	5.31	8.5532e-06
32×32	1.7113e-10	5.21	2.0807e-10	5.13	1.9021e-07	5.08	2.5521e-07
64×64	5.1213e-12	5.06	6.4133e-12	5.02	5.9026e-09	5.01	7.9601e-09
128×128	1.5949e-13	5.00	2.0054e-13	5.00	1.8412e-10	5.00	2.4890e-10

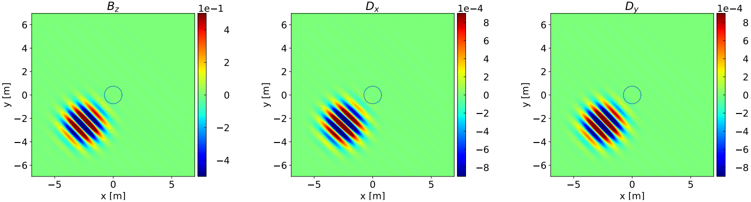
Table: Plane wave test, degree$=4$: convergence of error

$$\text{Error} = O(h^{k+1})$$

Refraction of Gaussian pulse by disc

$\epsilon_r = 5 - 4\tanh((\sqrt{x^2+y^2} - 0.75)/0.08) \in [1, 9]$, radius = 0.75 m

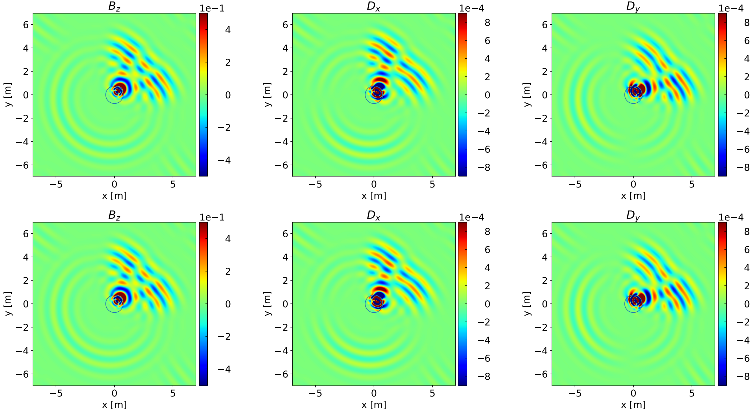
Initial condition, $\lambda = 1.5m$



Refraction of gaussian pulse by disc

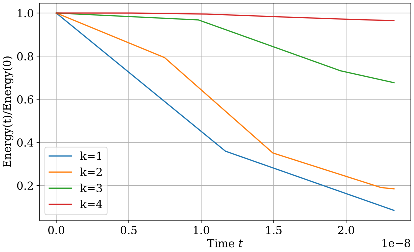
$\epsilon_r = 5 - 4\tanh((\sqrt{x^2+y^2} - 0.75)/0.08) \in [1, 9]$

$$t = 23.3\ ns$$

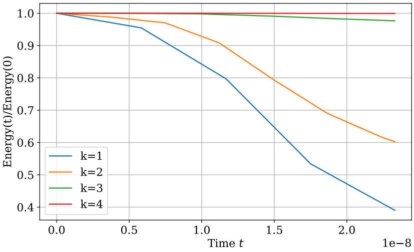


200×200 cells. Top row: $k = 3$, bottom row: $k = 4$

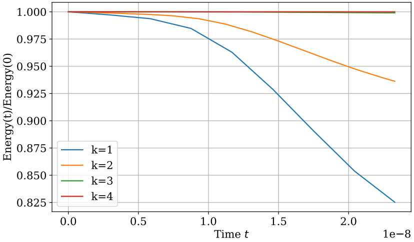
Refraction of gaussian pulse by disc: energy conservation



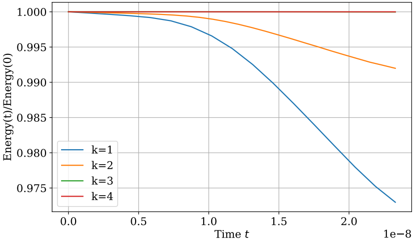
100×100



200×200



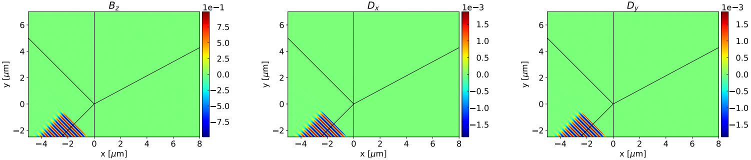
400×400



800×800

Refraction of plane wave: $\varepsilon_0 \leq \varepsilon \leq 2.25\varepsilon_0$

