A globally divergence-free discontinuous galerkin method for induction and related equations

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Maxwell Equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}$$

$\boldsymbol{B}$ = magnetic flux density
$\boldsymbol{E}$ = electric field

$\boldsymbol{D}$ = electric flux density
$\boldsymbol{H}$ = magnetic field
$\boldsymbol{J}$ = electric current density

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E} \qquad \mu, \varepsilon \in \mathbb{R}^{3\times 3} \text{ symmetric}$$

$$\varepsilon = \text{permittivity tensor}$$
$$\mu = \text{magnetic permeability tensor}$$
$$\sigma = \text{conductivity}$$

$$\nabla \cdot \boldsymbol{D} = \rho \quad \text{(electric charge density)}, \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Ideal MHD equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (pI + \rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{B} \otimes \boldsymbol{B}) &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot ((E + p)\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}) &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) &= 0
\end{aligned}
$$

Model problem

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = -\boldsymbol{M}$$

Divergence evolves according to

$$\frac{\partial}{\partial t}(\mathrm{div}\boldsymbol{B}) + \nabla \cdot \boldsymbol{M} = 0 \tag{1}$$

In 2-D

$$\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = -M_x, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = -M_y$$

Model problem

If $\boldsymbol{B}$ represents the magnetic field ($\boldsymbol{M} = 0$)

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$$

Magnetic monopoles do not exist

$$\nabla \cdot \boldsymbol{B} = 0$$

If

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{at} \quad t = 0$$

then

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) + \nabla \cdot \nabla \times \boldsymbol{E} = 0 \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{B} = 0 \quad t > 0$$

In 2-D, the induction equation can be written as

$$\frac{\partial B_x}{\partial t} + \frac{\partial E}{\partial y} = 0, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E}{\partial x} = 0, \qquad E = v_y B_x - v_x B_y$$

Some existing methods

- Constrained transport ([1] Evans & Hawley (1989))
  - $\nabla \cdot \boldsymbol{B} = 0$ implies $\boldsymbol{B} = \nabla \times \boldsymbol{A}$
  - Evolve $\boldsymbol{A}$
  - Compute $\boldsymbol{B}$ from $\boldsymbol{A}$
- Divergence-free reconstruction ([2] Balsara (2001))
- Globally divergence-free scheme ([3] Li et al. (2011))

Approximation of magnetic field

When dealing with problems where the vector field $\boldsymbol{B}$ must be divergence-free, it is natural to look for solutions in $H(\mathrm{div}, \Omega)$ which is defined as

$$H(\mathrm{div}, \Omega) = \{\boldsymbol{B} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}(\boldsymbol{B}) \in L^2(\Omega)\}$$

To approximate functions in $H(\mathrm{div}, \Omega)$ on a mesh $\mathcal{T}_h$, we need the following compatibility condition.

Theorem (See [4], Proposition 3.2.2)

Let $\boldsymbol{B}^h : \Omega \to \mathbb{R}^d$ be such that

1. $\boldsymbol{B}^h|_K \in \boldsymbol{H}^1(\Omega)$ *for all* $K \in \mathcal{T}_h$
2. *for each common face* $F = K_1 \cap K_2$, $K_1, K_2 \in \mathcal{T}_h$, *the trace of normal component* $\boldsymbol{n} \cdot \boldsymbol{B}^h|_{K_1}$ *and* $\boldsymbol{n} \cdot \boldsymbol{B}^h|_{K_2}$ *is the same.*

Then $\boldsymbol{B}^h \in H(div, \Omega)$. Conversely, if $\boldsymbol{B}^h \in H(div, \Omega)$ and (1) holds, then (2) is also satisfied.

