Divergence-free discontinuous Galerkin method for ideal compressible MHD equations

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Maxwell Equations

Linear hyperbolic system

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}$$

$\boldsymbol{B}$ = magnetic flux density

$\boldsymbol{E}$ = electric field

$\boldsymbol{D}$ = electric flux density

$\boldsymbol{H}$ = magnetic field

$\boldsymbol{J}$ = electric current density

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E} \qquad \mu, \varepsilon \in \mathbb{R}^{3\times 3} \text{ symmetric}$$

$$\begin{aligned} \varepsilon &= \text{permittivity tensor} \\ \mu &= \text{magnetic permeability tensor} \\ \sigma &= \text{conductivity} \end{aligned}$$

<u>Constraints</u>

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{D} = \rho \quad \text{(electric charge density)}, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Two fluid MHD

Non-linear hyperbolic system

Conservation laws for each species

$$\begin{aligned}\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha) &= 0 \\ \frac{\partial (\rho_\alpha \boldsymbol{v}_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha + p_\alpha I) &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha (\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}), \qquad \alpha = i, e \\ \frac{\partial \mathcal{E}_\alpha}{\partial t} + \nabla \cdot [(\mathcal{E}_\alpha + p_\alpha) \boldsymbol{v}_\alpha] &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha \boldsymbol{E} \cdot \boldsymbol{v}_\alpha\end{aligned}$$

$$\text{Total energy:} \qquad \mathcal{E}_\alpha = \frac{p_\alpha}{\gamma_\alpha - 1} + \frac{1}{2} \rho_\alpha |\boldsymbol{v}_\alpha|^2$$

Coupled with Maxwell's equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu_0 (\rho_i q_i \boldsymbol{v}_i + \rho_e q_e \boldsymbol{v}_e)$$

together with the <u>constraints</u>

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} (\rho_i q_i + \rho_e q_e)$$

Ideal compressible MHD equations

Nonlinear hyperbolic system

Compressible Euler equations with Lorentz force

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (P\mathrm{I} + \rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{B} \otimes \boldsymbol{B}) &= 0 \\
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot ((\mathcal{E} + P)\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}) &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) &= 0
\end{aligned}
$$

$$
P = p + \frac{1}{2}|\boldsymbol{B}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\boldsymbol{B}|^2
$$

Magnetic monopoles do not exist: $\implies \quad \nabla \cdot \boldsymbol{B} = 0$

Divergence constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\nabla \cdot \nabla \times}_{=0} \boldsymbol{E} = 0$$

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{B} = 0$$

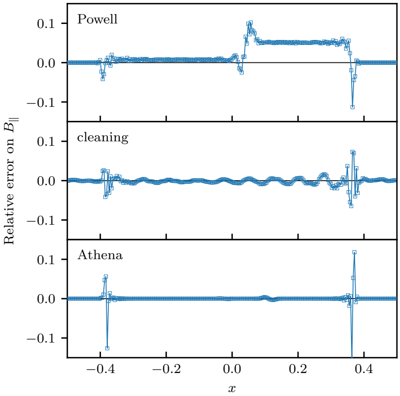
$$\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, 0) = 0 \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t)$$

Intrinsic property, not dynamical eqn

Lorentz force $\perp$ to $\boldsymbol{B}$

$$\nabla \cdot \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2} |\boldsymbol{B}|^2 I \right) = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + (\nabla \cdot \boldsymbol{B}) \boldsymbol{B}$$

Rotated shock tube



Guillet et al., MNRAS 2019

Discrete div-free $\Longrightarrow$ positivity (Kailiang Wu)

Objectives

- Based on conservation form of the equations
- Upwind-type schemes using Riemann solvers (Godunov approach)
- Divergence-free schemes for Maxwell's and compressible MHD
  - Cartesian grids at present
  - Divergence preserving schemes (RT)
- High order accurate
  - discontinuous-Galerkin FEM
- Non-oscillatory schemes for MHD
  - using limiters
  - div-free reconstruction using BDM[1]
- Explicit time stepping
  - local mass matrices
- Based on previous work for induction equation
  - J. Sci. Comp., Vol. 79, pp, 79-102, 2019

[1] Hazra et al., JCP, Vol. 394, 2019

Some existing methods

Exactly divergence-free methods

- Yee scheme (Yee (1966))
- Projection methods (Brackbill & Barnes (1980))
- Constrained transport (Evans & Hawley (1989))
- Divergence-free reconstruction (Balsara (2001))
- Globally divergence-free scheme (Li et al. (2011), Fu et al, (2018))

Approximate methods

- Locally divergence-free schemes (Cockburn, Li & Shu (2005))
- Godunov's symmetrized version of MHD (Powell, Gassner et al., C/K)
- Divergence cleaning methods (Dedner et al.)