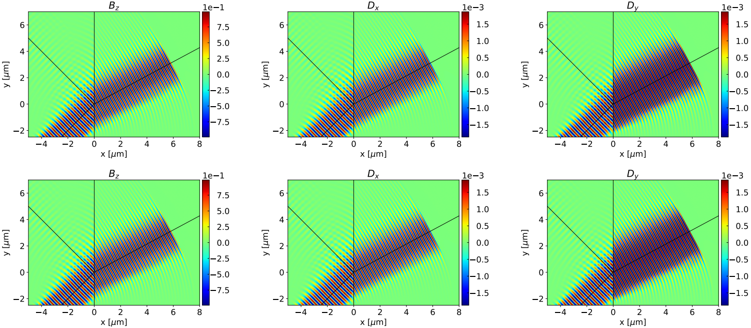
$\epsilon_r = 1.625 + 0.625\tanh(x/10^{-8})$, $\lambda = 0.5\mu m$



Initial condition

Refraction of plane wave: $\varepsilon_0 \leq \varepsilon \leq 2.25\varepsilon_0$

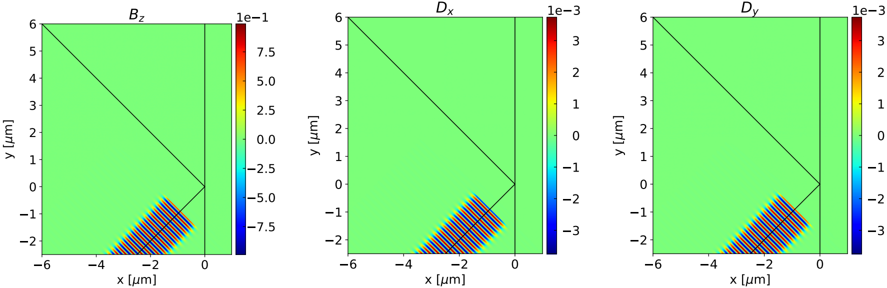
$\epsilon_r = 1.625 + 0.625 \tanh(x/10^{-8})$, $\lambda = 0.5\mu m$



650×475 cells. $t = 4 \times 10^{-14} s$. Top row: $k = 3$, bottom row: $k = 4$

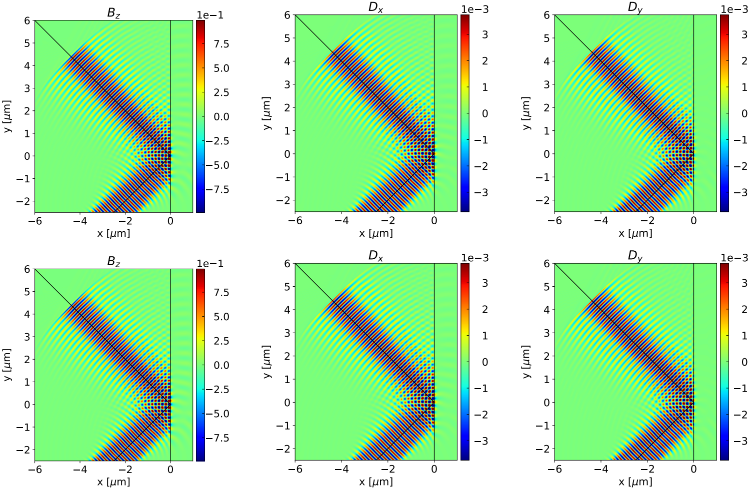
Total internal reflection of plane wave: $4\varepsilon_0 \geq \varepsilon \geq \varepsilon_0$

$\epsilon_r = 2.5 - 1.5\tanh(x/4\times10^{-8}),\ \lambda = 0.3\mu m$



Initial condition

Total internal reflection of plane wave: $4\varepsilon_0 \geq \varepsilon \geq \varepsilon_0$