Approximation of magnetic field

$P_k(x)$, $P_k(y)$: 1-D polynomials of degree at most k wrt the variables x, y respectively.
$Q_{r,s}(x, y)$: tensor product polynomials of degree r in the variable x and degree s in the variable y, i.e.,

$$Q_{r,s}(x, y) = \mathrm{span}\{x^i y^j,\ 0 \leq i \leq r,\ 0 \leq j \leq s\}$$

For $k \geq 0$, the Raviart-Thomas space of vector functions is defined as

$$\mathbf{RT}_k = Q_{k+1,k} \times Q_{k,k+1}, \qquad \dim(\mathbf{RT}_k) = 2(k+1)(k+2)$$

- For any $\boldsymbol{B}^h \in \mathbf{RT}_k$, we have

$$\mathrm{div}(\boldsymbol{B}^h) \in Q_{k,k}(x, y) =: Q_k(x, y)$$

Approximation of magnetic field

- The restriction of $\boldsymbol{B}^h = (B_x^h, B_y^h)$ to a face is a polynomial of degree k, i.e.,

$$B_x^h(\pm\Delta x/2, y) \in P_k(y), \qquad B_y^h(x, \pm\Delta y/2) \in P_k(x)$$

For doing the numerical computations, it is useful to map each cell to a reference cell.

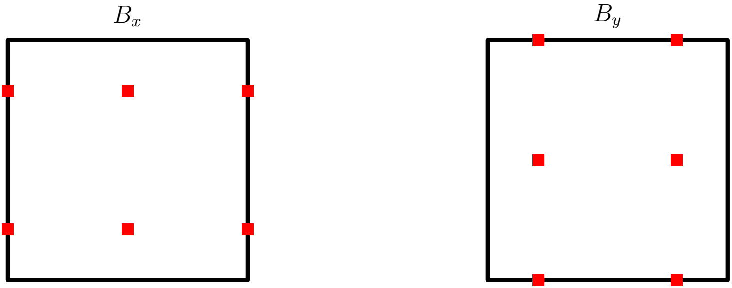
$$\{\xi_i, 0 \le i \le k+1\} = \text{Gauss-Lobatto-Legendre (GLL) nodes}$$

$$\{\hat{\xi}_i, 0 \le i \le k\} = \text{Gauss-Legendre (GL) nodes}$$

Let ϕ_i and $\hat{\phi}_i$ be the corresponding 1-D Lagrange polynomials. Then the magnetic field is given by

$$B_x^h(\xi, \eta) = \sum_{i=0}^{k+1} \sum_{j=0}^{k} (B_x)_{ij} \phi_i(\xi) \hat{\phi}_j(\eta), \quad B_y^h(\xi, \eta) = \sum_{i=0}^{k} \sum_{j=0}^{k+1} (B_y)_{ij} \hat{\phi}_i(\xi) \phi_j(\eta)$$

Approximation of magnetic field



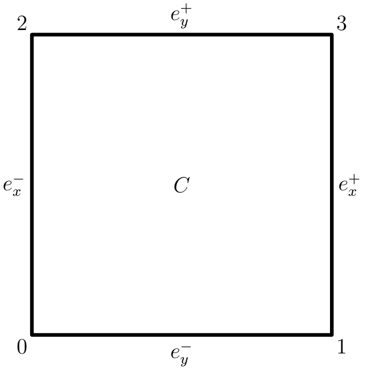
Location of dofs of Raviart-Thomas polynomial for $k = 1$

Our choice of nodes ensures that the normal component of the magnetic field is continuous on the cell faces.
We have the error estimates on Cartesian meshes [5], [6]

$$\|\boldsymbol{B} - \boldsymbol{B}^h\|_{\boldsymbol{L}^2(\Omega)} \le C h^{k+1} |\boldsymbol{B}|_{\boldsymbol{H}^{k+1}(\Omega)}$$

$$\|\mathrm{div}(\boldsymbol{B}) - \mathrm{div}(\boldsymbol{B}^h)\|_{\boldsymbol{L}^2(\Omega)} \le C h^{k+1} |\mathrm{div}(\boldsymbol{B})|_{\boldsymbol{H}^{k+1}(\Omega)}$$