MHD equations in 2-D

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} = 0$$

$$\mathcal{U} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z \end{bmatrix}, \quad \mathcal{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x (\boldsymbol{v} \cdot \mathfrak{B}) \\ 0 \\ -E_z \\ v_x B_z - v_z B_x \end{bmatrix}, \quad \mathcal{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y (\boldsymbol{v} \cdot \mathfrak{B}) \\ E_z \\ 0 \\ v_y B_z - v_z B_y \end{bmatrix}$$

where

$$\mathfrak{B} = (B_x, B_y, B_z), \qquad P = p + \frac{1}{2}|\mathfrak{B}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\mathfrak{B}|^2$$

E_z is the electric field in the z direction

$$E_z = -(\boldsymbol{v} \times \mathfrak{B})_z = v_y B_x - v_x B_y$$

Ideal MHD in one dimension

Divergence constraint
$$\frac{\partial B_x}{\partial x} = 0 \qquad \Longrightarrow \qquad B_x = \text{constant}$$

Conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0 \qquad \boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \mathcal{E} \\ B_y \\ B_z \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \rho u \\ P + \rho u^2 - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ (\mathcal{E} + P) u - (\boldsymbol{v} \cdot \boldsymbol{B}) B_x \\ u B_y - v B_x \\ u B_z - w B_x \end{bmatrix}$$

Flux jacobian matrix

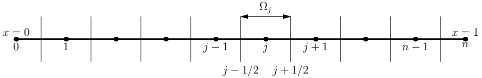
$$A = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$$
has seven real eigenvalues and eigenvectors

$$u - c_f \le u - c_a \le u - c_s \le u \le u + c_s \le u + c_a \le u + c_f$$

$$c_a = \frac{|B_x|}{\sqrt{\rho}} \qquad a = \sqrt{\frac{\gamma p}{\rho}} \qquad c_{f/s} = \sqrt{\frac{1}{2}\left[a^2 + |\boldsymbol{b}|^2 \pm \sqrt{(a^2 + |\boldsymbol{b}|^2)^2 - 4a^2 b_x^2}\right]} \qquad \boldsymbol{b} = \frac{\boldsymbol{B}}{\sqrt{\rho}}$$

Alfven speed — Sound speed — Fast/slow magnetosonic speeds

Finite volume method

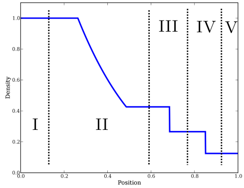


Weak solution: Satisfy conservation law on each finite volume

$$\Delta x \frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t} + \boldsymbol{F}_{j+\frac{1}{2}} - \boldsymbol{F}_{j-\frac{1}{2}} = 0$$

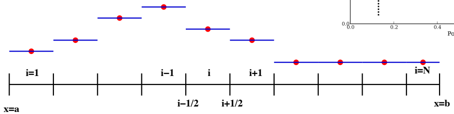
Basic unknown:
mean value in each cell

$$\boldsymbol{U}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \boldsymbol{U}(x, t_n) \mathrm{d}x$$



What is the flux ?

Riemann problem at each cell face

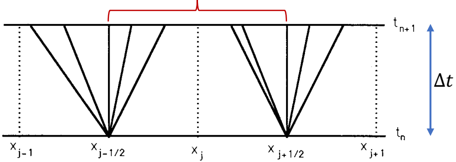


$$\boldsymbol{U}(x, t_n) = \begin{cases} \boldsymbol{U}_j^n & x < x_{j+\frac{1}{2}} \\ \boldsymbol{U}_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases} \Rightarrow \boldsymbol{U}(x,t) = \boldsymbol{U}_R \left(\frac{x - x_{i+\frac{1}{2}}}{t - t_n}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^{n+1} \right), \qquad t > t_n$$

Self-similar solution of RP

Finite volume method

Evolve waves for small time Δt (CFL condition)



Average solution at new time level

$$\boldsymbol{U}_j^{n+1} = \frac{1}{\Delta x}\left[\int_{x_{j-\frac{1}{2}}}^{x_j} \boldsymbol{U}_R\left(\frac{x - x_{j-\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right) \mathrm{d}x + \int_{x_j}^{x_{j+\frac{1}{2}}} \boldsymbol{U}_R\left(\frac{x - x_{j+\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n\right) \mathrm{d}x\right]$$

Finite volume form

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}[\boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)) - \boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n))]$$

RP→Evolve→Average: Godunov finite volume scheme

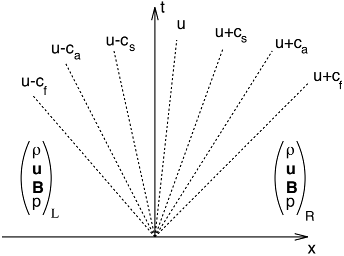
MHD Riemann problem

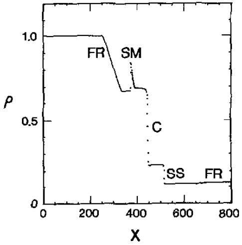
7 waves

$u \pm c_f$	fast shock/rarefaction wave
$u \pm c_a$	rotational discontinuity
$u \pm c_s$	slow shock/rarefaction wave
u	contact discontinuity

Neither genuinely non-linear nor linearly degenerate

Linearly degenerate





Brio-Wu

$c_s \leq c_a \leq c_f$: Wave speeds can coincide → non-strictly hyperbolic
Non-regular waves: compound waves, over-compressive intermediate shocks possible
Riemann solution is not always unique

MHD in multi-dimensions

x-direction Riemann problem

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} = 0, \quad \Longrightarrow \quad \frac{\partial \mathcal{U}}{\partial t} + \mathcal{A}_x \frac{\partial \mathcal{U}}{\partial x} = 0, \quad \mathcal{A}_x = \mathcal{F}_x'(\mathcal{U})$$

$\mathcal{A}_x$: 8 real eigenvalues, one zero, 7 lin. ind. eigenvectors only !!!