350×425 cells. $t = 5 \times 10^{-14} s$. Top row: $k = 3$, bottom row: $k = 4$

DG scheme for MHD using Raviart-Thomas polynomials[2]

[2] Based on J. Sci. Comp., 2019, `https://doi.org/10.1007/s10915-018-0841-4`

MHD equations in 2-D

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} = 0$$

$$\mathcal{U} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z \end{bmatrix}, \quad \mathcal{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x (\boldsymbol{v} \cdot \mathfrak{B}) \\ 0 \\ -E_z \\ v_x B_z - v_z B_x \end{bmatrix}, \quad \mathcal{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y (\boldsymbol{v} \cdot \mathfrak{B}) \\ E_z \\ 0 \\ v_y B_z - v_z B_y \end{bmatrix}$$

Split into two parts

$$\boldsymbol{U} = [\rho,\ \rho \boldsymbol{v},\ \mathcal{E},\ B_z]^\top, \qquad \boldsymbol{B} = (B_x, B_y)$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \boldsymbol{B}) = 0, \qquad \frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0$$

MHD equations in 2-D

E_z is the electric field in the z direction

$$E_z = -(\boldsymbol{v} \times \boldsymbol{B})_z = v_y B_x - v_x B_y$$

The fluxes $\boldsymbol{F} = (\boldsymbol{F}_x, \boldsymbol{F}_y)$ are of the form

$$\boldsymbol{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x (\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ v_x B_z - v_z B_x \end{bmatrix}, \qquad \boldsymbol{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y (\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ v_y B_z - v_z B_y \end{bmatrix}$$

where

$$\boldsymbol{\mathfrak{B}} = (B_x, B_y, B_z), \qquad P = p + \frac{1}{2}|\boldsymbol{\mathfrak{B}}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\boldsymbol{\mathfrak{B}}|^2$$

Approximation spaces

Hydrodynamic variables

$$\boldsymbol{U}(\xi, \eta) = \sum_{i=0}^{k} \sum_{j=0}^{k} \boldsymbol{U}_{ij} \phi_i(\xi) \phi_j(\eta) \in \mathbb{Q}_{k,k}$$

Normal component of $\boldsymbol{B}$ on faces

$$\text{on vertical faces} : b_x(\eta) = \sum_{j=0}^{k} a_j \phi_j(\eta) \in \mathbb{P}_k(\eta)$$

$$\text{on horizontal faces} : b_y(\xi) = \sum_{j=0}^{k} b_j \phi_j(\xi) \in \mathbb{P}_k(\xi)$$

Approximation spaces

For $k \geq 1$,define certain *cell moments*

$$\alpha_{ij} = \alpha_{ij}(B_x) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta)\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta, \quad 0 \leq i \leq k-1, \quad 0 \leq j \leq k$$

$$\beta_{ij} = \beta_{ij}(B_y) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta)\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta, \quad 0 \leq i \leq k, \quad 0 \leq j \leq k-1$$

$$m_{ij} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)\phi_j(\eta)]^2 \mathsf{d}\xi\mathsf{d}\eta = m_i m_j$$

α_{00}, β_{00} are cell averages of B_x, B_y

Solution variables

$$\{\boldsymbol{U}(\xi,\eta)\}, \qquad \{b_x(\eta)\}, \qquad \{b_y(\xi)\}, \qquad \{\alpha,\beta\}$$

RT reconstruction

Given $b_x^\pm(\eta) \in \mathbb{P}_k$ and $b_y^\pm(\xi) \in \mathbb{P}_k$, and also the set of cell moments

$$\{\alpha_{ij},\ 0 \le i \le k-1,\ 0 \le j \le k\}, \qquad \{\beta_{ij},\ 0 \le i \le k,\ 0 \le j \le k-1\}$$

Find $B_x \in \mathbb{Q}_{k+1,k}$ and $B_y \in \mathbb{Q}_{k,k+1}$ such that

$$B_x(\pm\tfrac{1}{2}, \eta) = b_x^\pm(\eta), \quad \eta \in [-\tfrac{1}{2}, \tfrac{1}{2}], \qquad B_y(\xi, \pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \xi \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

$$\frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta)\phi_i(\xi)\phi_j(\eta) \mathrm{d}\xi \mathrm{d}\eta = \alpha_{ij}, \qquad 0 \le i \le k-1, \quad 0 \le j \le k$$

$$\frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta)\phi_i(\xi)\phi_j(\eta) \mathrm{d}\xi \mathrm{d}\eta = \beta_{ij}, \qquad 0 \le i \le k, \quad 0 \le j \le k-1$$

1. There is a unique solution.
2. If the data comes from a divergence-free field, then the reconstructed field is also divergence-free.