Moments



The edge moments are given by

$$\int_{e_x^{\mp}} B_x^h \phi \mathsf{d}y \qquad \forall \phi \in P_k(y)$$

and

$$\int_{e_y^{\mp}} B_y^h \phi \mathsf{d}x \qquad \forall \phi \in P_k(x)$$

The cell moments are given by

$$\int_C B_x^h \psi \mathsf{d}x \mathsf{d}y \qquad \forall \psi \in \partial_x Q_k(x, y) := Q_{k-1,k}(x, y)$$

Moments

and

$$\int_C B_y^h \psi \mathsf{d}x\mathsf{d}y \qquad \forall \psi \in \partial_y Q_k(x,y) := Q_{k,k-1}(x,y)$$

Note that

$$\dim P_k(x) = \dim P_k(y) = k+1$$

and

$$\dim \partial_x Q_k(x,y) = \dim \partial_y Q_k(x,y) = k(k+1)$$

so that we have in total

$$4(k+1) + 2k(k+1) = 2(k+1)(k+2) = \dim \mathbf{RT}_k$$

The moments on the edges $e_x^{\mp}$ uniquely determine the restriction of B_x^h on those edges, and similarly the moments on $e_y^{\mp}$ uniquely determine the restriction of B_y^h on the corresponding edges. This ensures continuity of the normal component of $\boldsymbol{B}^h$ on all the edges.

Theorem

If all the moments are zero for any cell C, then $\boldsymbol{B}^h \equiv 0$ inside that cell.

Proof: The edge moments being zero implies that

$$B_x^h \equiv 0 \quad \text{on} \quad e_x^{\mp} \qquad \text{and} \qquad B_y^h \equiv 0 \quad \text{on} \quad e_y^{\mp}$$

Now take $\psi = \partial_x \phi$ for some $\phi \in Q_k$ in the cell moment equation of B_x^h and perform an integration by parts

$$-\int_C \frac{\partial B_x^h}{\partial x} \phi \mathsf{d}x \mathsf{d}y - \int_{e_x^-} B_x^h \phi \mathsf{d}y + \int_{e_x^+} B_x^h \phi \mathsf{d}y = 0$$

and hence

$$\int_C \frac{\partial B_x^h}{\partial x} \phi \mathsf{d}x \mathsf{d}y = 0 \qquad \forall \phi \in Q_k$$

Since $\frac{\partial B_x^h}{\partial x} \in Q_k$, this implies that $\frac{\partial B_x^h}{\partial x} \equiv 0$ and hence $B_x^h \equiv 0$. Similarly, we conclude that $B_y^h \equiv 0$. □

Theorem

Let $\boldsymbol{B}^h \in \mathbf{RT}_k$ *satisfy the moments*

$$\int_{e_x^{\mp}} B_x^h \phi\, dy = \int_{e_x^{\mp}} B_x \phi\, dy \qquad \forall \phi \in P_k(y) \tag{2}$$

$$\int_{e_y^{\mp}} B_y^h \phi\, dx = \int_{e_y^{\mp}} B_y \phi\, dx \qquad \forall \phi \in P_k(x) \tag{3}$$

$$\int_C B_x^h \psi\, dx\, dy = \int_C B_x \psi\, dx\, dy \qquad \forall \psi \in \partial_x Q_k(x,y) \tag{4}$$

$$\int_C B_y^h \psi\, dx\, dy = \int_C B_y \psi\, dx\, dy \qquad \forall \psi \in \partial_y Q_k(x,y) \tag{5}$$

for a given vector field $\boldsymbol{B} \in H(div, \Omega)$. *If* $div(\boldsymbol{B}) \equiv 0$ *then* $div(\boldsymbol{B}^h) \equiv 0$.