In the Riemann problem, $(B_x)_L \neq (B_x)_R$

Modify the MHD equations (Godunov, Powell)

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} + \Phi \nabla \cdot \boldsymbol{B} = 0$$

8 real eigenvalues and 8 lin. ind. eigenvectors

- Build approximate Riemann solver (Powell et al.)
- Build entropy stable schemes (C/Klingenberg, Winters et al.)

BUT: not divergence-free, not conservative

MHD equations in 2-D

Split into two parts

$$\boldsymbol{U} = [\rho,\ \rho\boldsymbol{v},\ \mathcal{E},\ B_z]^\top, \qquad \boldsymbol{B} = (B_x, B_y)$$

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \boldsymbol{B}) = 0, \qquad \frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0}$$

The fluxes $\boldsymbol{F} = (\boldsymbol{F}_x, \boldsymbol{F}_y)$ are of the form

$$\boldsymbol{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x (\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ v_x B_z - v_z B_x \end{bmatrix}, \qquad \boldsymbol{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y (\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ v_y B_z - v_z B_y \end{bmatrix}$$

Constraint preserving finite difference

Store magnetic field on the faces: $(B_x)_{i+\frac{1}{2},j}$, $(B_y)_{i,j+\frac{1}{2}}$

$$\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0 \quad \Longrightarrow \quad \frac{\mathsf{d}}{\mathsf{d}t}(B_x)_{i+\frac{1}{2},j} + \frac{(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} = 0$$

$$\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\mathsf{d}}{\mathsf{d}t}(B_y)_{i,j+\frac{1}{2}} - \frac{(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} = 0$$

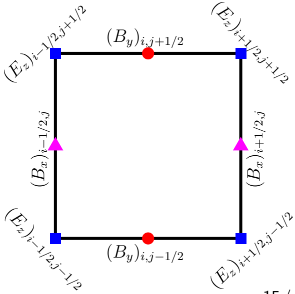
Measure divergence at cell center

$$\nabla_h \cdot \boldsymbol{B}_{i,j} = \frac{(B_x)_{i+\frac{1}{2},j} - (B_x)_{i-\frac{1}{2},j}}{\Delta x} + \frac{(B_y)_{i,j+\frac{1}{2}} - (B_y)_{i,j-\frac{1}{2}}}{\Delta y}$$

Then

$$\frac{\mathsf{d}}{\mathsf{d}t}\nabla_h \cdot \boldsymbol{B}_{i,j} = 0$$

The corner fluxes cancel one another !!!



Approximation of magnetic field

$$\boldsymbol{B}_h \in V_h = \text{some finite element space}$$

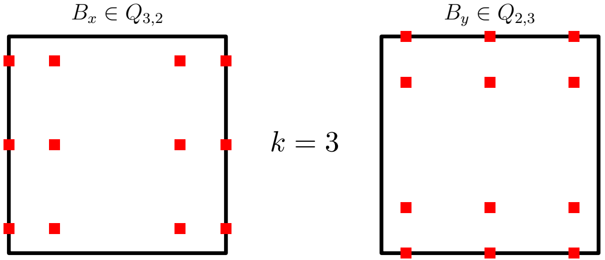
If we want $\nabla \cdot \boldsymbol{B}_h = 0$, it is natural to look for approximations in

$$\boldsymbol{B}_h \in V_h \subset H(div, \Omega) = \{\boldsymbol{B} \in L^2(\Omega) : \mathrm{div}(\boldsymbol{B}) \in L^2(\Omega)\}$$

Conformal approximation of functions in $H(div, \Omega)$ on a mesh $\mathcal{T}_h$ with piecewise polynomials, we need

$$\boxed{\boldsymbol{B} \cdot \boldsymbol{n} \text{ continuous across element faces}}$$

Possible options: Brezzi-Douglas-Marini, Raviart-Thomas, etc.



Approximation spaces: Degree $k \geq 0$

Map cell K to reference cell $\hat{K} = [-\frac{1}{2}, +\frac{1}{2}] \times [-\frac{1}{2}, +\frac{1}{2}]$

$$\mathbb{P}_r(\xi) = \text{span}\{1, \xi, \xi^2, \ldots, \xi^r\}, \quad \mathbb{Q}_{r,s}(\xi, \eta) = \mathbb{P}_r(\xi) \otimes \mathbb{P}_s(\eta)$$

Hydrodynamic variables in each cell

$$\boldsymbol{U}(\xi, \eta) = \sum_{i=0}^{k} \sum_{j=0}^{k} \boldsymbol{U}_{ij} \phi_i(\xi) \phi_j(\eta) \in \mathbb{Q}_{k,k}$$

Normal component of $\boldsymbol{B}$ on faces

$$\text{on vertical faces} : b_x(\eta) = \sum_{j=0}^{k} a_j \phi_j(\eta) \in \mathbb{P}_k(\eta)$$

$$\text{on horizontal faces} : b_y(\xi) = \sum_{j=0}^{k} b_j \phi_j(\xi) \in \mathbb{P}_k(\xi)$$

$\{\phi_i(\xi)\}$ are **orthogonal polynomials** on $[-\frac{1}{2}, +\frac{1}{2}]$, with degree$(\phi_i) = i$.