DG for $\boldsymbol{B}$ on faces

On every vertical face of the mesh

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_x}{\partial t}\phi_i \mathsf{d}\eta - \frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\eta}\mathsf{d}\eta + \frac{1}{\Delta y}[\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

On every horizontal face of the mesh

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_y}{\partial t}\phi_i \mathsf{d}\xi + \frac{1}{\Delta x}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\xi}\mathsf{d}\xi - \frac{1}{\Delta x}[\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

$\hat{E}_z$: on face, 1-D Riemann solver

$\tilde{E}_z$: at vertex, 2-D Riemann solver

DG for $\boldsymbol{B}$ on cells

$$
\begin{aligned}
m_{ij} \frac{\mathsf{d}\alpha_{ij}}{\mathsf{d}t} &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial B_x}{\partial t} \phi_i(\xi)\phi_j(\eta) \mathsf{d}\xi \mathsf{d}\eta \\
&= -\frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial E_z}{\partial \eta} \phi_i(\xi)\phi_j(\eta) \mathsf{d}\xi \mathsf{d}\eta \\
&= -\frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\hat{E}_z(\xi, \tfrac{1}{2})\phi_i(\xi)\phi_j(\tfrac{1}{2}) - \hat{E}_z(\xi, -\tfrac{1}{2})\phi_i(\xi)\phi_j(-\tfrac{1}{2})] \mathsf{d}\xi \\
&\quad + \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} E_z(\xi, \eta) \phi_i(\xi) \phi_j'(\eta) \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \le i \le k-1, \quad 0 \le j \le k
\end{aligned}
$$

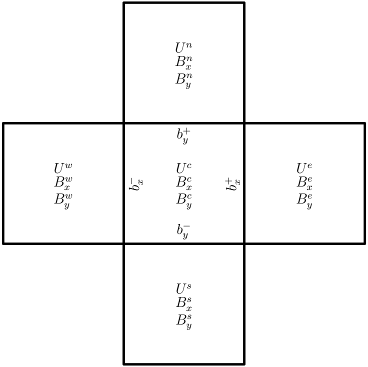
Not a Galerkin method, test functions ($\mathbb{Q}_{k-1,k}$) different from trial functions ($\mathbb{Q}_{k+1,k}$)

DG for $\boldsymbol{U}$ on cells

For $\Phi_i \in \mathbb{Q}_{k,k}$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial \boldsymbol{U}^c}{\partial t} \Phi_i(\xi,\eta) \mathsf{d}\xi \mathsf{d}\eta - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[\frac{1}{\Delta x} \boldsymbol{F}_x \frac{\partial \Phi_i}{\partial \xi} + \frac{1}{\Delta y} \boldsymbol{F}_y \frac{\partial \Phi_i}{\partial \eta} \right] \mathsf{d}\xi \mathsf{d}\eta$$
$$+ \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_x^+ \Phi_i(\tfrac{1}{2}, \eta) \mathsf{d}\eta - \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_x^- \Phi_i(-\tfrac{1}{2}, \eta) \mathsf{d}\eta$$
$$+ \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_y^+ \Phi_i(\xi, \tfrac{1}{2}) \mathsf{d}\xi - \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_y^- \Phi_i(\xi, -\tfrac{1}{2}) \mathsf{d}\xi = 0$$