Proof: We choose $\psi = \partial_x \phi$ and $\psi = \partial_y \phi$ for some $\phi \in Q_k(x, y)$ respectively in the two cell moment equations (4), (5). Adding these two equations together, we get

$$\int_C (B_x^h \partial_x \phi + B_y^h \partial_x \phi) \mathsf{d}x \mathsf{d}y = \int_C (B_x \partial_x \phi + B_y \partial_y \phi) \mathsf{d}x \mathsf{d}y$$

Performing integration by parts on both sides

$$-\int_C \operatorname{div}(\boldsymbol{B}^h) \phi \mathsf{d}x \mathsf{d}y + \int_{\partial C} \phi \boldsymbol{B}^h \cdot \boldsymbol{n} \mathsf{d}s = -\int_C \operatorname{div}(\boldsymbol{B}) \phi \mathsf{d}x \mathsf{d}y + \int_{\partial C} \phi \boldsymbol{B} \cdot \boldsymbol{n} \mathsf{d}s$$

Note that ϕ restricted to ∂C is a one dimensional polynomial of degree k and the edge moments of $\boldsymbol{B}^h$ and $\boldsymbol{B}$ agree with one another by equations (2), (3). Hence we get

$$\int_C \operatorname{div}(\boldsymbol{B}^h) \phi \mathsf{d}x \mathsf{d}y = \int_C \operatorname{div}(\boldsymbol{B}) \phi \mathsf{d}x \mathsf{d}y \qquad \forall \phi \in Q_k(x, y)$$

If $\mathrm{div}(\boldsymbol{B}) \equiv 0$ then

$$\int_C \mathrm{div}(\boldsymbol{B}^h)\phi \mathsf{d}x\mathsf{d}y = 0 \qquad \forall \phi \in Q_k(x,y)$$

Since $\mathrm{div}(\boldsymbol{B}^h) \in Q_k(x,y)$ this implies that $\mathrm{div}(\boldsymbol{B}^h) \equiv 0$ everywhere inside the cell C. □

Remark: The proof makes use of integration by parts for which the quadrature must be exact. The integrals involving $\boldsymbol{B}^h$ can be evaluated exactly using Gauss quadrature of sufficient accuracy. This is not the case for the integrals involving $\boldsymbol{B}$ since it can be an arbitrary nonlinear function. When $\mathrm{div}(\boldsymbol{B}) = 0$, we have $\boldsymbol{B} = (\partial_y \Phi, -\partial_x \Phi)$ for some smooth function Φ. We can approximate Φ by $\Phi_h \in Q_{k+1}$ and compute the projections using $(\partial_y \Phi_h, -\partial_x \Phi_h)$ in which case the integrations can be performed exactly.

Example: $\mathbf{RT}_0$

$$B_x^h(x,y) = a_0 + a_1 x, \qquad B_y^h(x,y) = b_0 + b_1 y$$

In this case we have only the edge moments. The polynomial test function spaces needed to specify the edge moments are

$$P_0(x) = \text{span}\{1\}, \qquad P_0(y) = \text{span}\{1\}$$

and the four moments corresponding to the four faces are

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} B_x^h(-1/2,y)\mathsf{d}y = \alpha_1 \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} B_x^h(1/2,y)\mathsf{d}y = \alpha_2$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} B_y^h(x,-1/2)\mathsf{d}x = \beta_1 \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} B_y^h(x,1/2)\mathsf{d}x = \beta_2$$

The solution is given by

$$a_0 = \tfrac{1}{2}(\alpha_1 + \alpha_2), \qquad a_1 = \alpha_2 - \alpha_1$$

$$b_0 = \tfrac{1}{2}(\beta_1 + \beta_2), \qquad b_1 = \beta_2 - \beta_1$$

Example: $\mathbf{RT}_1$

$$
\begin{aligned}
B_x^h(x,y) &= a_0 + a_1 x + a_2 y + a_3 xy + a_4(x^2 - \tfrac{1}{12}) + a_5(x^2 - \tfrac{1}{12})y \\
B_y^h(x,y) &= b_0 + b_1 x + b_2 y + b_3 xy + b_4(y^2 - \tfrac{1}{12}) + b_5 x(y^2 - \tfrac{1}{12})
\end{aligned}
$$