Approximation spaces: Degree $k \geq 0$

For $k \geq 1$,define certain **cell moments**

$$\alpha_{ij} = \alpha_{ij}(B_x) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k-1,k}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k-1, \quad 0 \leq j \leq k$$

$$\beta_{ij} = \beta_{ij}(B_y) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k,k-1}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k, \quad 0 \leq j \leq k-1$$

$$m_{ij} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)\phi_j(\eta)]^2 \mathsf{d}\xi \mathsf{d}\eta = m_i m_j, \quad m_i = \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)]^2 \mathsf{d}\xi$$

α_{00}, β_{00} are **cell averages** of B_x, B_y

<u>Solution variables</u>

$$\{\boldsymbol{U}(\xi,\eta)\}, \qquad \{b_x(\eta)\}, \qquad \{b_y(\xi)\}, \qquad \{\alpha,\beta\}$$

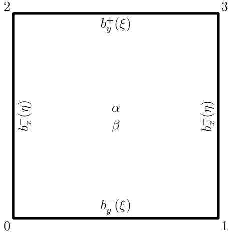
The set $\{b_x, b_y, \alpha, \beta\}$ are the dofs for the **Raviart-Thomas** space.

RT reconstruction: $b_x^\pm(\eta), b_y^\pm(\xi), \alpha, \beta \to \boldsymbol{B}(\xi,\eta)$

Given $b_x^\pm(\eta) \in \mathbb{P}_k$ and $b_y^\pm(\xi) \in \mathbb{P}_k$,
and set of cell moments

$$\{\alpha_{ij},\ 0 \le i \le k-1,\ 0 \le j \le k\}$$

$$\{\beta_{ij},\ 0 \le i \le k,\ 0 \le j \le k-1\}$$



Find $B_x \in \mathbb{Q}_{k+1,k}$ and $B_y \in \mathbb{Q}_{k,k+1}$ such that

$$B_x(\pm\tfrac{1}{2},\eta) = b_x^\pm(\eta), \quad \eta \in [-\tfrac{1}{2},\tfrac{1}{2}], \qquad B_y(\xi,\pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \xi \in [-\tfrac{1}{2},\tfrac{1}{2}]$$

$$\frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta)\phi_i(\xi)\phi_j(\eta) \mathrm{d}\xi \mathrm{d}\eta = \alpha_{ij}, \qquad 0 \le i \le k-1, \quad 0 \le j \le k$$

$$\frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta)\phi_i(\xi)\phi_j(\eta) \mathrm{d}\xi \mathrm{d}\eta = \beta_{ij}, \qquad 0 \le i \le k, \quad 0 \le j \le k-1$$

(1) $\exists$ unique solution. (2) $\boldsymbol{B} \cdot \boldsymbol{n}$ continuous.
(3) Data div-free $\implies$ reconstructed $\boldsymbol{B}$ is div-free.

DG scheme for $\boldsymbol{B}$ on faces

On every vertical face of the mesh: $\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_x}{\partial t} \phi_i \mathsf{d}\eta - \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\eta} \mathsf{d}\eta + \frac{1}{\Delta y} [\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

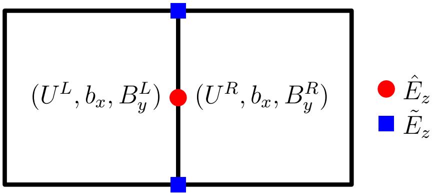
On every horizontal face of the mesh: $\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_y}{\partial t} \phi_i \mathsf{d}\xi + \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\xi} \mathsf{d}\xi - \frac{1}{\Delta x} [\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

Numerical fluxes

$\hat{E}_z$: on face, 1-D Riemann solver

$\tilde{E}_z$: at vertex, 2-D Riemann solver



DG scheme for $\boldsymbol{B}$ on cells

$$
\begin{aligned}
m_{ij} \frac{\mathsf{d}\alpha_{ij}}{\mathsf{d}t} &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial B_x}{\partial t} \phi_i(\xi)\phi_j(\eta) \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \le i \le k-1, \quad 0 \le j \le k \\
&= -\frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial E_z}{\partial \eta} \phi_i(\xi)\phi_j(\eta) \mathsf{d}\xi \mathsf{d}\eta \\
&= -\frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\hat{E}_z(\xi, \tfrac{1}{2})\phi_i(\xi)\phi_j(\tfrac{1}{2}) - \hat{E}_z(\xi, -\tfrac{1}{2})\phi_i(\xi)\phi_j(-\tfrac{1}{2})] \mathsf{d}\xi \\
&\quad + \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} E_z(\xi, \eta)\phi_i(\xi)\phi_j'(\eta) \mathsf{d}\xi \mathsf{d}\eta
\end{aligned}
$$

Numerical fluxes

$$\hat{E}_z : \text{on face, 1-D Riemann solver}$$

Not a Galerkin method, test functions ($\mathbb{Q}_{k-1,k}$) different from trial functions ($\mathbb{Q}_{k+1,k}$)