DG for $\boldsymbol{U}$ on cells



$$\boldsymbol{F}_x = \boldsymbol{F}_x(\boldsymbol{U}^c, B_x^c, B_y^c), \qquad \boldsymbol{F}_y = \boldsymbol{F}_y(\boldsymbol{U}^c, B_x^c, B_y^c)$$

$$\hat{\boldsymbol{F}}_x^+ = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^c, b_x^+, B_y^c), (\boldsymbol{U}^e, b_x^+, B_y^e)), \qquad \hat{\boldsymbol{F}}_x^- = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^w, b_x^-, B_y^w), (\boldsymbol{U}^c, b_x^-, B_y^c))$$

$$\hat{\boldsymbol{F}}_y^+ = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^c, B_x^c, b_y^+), (\boldsymbol{U}^n, B_x^n, b_y^+)), \qquad \hat{\boldsymbol{F}}_y^- = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^s, B_x^s, b_y^-), (\boldsymbol{U}^c, B_x^c, b_y^-))$$

Constraints on $\boldsymbol{B}$

Definition (Strongly divergence-free)

We will say that a vector field $\boldsymbol{B}$ defined on a mesh is strongly divergence-free if

1. $\nabla \cdot \boldsymbol{B} = 0$ in each cell K
2. $\boldsymbol{B} \cdot \boldsymbol{n}$ is continuous at each face F

Theorem

The DG scheme satisfies

$$\frac{d}{dt} \int_K (\nabla \cdot \boldsymbol{B}) \phi dx dy = 0, \qquad \forall \phi \in \mathbb{Q}_{k,k}$$

and since $\nabla \cdot \boldsymbol{B} \in \mathbb{Q}_{k,k} \implies \nabla \cdot \boldsymbol{B} =$ *constant wrt time. If* $\nabla \cdot \boldsymbol{B} = 0$ *everywhere at the initial time, then this is true at any future time also.*

Constraints on $\boldsymbol{B}$

But: Applying a limiter in a post-processing step destroys div-free property !!!

Definition (Weakly divergence-free)

We will say that a vector field $\boldsymbol{B}$ defined on a mesh is weakly divergence-free if

1. $\int_{\partial K} \boldsymbol{B} \cdot \boldsymbol{n} \mathrm{d}s = 0$ for each cell K in the mesh.
2. $\boldsymbol{B} \cdot \boldsymbol{n}$ is continuous at each face F in the mesh

Theorem

The DG scheme satisfies

$$\frac{d}{dt} \int_{\partial K} \boldsymbol{B} \cdot \boldsymbol{n} ds = 0$$

Constraints on $\boldsymbol{B}$

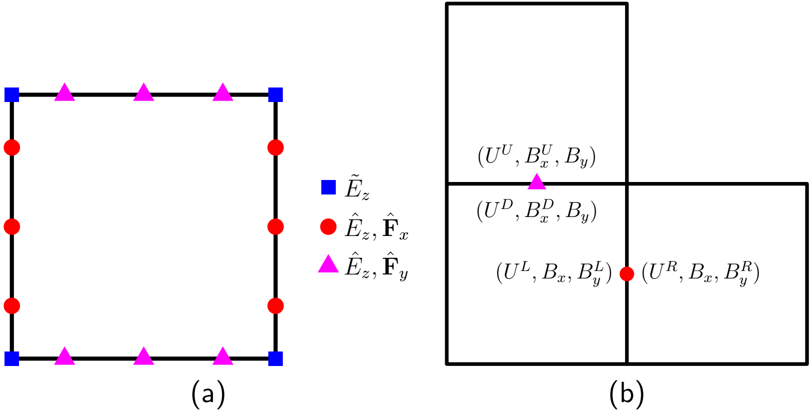
$$\int_{\partial K} \boldsymbol{B} \cdot \boldsymbol{n} \mathrm{d}s = (a_0^+ - a_0^-)\Delta y + (b_0^+ - b_0^-)\Delta x$$

where $a_0^\pm$ are face averages of B_x on right/left faces and $b_0^\pm$ are face averages of B_y on top/bottom faces respectively.

Corollary

If the limiting procedure preserves the mean value of $\boldsymbol{B} \cdot \boldsymbol{n}$ stored on the faces, then the DG scheme with limiter yields weakly divergence-free solutions.