The polynomial test function spaces needed to specify the moments (2)-(5) are

$$P_1(x) = \mathrm{span}\{1, x\}, \qquad P_1(y) = \mathrm{span}\{1, y\}$$

$$\partial_x Q_1(x,y) = \mathrm{span}\{1, y\}, \qquad \partial_y Q_1(x,y) = \mathrm{span}\{1, x\}$$

Example: $\mathbf{RT}_2$

The polynomial test function spaces needed to specify the edge moments are

$$P_2(x) = \text{span}\{1, x, x^2 - \tfrac{1}{12}\}, \qquad P_2(y) = \text{span}\{1, y, y^2 - \tfrac{1}{12}\}$$

while those needed for the cell moments are given by

$$\partial_x Q_2(x, y) = \text{span}\{1, x, y, xy, y^2 - \tfrac{1}{12}, x(y^2 - \tfrac{1}{12})\}$$

$$\partial_y Q_2(x, y) = \text{span}\{1, x, y, xy, x^2 - \tfrac{1}{12}, (x^2 - \tfrac{1}{12})y\}$$

DG scheme for the induction equation

Constructing $\boldsymbol{B}^h$ from the edge and cell moments allowed us to get divergence-free approximation.

We will construct a scheme to evolve the same moments in time.

Edge moments are evolved by

$$\int_{e_x^\mp} \frac{\partial B_x^h}{\partial t}\phi \mathsf{d}y - \int_{e_x^\mp} \hat{E}\frac{\partial \phi}{\partial y}\mathsf{d}y + [\tilde{E}\phi]_{e_x^\mp} = -\int_{e_x^\mp} \hat{M}_x \phi \mathsf{d}y, \quad \forall \phi \in P_k(y) \quad (6)$$

$$\int_{e_y^\mp} \frac{\partial B_y^h}{\partial t}\phi \mathsf{d}x + \int_{e_y^\mp} \hat{E}\frac{\partial \phi}{\partial x}\mathsf{d}x - [\tilde{E}\phi]_{e_y^\mp} = -\int_{e_y^\mp} \hat{M}_y \phi \mathsf{d}y, \quad \forall \phi \in P_k(x) \quad (7)$$

where

$$\hat{E} = \text{numerical flux from a 1-D Riemann solver required on the faces}$$

$$\tilde{E} = \text{numerical flux from a multi-D Riemann solver needed at vertices}$$

DG scheme for the induction equation

$$[\tilde{E}\phi]_{e_x^-} = (\tilde{E}\phi)_2 - (\tilde{E}\phi)_0, \qquad [\tilde{E}\phi]_{e_x^+} = (\tilde{E}\phi)_3 - (\tilde{E}\phi)_1$$

$$[\tilde{E}\phi]_{e_y^-} = (\tilde{E}\phi)_1 - (\tilde{E}\phi)_0, \qquad [\tilde{E}\phi]_{e_y^+} = (\tilde{E}\phi)_3 - (\tilde{E}\phi)_2$$

The cells moments are evolved by the following standard DG scheme

$$\int_C \frac{\partial B_x^h}{\partial t}\psi \mathsf{d}x\mathsf{d}y - \int_C E\frac{\partial \psi}{\partial y}\mathsf{d}x\mathsf{d}y + \int_{\partial C}\hat{E}\psi n_y \mathsf{d}s = -\int_C M_x \psi \mathsf{d}x\mathsf{d}y, \quad \forall \psi \in \partial_x Q_k(x,y) \tag{8}$$

$$\int_C \frac{\partial B_y^h}{\partial t}\psi \mathsf{d}x\mathsf{d}y + \int_C E\frac{\partial \psi}{\partial x}\mathsf{d}x\mathsf{d}y - \int_{\partial C}\hat{E}\psi n_x \mathsf{d}s = -\int_C M_y \psi \mathsf{d}x\mathsf{d}y, \quad \forall \psi \in \partial_y Q_k(x,y) \tag{9}$$

Note that the same 1-D numerical flux $\hat{E}$ is used in both the edge and cell moment equations whereas the vertex numerical flux $\tilde{E}$ is needed only in the edge moment equations.