DG scheme for $\boldsymbol{U}$ on cells

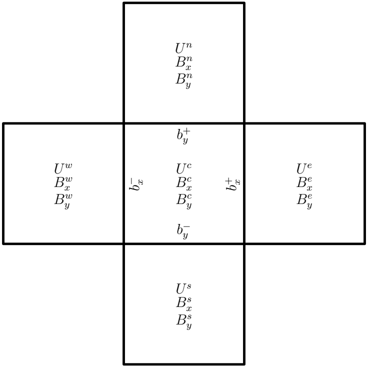
For each test function $\Phi(\xi,\eta) = \phi_i(\xi)\phi_j(\eta) \in \mathbb{Q}_{k,k}$

$$\begin{aligned}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial \boldsymbol{U}^c}{\partial t}\Phi(\xi,\eta)\mathsf{d}\xi\mathsf{d}\eta - \int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[\frac{1}{\Delta x}\boldsymbol{F}_x\frac{\partial\Phi}{\partial\xi} + \frac{1}{\Delta y}\boldsymbol{F}_y\frac{\partial\Phi}{\partial\eta}\right]\mathsf{d}\xi\mathsf{d}\eta \\ + \frac{1}{\Delta x}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\hat{\boldsymbol{F}}_x^+\Phi(\tfrac{1}{2},\eta)\mathsf{d}\eta - \frac{1}{\Delta x}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\hat{\boldsymbol{F}}_x^-\Phi(-\tfrac{1}{2},\eta)\mathsf{d}\eta \\ + \frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\hat{\boldsymbol{F}}_y^+\Phi(\xi,\tfrac{1}{2})\mathsf{d}\xi - \frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\hat{\boldsymbol{F}}_y^-\Phi(\xi,-\tfrac{1}{2})\mathsf{d}\xi = 0\end{aligned}$$

Numerical fluxes

$$\hat{\boldsymbol{F}}_x^\pm, \hat{\boldsymbol{F}}_y^\pm : \text{on face, 1-D Riemann solver}$$

DG scheme for $\boldsymbol{U}$ on cells



$$\boldsymbol{F}_x = \boldsymbol{F}_x(\boldsymbol{U}^c, B_x^c, B_y^c), \qquad \boldsymbol{F}_y = \boldsymbol{F}_y(\boldsymbol{U}^c, B_x^c, B_y^c)$$

$$\hat{\boldsymbol{F}}_x^+ = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^c, b_x^+, B_y^c), (\boldsymbol{U}^e, b_x^+, B_y^e)), \qquad \hat{\boldsymbol{F}}_x^- = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^w, b_x^-, B_y^w), (\boldsymbol{U}^c, b_x^-, B_y^c))$$

$$\hat{\boldsymbol{F}}_y^+ = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^c, B_x^c, b_y^+), (\boldsymbol{U}^n, B_x^n, b_y^+)), \qquad \hat{\boldsymbol{F}}_y^- = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^s, B_x^s, b_y^-), (\boldsymbol{U}^c, B_x^c, b_y^-))$$

Constraints on $\boldsymbol{B}$

Definition (Globally divergence-free)

A vector field $\boldsymbol{B}$ defined on a mesh is strongly divergence-free if

1. $\nabla \cdot \boldsymbol{B} = 0$ in each cell $K \in \mathcal{T}_h$
2. $\boldsymbol{B} \cdot \boldsymbol{n}$ is continuous at each face $F \in \mathcal{T}_h$

Theorem

(1) The DG scheme satisfies

$$\frac{d}{dt} \int_K (\nabla \cdot \boldsymbol{B}) \phi dx dy = 0, \qquad \forall \phi \in \mathbb{Q}_{k,k}$$

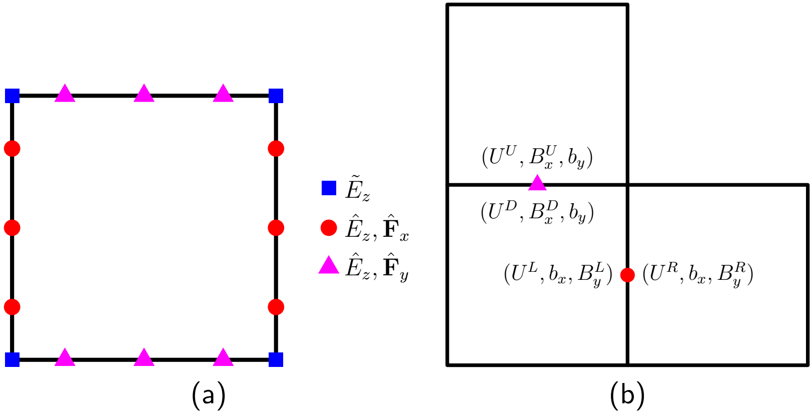
and since $\nabla \cdot \boldsymbol{B} \in \mathbb{Q}_{k,k} \implies \nabla \cdot \boldsymbol{B} =$ *constant wrt time.*

(2) If $\nabla \cdot \boldsymbol{B} \equiv 0$ *at* $t = 0 \implies \nabla \cdot \boldsymbol{B} \equiv 0$ *for* $t > 0$

But: Applying a limiter in a post-processing step destroys div-free property !!!

Numerical fluxes

$(k+1)$-point Gauss-Legendre quadrature on faces



(a) Face quadrature points and numerical fluxes. (b) 1-D Riemann problems at a vertical and horizontal face of a cell

Numerical fluxes

To estimate $\hat{\boldsymbol{F}}_x$, $\hat{E}_z$, solve 1-D Riemann problem at each face quadrature point

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} = 0, \qquad \mathcal{U}(x,0) = \begin{cases} \mathcal{U}^L = \mathcal{U}(\boldsymbol{U}^L, b_x, B_y^L) & x < 0 \\ \mathcal{U}^R = \mathcal{U}(\boldsymbol{U}^R, b_x, B_y^R) & x > 0 \end{cases}$$