Numerical fluxes



(a) Face quadrature points and numerical fluxes. (b) 1-D Riemann problems at a vertical and horizontal face of a cell

Numerical fluxes

Solve 1-D Riemann problem

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} = 0, \qquad \mathcal{U}(x,0) = \begin{cases} \mathcal{U}^L = \mathcal{U}(\boldsymbol{U}^L, B_x, B_y^L) & x < 0 \\ \mathcal{U}^R = \mathcal{U}(\boldsymbol{U}^R, B_x, B_y^R) & x > 0 \end{cases}$$

$$\hat{\boldsymbol{F}}_x = \begin{bmatrix} (\hat{\mathcal{F}}_x)_1 \\ (\hat{\mathcal{F}}_x)_2 \\ (\hat{\mathcal{F}}_x)_3 \\ (\hat{\mathcal{F}}_x)_4 \\ (\hat{\mathcal{F}}_x)_5 \\ (\hat{\mathcal{F}}_x)_8 \end{bmatrix}, \qquad \hat{E}_z = -(\hat{\mathcal{F}}_x)_7$$

HLL Riemann solver in 1-D

Include only slowest and fastest waves: $S_L < S_R$

Intermediate state from conservation law

$$\mathcal{U}^* = \frac{S_R \mathcal{U}^R - S_L \mathcal{U}^L - (\mathcal{F}_x^R - \mathcal{F}_x^L)}{S_R - S_L}$$

Flux obtained by satisfying conservation law over half Riemann fan

$$\mathcal{F}_x^* = \frac{S_R \mathcal{F}_x^L - S_L \mathcal{F}_x^R + S_L S_R (\mathcal{U}^R - \mathcal{U}^L)}{S_R - S_L}$$

Numerical flux is given by

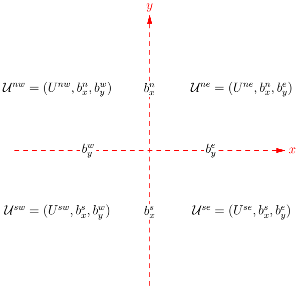
$$\hat{\mathcal{F}}_x = \begin{cases} \mathcal{F}_x^L & S_L > 0 \\ \mathcal{F}_x^R & S_R < 0 \\ \mathcal{F}_x^* & \text{otherwise} \end{cases}$$

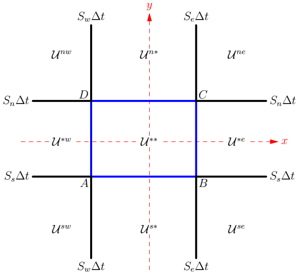
HLL Riemann solver in 1-D

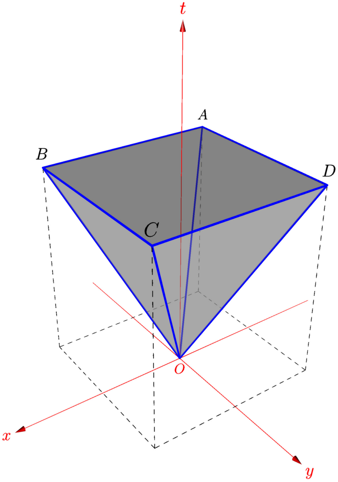
The electric field is obtained from the seventh component of the numerical flux

$$\hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = -(\hat{\mathcal{F}}_x)_7 = \begin{cases} E_z^L & S_L > 0 \\ E_z^R & S_R < 0 \\ \frac{S_R E_z^L - S_L E_z^R - S_L S_R (B_y^R - B_y^L)}{S_R - S_L} & \text{otherwise} \end{cases}$$

2-D Riemann problem







2-D Riemann problem

Strongly interating state

$$B_x^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_x^{ne} - 2S_n S_w B_x^{nw} + 2S_s S_w B_x^{sw} - 2S_s S_e B_x^{se}$$
$$- S_e(E_z^{ne} - E_z^{se}) + S_w(E_z^{nw} - E_z^{sw}) - (S_e - S_w)(E_z^{n*} - E_z^{s*}) \Big]$$

$$B_y^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_y^{ne} - 2S_n S_w B_y^{nw} + 2S_s S_w B_y^{sw} - 2S_s S_e B_y^{se}$$
$$+ S_n(E_z^{ne} - E_z^{nw}) - S_s(E_z^{se} - E_z^{sw}) + (S_n - S_s)(E_z^{*e} - E_z^{*w}) \Big]$$