Theorem

Assuming that $\boldsymbol{M} = 0$, the DG scheme (6)-(9) preserves the divergence of the magnetic field.

Proof: For any $\phi \in Q_k(x, y)$ take $\psi = \partial_x \phi$ and $\psi = \partial_y \phi$ in the two cell moment equations respectively and add them together to obtain

$$\int_C \left[\frac{\partial B_x^h}{\partial t} \partial_x \phi + \frac{\partial B_y^h}{\partial t} \partial_y \phi \right] \mathrm{d}x \mathrm{d}y - \int_{\partial C} \hat{E} (n_x \partial_y \phi - n_y \partial_x \phi) \mathrm{d}s = 0$$

Note that two of the cell integrals cancel since $\partial_x \partial_y \phi = \partial_y \partial_x \phi$. Performing an integration by parts in the first term, we obtain

$$- \int_C \phi \frac{\partial}{\partial t} \mathrm{div}(\boldsymbol{B}^h) \mathrm{d}x \mathrm{d}y + \int_{\partial C} \phi \frac{\partial}{\partial t} (\boldsymbol{B}^h \cdot \boldsymbol{n}) \mathrm{d}s - \int_{\partial C} \hat{E} (n_x \partial_y \phi - n_y \partial_x \phi) \mathrm{d}s = 0$$

Now, let us concentrate on the last two terms which can be re-arranged as follows

$$\begin{aligned}
&= \int_{e_x^+} \phi \frac{\partial B_x^h}{\partial t} \mathsf{d}y - \int_{e_x^-} \phi \frac{\partial B_x^h}{\partial t} \mathsf{d}y + \int_{e_y^+} \phi \frac{\partial B_y^h}{\partial t} \mathsf{d}x - \int_{e_y^-} \phi \frac{\partial B_y^h}{\partial t} \mathsf{d}x \\
&\quad - \int_{e_x^+} \hat{E} \partial_y \phi \mathsf{d}y + \int_{e_x^-} \hat{E} \partial_y \phi \mathsf{d}y + \int_{e_y^+} \hat{E} \partial_x \phi \mathsf{d}x - \int_{e_y^-} \hat{E} \partial_x \phi \mathsf{d}x \\
&= -[\tilde{E}\phi]_{e_x^+} + [\tilde{E}\phi]_{e_x^-} + [\tilde{E}\phi]_{e_y^+} - [\tilde{E}\phi]_{e_y^-} \\
&= -(\tilde{E}\phi)_3 + (\tilde{E}\phi)_1 + (\tilde{E}\phi)_2 - (\tilde{E}\phi)_0 + (\tilde{E}\phi)_3 - (\tilde{E}\phi)_2 - (\tilde{E}\phi)_1 + (\tilde{E}\phi)_0 \\
&= 0
\end{aligned}$$

where we used the edge moment equations since the restriction of ϕ on each edge is a one dimensional polynomial of degree k. Hence we have

$$\int_C \phi \frac{\partial}{\partial t} \mathrm{div}(\boldsymbol{B}^h) \mathsf{d}x \mathsf{d}y = 0 \qquad \forall \phi \in Q_k(x, y)$$

Since $\mathrm{div}(\boldsymbol{B}^h) \in Q_k(x, y)$ we conclude that the divergence is preserved by the numerical scheme. □

Remark: The above proof required integration by parts in the terms involving the time derivative. These integrals can be computed exactly using Gauss quadrature of sufficient order. The other cell integral in the DG scheme can be computed using any quadrature rule of sufficient order and need not be exact. All the edge integrals which involve the numerical flux $\hat{E}$ appearing in the edge moment and cell moment evolution equations must be computed with the same rule and it is not necessary to be exact for the above proof to hold. However, from an accuracy point of view, these quadratures must be of a sufficiently high order to obtain optimal error estimates.