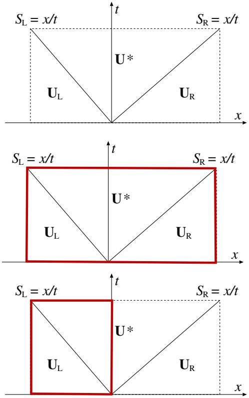
$$\hat{\boldsymbol{F}}_x = \begin{bmatrix} (\hat{\mathcal{F}}_x)_1 \\ (\hat{\mathcal{F}}_x)_2 \\ (\hat{\mathcal{F}}_x)_3 \\ (\hat{\mathcal{F}}_x)_4 \\ (\hat{\mathcal{F}}_x)_5 \\ (\hat{\mathcal{F}}_x)_8 \end{bmatrix}, \qquad \hat{E}_z = -(\hat{\mathcal{F}}_x)_7$$

Riemann problem can lead to 7 waves !!! Solve approximately.

HLL solver in 1-D: slowest and fastest waves: $S_L < S_R$

- Intermediate state from conservation law

$$\mathcal{U}^* = \frac{S_R \mathcal{U}^R - S_L \mathcal{U}^L - (\mathcal{F}_x^R - \mathcal{F}_x^L)}{S_R - S_L}$$



- Satisfying conservation law over half Riemann fan

$$\mathcal{F}_x^* = \frac{S_R \mathcal{F}_x^L - S_L \mathcal{F}_x^R + S_L S_R (\mathcal{U}^R - \mathcal{U}^L)}{S_R - S_L}$$

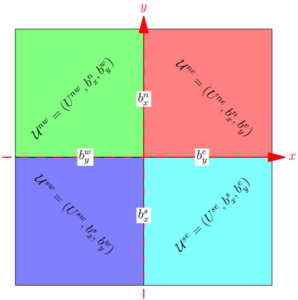
- Numerical flux is given by

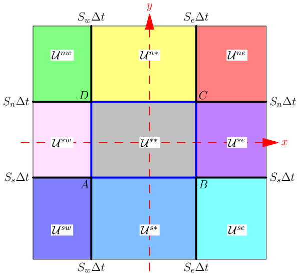
$$\hat{\mathcal{F}}_x = \begin{cases} \mathcal{F}_x^L & S_L > 0 \\ \mathcal{F}_x^R & S_R < 0 \\ \mathcal{F}_x^* & \text{otherwise} \end{cases}$$

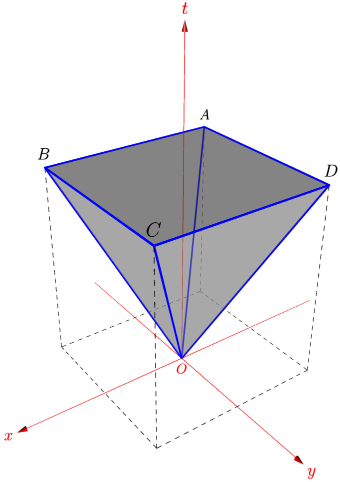
- Electric field from 7'th component

$$\hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = -(\hat{\mathcal{F}}_x)_7 = \begin{cases} E_z^L & S_L > 0 \\ E_z^R & S_R < 0 \\ \frac{S_R E_z^L - S_L E_z^R - S_L S_R (B_y^R - B_y^L)}{S_R - S_L} & \text{otherwise} \end{cases}$$

2-D Riemann problem







2-D Riemann problem

Strongly interacting state

$$B_x^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_x^{ne} - 2S_n S_w B_x^{nw} + 2S_s S_w B_x^{sw} - 2S_s S_e B_x^{se} - S_e(E_z^{ne} - E_z^{se}) + S_w(E_z^{nw} - E_z^{sw}) - (S_e - S_w)(E_z^{n*} - E_z^{s*}) \Big]$$

$$B_y^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_y^{ne} - 2S_n S_w B_y^{nw} + 2S_s S_w B_y^{sw} - 2S_s S_e B_y^{se} + S_n(E_z^{ne} - E_z^{nw}) - S_s(E_z^{se} - E_z^{sw}) + (S_n - S_s)(E_z^{*e} - E_z^{*w}) \Big]$$

Jump conditions b/w $**$ and $\{n*, s*, *e, *w\}$

$$E_z^{**} = E_z^{n*} - S_n(B_x^{n*} - B_x^{**})$$
$$E_z^{**} = E_z^{s*} - S_s(B_x^{s*} - B_x^{**}) \qquad \text{4 equations}$$
$$E_z^{**} = E_z^{*e} + S_e(B_y^{*e} - B_y^{**}) \qquad \text{1 unknown}$$
$$E_z^{**} = E_z^{*w} + S_w(B_y^{*w} - B_y^{**})$$

2-D Riemann problem

Over-determined, least-squares solution (Vides et al.)