Jump conditions across four waves

$$E_z^{**} = E_z^{n*} - S_n(B_x^{n*} - B_x^{**})$$
$$E_z^{**} = E_z^{s*} - S_s(B_x^{s*} - B_x^{**})$$
$$E_z^{**} = E_z^{*e} + S_e(B_y^{*e} - B_y^{**})$$
$$E_z^{**} = E_z^{*w} + S_w(B_y^{*w} - B_y^{**})$$

2-D Riemann problem

Over-determined, least-squares solution, average the four values

$$E_z^{**} = \frac{1}{4}(E_z^{n*} + E_z^{s*} + E_z^{*e} + E_z^{*w}) - \frac{1}{4}S_n(B_x^{n*} - B_x^{**}) - \frac{1}{4}S_s(B_x^{s*} - B_x^{**}) + \frac{1}{4}S_e(B_y^{*e} - B_y^{**}) + \frac{1}{4}S_w(B_y^{*w} - B_y^{**})$$

Consistency with 1-D solver

$$\mathcal{U}^{nw} = \mathcal{U}^{sw} = \mathcal{U}^L, \qquad \mathcal{U}^{ne} = \mathcal{U}^{se} = \mathcal{U}^R$$

then

$$E_z^{**} = \hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = \text{1-D HLL}$$

A 3-wave, 1-D consistent HLLC solver can also be derived.

Limiting procedure

Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

1. Perform RT reconstruction $\implies \boldsymbol{B}(\xi, \eta)$. Apply TVD limiter in characteristic variables to $\{\boldsymbol{U}(\xi, \eta), \boldsymbol{B}(\xi, \eta)\}$.
2. On each face, use limited left/right $\boldsymbol{B}(\xi, \eta)$ to limit b_x, b_y

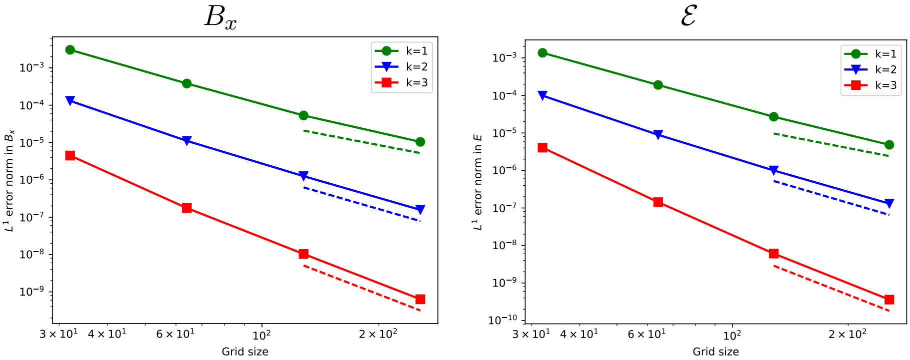
$$b_x(\eta) \leftarrow \operatorname{minmod}\left(b_x(\eta), B_x^L(\tfrac{1}{2}, \eta), B_x^R(-\tfrac{1}{2}, \eta)\right)$$

   Do not change mean value on faces.

3. Restore divergence-free property using divergence-free-reconstruction
   1. Strongly divergence-free: need to reset cell averages α_{00}, β_{00}
   2. Weakly divergence-free: α_{00}, β_{00} are not changed

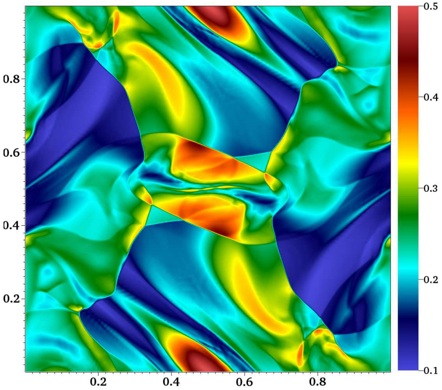
$$\nabla \cdot \boldsymbol{B} = d_1\phi_1(\xi) + d_2\phi_1(\eta)$$