Remark: The preservation of divergence does not rely on the specific form of the fluxes $\tilde{E}$, $\hat{E}$ but only on the fact that we have a unique flux $\tilde{E}$ at all the vertices, and that we use the same 1-D numerical flux $\hat{E}$ in both the edge and cell moment equations.

Theorem

The divergence evolves consistently with equation (1) in the sense that

$$\int_C \phi \frac{\partial}{\partial t} div(\boldsymbol{B}^h)\, dx\, dy - \int_C \boldsymbol{M} \cdot \nabla \phi\, dx\, dy + \int_{\partial C} \phi \hat{\boldsymbol{M}} \cdot \boldsymbol{n}\, ds = 0, \qquad \forall \phi \in Q_k$$

Since $div(\boldsymbol{B}^h) \in Q_k$*, we can expect* $div(\boldsymbol{B}^h)$ *to be accurate to* $O(h^{k+1})$*.*

Numerical fluxes

Using the zero divergence condition, rewrite the induction equation

$$\frac{\partial B_x}{\partial t}+\boldsymbol{v}\cdot\nabla B_x+B_x\frac{\partial v_y}{\partial y}-B_y\frac{\partial v_x}{\partial y}=0,\quad \frac{\partial B_y}{\partial t}+\boldsymbol{v}\cdot\nabla B_y+B_y\frac{\partial v_x}{\partial x}-B_x\frac{\partial v_y}{\partial x}=0$$

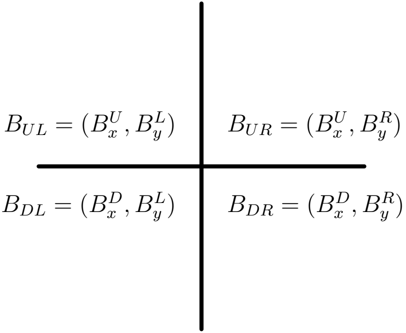
The characteristics are the integral curves of $\boldsymbol{v}$. The 1-D numerical flux is given by

$$\hat{E}=\begin{cases} E_L & \text{if } \boldsymbol{v}\cdot\boldsymbol{n}>0 \\ E_R & \text{otherwise}\end{cases}$$

For example, acorss the face $e_x^{\mp}$, the flux is given by

$$\hat{E}=\begin{cases} v_y B_x - v_x B_y^L & \text{if } v_x>0 \\ v_y B_x - v_x B_y^R & \text{otherwise}\end{cases}$$

Numerical fluxes



The corner flux is given by

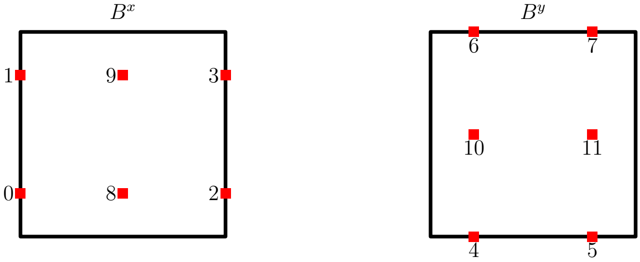
$$\tilde{E} = \begin{cases} E_{DL} & \text{if } v_x > 0,\ v_y > 0 \\ E_{UL} & \text{if } v_x > 0,\ v_y < 0 \\ E_{DR} & \text{if } v_x < 0,\ v_y > 0 \\ E_{UR} & \text{if } v_x < 0,\ v_y < 0 \end{cases}$$

which can be written in compact form as

$$\tilde{E} = \frac{v_y}{2}(B_x^U + B_x^D) - \frac{v_x}{2}(B_y^L + B_y^R) - \frac{|v_y|}{2}\left(B_x^U - B_x^D\right) + \frac{|v_x|}{2}\left(B_y^R - B_y^L\right)$$

At inflow boundaries where $\boldsymbol{v} \cdot \boldsymbol{n} < 0$, the flux is computed from the specified boundary values, while at outflow boundaries, the flux is determined from the interior solution.