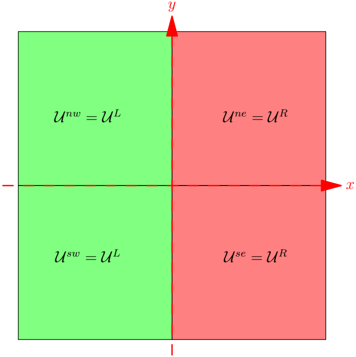
$$E_z^{**} = \frac{1}{4}(E_z^{n*} + E_z^{s*} + E_z^{*e} + E_z^{*w}) - \frac{1}{4}S_n(B_x^{n*} - B_x^{**}) - \frac{1}{4}S_s(B_x^{s*} - B_x^{**}) + \frac{1}{4}S_e(B_y^{*e} - B_y^{**}) + \frac{1}{4}S_w(B_y^{*w} - B_y^{**})$$

Consistency with 1-D solver

$$\mathcal{U}^{nw} = \mathcal{U}^{sw} = \mathcal{U}^{L}$$
$$\mathcal{U}^{ne} = \mathcal{U}^{se} = \mathcal{U}^{R}$$

then

$$E_z^{**} = \hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = \text{1-D HLL}$$



HLLC Riemann solver

1-D solver

- Slowest and fastest waves S_L, S_R, and contact wave $S_M = u_*$
- Two intermediate states: $\mathcal{U}^{*L}$, $\mathcal{U}^{*R}$
- No unique way to satisfy all jump conditions: Gurski (2004), Li (2005)
- Common value of magnetic field $\boldsymbol{B}^{*L} = \boldsymbol{B}^{*R}$
- Common electric field $E_z^{*L} = E_z^{*R}$, same as in HLL

2-D solver

- Electric field estimate E_z^{**} same as HLL
- Consistent with 1-D solver

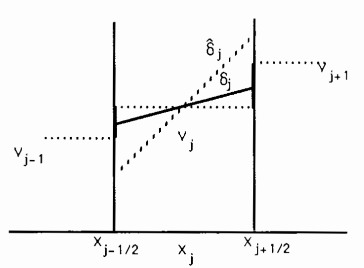
Limiting procedure

Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

1. Perform RT reconstruction $\implies \boldsymbol{B}(\xi, \eta)$.
2. Apply TVD limiter in characteristic variables to $\{\boldsymbol{U}(\xi, \eta), \boldsymbol{B}(\xi, \eta)\}$.
3. On each face, use limited left/right $\boldsymbol{B}(\xi, \eta)$ to limit b_x, b_y

$$b_x(\eta) \leftarrow \text{minmod}\left(b_x(\eta), B_x^L(\tfrac{1}{2}, \eta), B_x^R(-\tfrac{1}{2}, \eta)\right)$$

   Do not change mean value on faces.
4. Restore divergence-free property using divergence-free-reconstruction[2]



[2] Hazra et al., JCP, Vol. 394, 2019

Divergence-free reconstruction

For each cell, find $\boldsymbol{B}(\xi,\eta)$ such that

$$B_x(\pm\tfrac{1}{2},\eta) = b_x^{\pm}(\eta), \quad \forall \eta \in [-\tfrac{1}{2},+\tfrac{1}{2}]$$

$$B_y(\xi,\pm\tfrac{1}{2}) = b_y^{\pm}(\xi), \quad \forall \xi \in [-\tfrac{1}{2},+\tfrac{1}{2}]$$

$$\nabla\cdot\boldsymbol{B}(\xi,\eta) = 0, \qquad \forall (\xi,\eta) \in [-\tfrac{1}{2},\tfrac{1}{2}]\times[-\tfrac{1}{2},\tfrac{1}{2}]$$

We look for $\boldsymbol{B}$ in (Brezzi & Fortin, Section III.3.2)

$$\mathrm{BDM}(k) = \mathbb{P}_k^2 \oplus \nabla\times(x^{k+1}y) \oplus \nabla\times(xy^{k+1})$$

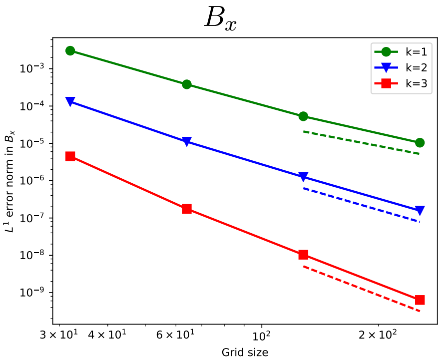
- For $k = 0,1,2$, we can solve the above problem
- For $k \geq 3$, we need additional information
  - $k=3$: $b_{10} - a_{01} = \omega_1 = \nabla\times\boldsymbol{B}(0,0)$
  - $k=4$: ω_1 and $b_{20} - a_{11} = \omega_2 \approx \frac{\partial}{\partial x}\nabla\times\boldsymbol{B}$, $b_{11} - a_{02} = \omega_3 \approx \frac{\partial}{\partial y}\nabla\times\boldsymbol{B}$
  - ω_1, etc. are known from α,β
- For more details, see Hazra et al., JCP, Vol. 394, 2019

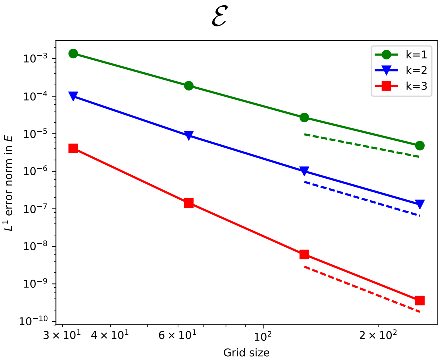
Algorithm 1: Constraint preserving scheme for ideal compressible MHD