Smooth vortex

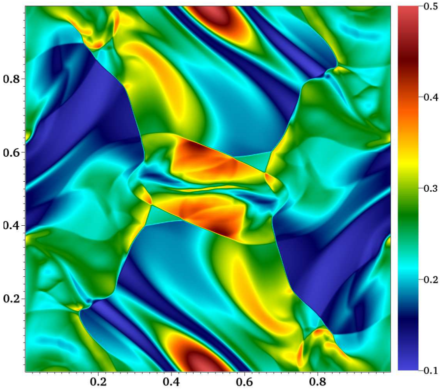


Orszag-Tang test

Density, $t = 0.5$, 512×512 cells

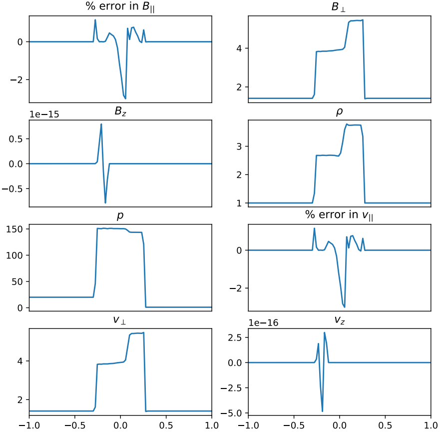


Weakly div-free

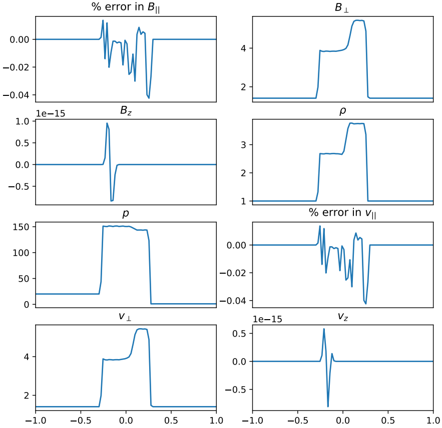


Strongly div-free

Rotated shock tube: 128 cells, HLL



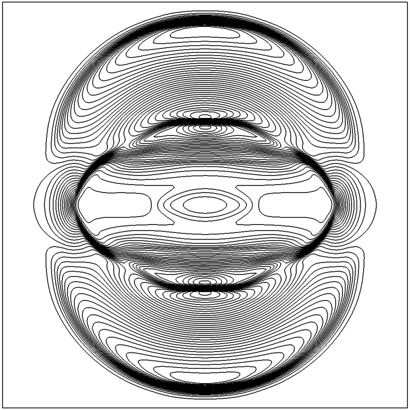
weakly div-free



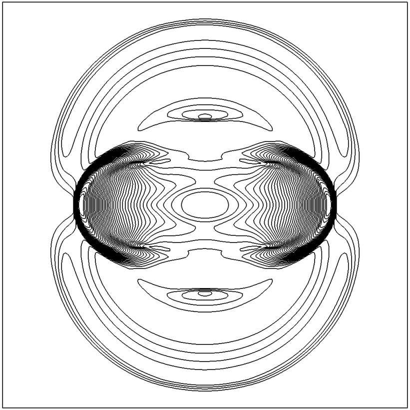
strongly div-free

Blast wave: 200×200 cells

$$\rho = 1, \quad \boldsymbol{v} = (\ 0,\ 0,\ 0), \quad \mathfrak{B} = \frac{1}{\sqrt{4\pi}}(100,\ 0,\ 0), \quad p = \begin{cases} 1000 & r < 0.1 \\ 0.1 & r > 0.1 \end{cases}$$



$B_x^2 + B_y^2$



$v_x^2 + v_y^2$

Summary

- Reconstruction of $\boldsymbol{B}$ using div and curl
- Div-free DG scheme using RT basis
- Div-free limiting needs to ensure strong div-free condition
- Multi-D Riemann solvers essential
  - consistency with 1-d solver is non-trivial; ok for HLL and HLLC (3-wave)
- Extension to 3-D seems *easy*, also AMR
- Extension to unstructured grids (use Piola transform)
- Limiters are still major obstacle for high order
  - WENO-type ideas
  - Machine learning ideas (Ray & Hesthaven)
- Extension to resistive case: $\boldsymbol{B}_t + \nabla \times \boldsymbol{E} = -\nabla \times (\eta \boldsymbol{J})$, $\boldsymbol{J} = \nabla \times \boldsymbol{B}$

$$\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial y}(E_z + \eta J_z) = 0, \ \frac{\partial B_y}{\partial t} - \frac{\partial}{\partial x}(E_z - \eta J_z) = 0, \ J_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$