Cell mass matrix



$$
\begin{bmatrix}
M^x & 0 & 0 & 0 & 0 & 0 \\
0 & M^x & 0 & 0 & 0 & 0 \\
0 & 0 & M^y & 0 & 0 & 0 \\
0 & 0 & 0 & M^y & 0 & 0 \\
N_l^x & N_r^x & 0 & 0 & Q^x & 0 \\
0 & 0 & N_b^y & N_t^y & 0 & Q^y
\end{bmatrix}
$$

Numerical Results

- Edge quadrature using $(k+2)$-point GL rule
- Cell quadrature using $(k+2) \times (k+2)$-point GL rule
- Time integration by 3-stage, 3-rd order SSPRK
- Time step

$$\Delta t < \frac{1}{(2k+1)\max\left(\frac{|v_x|}{\Delta x} + \frac{|v_y|}{\Delta y}\right)}$$

- Code written using `deal.II` library

Approximation property

Take $\boldsymbol{B} = (\partial_x \Phi, -\partial_x \Phi)$ where

$$\Phi(x, y) = \sin(2\pi x)\sin(2\pi y), \qquad (x, y) \in [0, 1] \times [0, 1]$$

h	$\|B - B_h\|_{L^2(\Omega)}$		$\|div(B_h)\|_{L^2(\Omega)}$
0.1250	1.0189e-01	-	3.7147e-14
0.0625	2.5519e-02	2.00	9.5162e-14
0.0312	6.3826e-03	2.00	3.7880e-13
0.0156	1.5958e-03	2.00	1.4840e-12
0.0078	3.9896e-04	2.00	5.8016e-12

Table: Error convergence for $k = 1$

h	$\|B - B_h\|_{L^2(\Omega)}$		$\|div(B_h)\|_{L^2(\Omega)}$
0.1250	6.7521e-03	-	1.3265e-13
0.0625	8.4659e-04	3.00	3.7389e-13
0.0312	1.0590e-04	3.00	1.3266e-12
0.0156	1.3241e-05	3.00	5.2716e-12
0.0078	1.6552e-06	3.00	2.0924e-11

Table: Error convergence for $k = 2$

Smooth test case

The initial condition is given by $\boldsymbol{B}_0 = (\partial_y \Phi, -\partial_x \Phi)$ where

$$\Phi(x, y) = \frac{1}{10} \exp[-20((x - 1/2)^2 + y^2)]$$

and the velocity field is $\boldsymbol{v} = (y, -x)$. The exact solution is a pure rotation of the initial condition and is given by

$$\boldsymbol{B}(\boldsymbol{r}, t) = R(t)\boldsymbol{B}_0(R(-t)\boldsymbol{r}), \qquad R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

We compute the numerical solution on the computational domain $[-1, +1] \times [-1, +1]$ upto a final time of $T = 2\pi$ at which time the solution comes back to the initial condition.

Animation

Discontinuous test case

We take the potential

$$\Phi(x,y) = \begin{cases} 2y - 2x & x > y \\ 0 & \text{otherwise} \end{cases}$$

and the velocity field is $\boldsymbol{v} = (1,2)$. This leads to a discontinuous magnetic field with the discontinuity along the line $x = y$

$$\boldsymbol{B}_0 = \begin{cases} (2,2) & x > y \\ (0,0) & x < y \end{cases}$$

The exact solution is given by

$$\boldsymbol{B}(x,y,t) = \boldsymbol{B}_0(x - t, y - 2t)$$

Animation

Summary

- DG scheme on Cartesian meshes
- Globally divergence-free solutions
- Arbitrary orders possible
- Local mass matrices: good for explicit time-stepping

Ongoing work

- Adaptive mesh refinement
- Limiters
- Application to
  - Maxwell equations (CED)
  - Magnetohydrodynamics (MHD)

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Thank You

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