Allocate memory for all variables;
Set initial condition for $\boldsymbol{U}, b_x, b_y, \alpha, \beta$;
Loop over cells and reconstruct B_x, B_y;
Set time counter $t = 0$;
while $t < T$ **do**
    Copy current solution into old solution;
    Compute time step Δt;
    for *each RK stage* **do**
        Loop over vertices and compute vertex flux;
        Loop over faces and compute all face integrals;
        Loop over cells and compute all cell integrals;
        Update solution to next stage;
        Loop over cells and do RT reconstruction $(b_x, b_y, \alpha, \beta) \rightarrow \boldsymbol{B}$;
        Loop over cells and apply limiter on $\boldsymbol{U}, \boldsymbol{B}$;
        Loop over faces and limit solution b_x, b_y;
        Loop over faces and perform div-free reconstruction;
        Apply positivity limiter;
    end
    $t = t + \Delta t$;
end

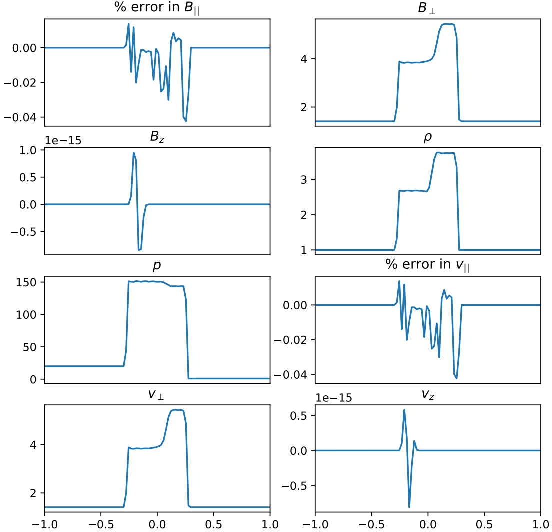
Numerical Results

Smooth vortex



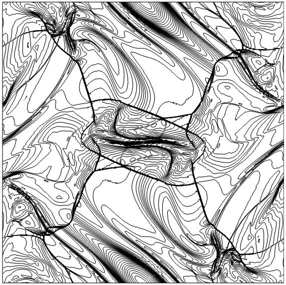


Rotated shock tube: $k = 1$, 128 cells, HLL

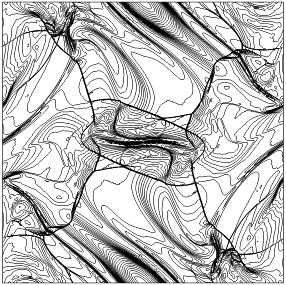


Orszag-Tang test

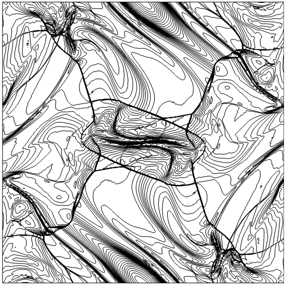
Density, $t = 0.5$, $k = 3$, 512×512 cells



LxF



HLL

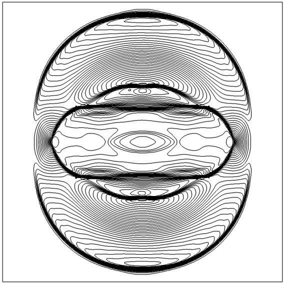


HLLC

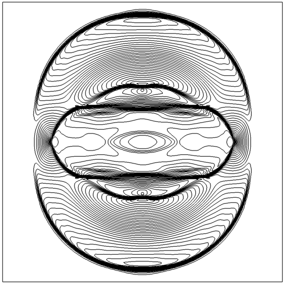
Blast wave: 200×200 cells

$$\rho = 1, \quad \boldsymbol{v} = (\ 0,\ 0,\ 0), \quad \mathfrak{B} = \frac{1}{\sqrt{4\pi}}(100,\ 0,\ 0), \quad p = \begin{cases} 1000 & r < 0.1 \\ 0.1 & r > 0.1 \end{cases}$$

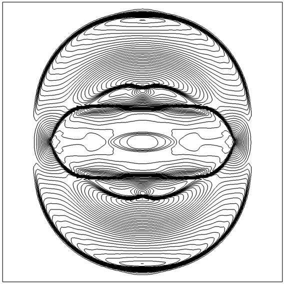
Contours of $\frac{1}{2}(B_x^2 + B_y^2)$



LxF



HLL



HLLC

Summary

- Div-free DG scheme using RT basis for $\boldsymbol{B}$
- Multi-D Riemann solvers essential
  - consistency with 1-d solver is not automatic; ok for HLL (2-wave) and HLLC (3-wave); what about HLLD (5-wave) ?
- Div-free limiting needs to ensure strong div-free condition
  - Reconstruction of $\boldsymbol{B}$ using div=0 and curl=given
- Extension to 3-D seems *easy*, also AMR
- Extension to unstructured grids (use Piola transform)
- Limiters are still major obstacle for high order
  - WENO-type ideas
  - Machine learning ideas (Ray & Hesthaven)
- No proof of positivity limiter for div-free scheme
  - Not a fully discontinuous solution
- Extension to resistive case: $\boldsymbol{B}_t + \nabla \times \boldsymbol{E} = -\nabla \times (\eta \boldsymbol{J})$, $\boldsymbol{J} = \nabla \times \boldsymbol{B}$

$$\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial y}(E_z + \eta J_z) = 0, \ \frac{\partial B_y}{\partial t} - \frac{\partial}{\partial x}(E_z - \eta J_z) = 0, \ J_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

Joint work with
Rakesh Kumar, TIFR-CAM (MHD)
Arijit Hazra, TIFR-CAM (Maxwell)

Thank You