Well-balanced schemes for Euler equations with gravity

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Euler equations with gravity

Flow properties

\[ \rho = \text{density}, \quad u = \text{velocity} \]

\[ p = \text{pressure}, \quad E = \text{total energy} \]

Gravitational potential \( \phi \); force per unit volume of fluid

\[ -\rho \nabla \phi \]

System of conservation laws

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \]

\[ \frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (p + \rho u^2) = -\rho \frac{\partial \phi}{\partial x} \]

\[ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x} (E + p) u = -\rho u \frac{\partial \phi}{\partial x} \]
**Euler equations with gravity**

Perfect gas assumption

\[ p = (\gamma - 1) \left[ E - \frac{1}{2} \rho u^2 \right], \quad \gamma = \frac{c_p}{c_v} > 1 \]

In compact notation

\[
\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = - \begin{bmatrix} 0 \\ \rho \\ \rho u \end{bmatrix} \frac{\partial \phi}{\partial x}
\]

where

\[
q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E + p)u \end{bmatrix}
\]
Hydrostatic solutions

- Fluid at rest
  \[ u_e = 0 \]

- Mass and energy equation satisfied

- Momentum equation
  \[ \frac{dp_e}{dx} = -\rho_e \frac{d\phi}{dx} \quad (1) \]

- Need additional assumptions to solve this equation
- Assume ideal gas and some temperature profile \( T_e(x) \)

\[ p_e(x) = \rho_e(x)RT_e(x), \quad R = \text{gas constant} \]

integrate (1) to obtain

\[ p_e(x) = p_0 \exp \left( - \int_{x_0}^{x} \frac{\phi'(s)}{RT_e(s)} ds \right) \]
Hydrostatic solutions

- If the hydrostatic state is *isothermal*, i.e., $T_e(x) = T_e = \text{const}$, then

$$p_e(x) \exp \left( \frac{\phi(x)}{RT_e} \right) = \text{const} \quad (2)$$

Density

$$\rho_e(x) = \frac{p_e(x)}{RT_e}$$

- If the hydrostatic solution is *polytropic* then we have following relations

$$p_e \rho_e^{-\nu} = \text{const}, \quad p_e T_e^{\frac{-\nu}{\nu - 1}} = \text{const}, \quad \rho_e T_e^{\frac{-1}{\nu - 1}} = \text{const} \quad (3)$$

where $\nu > 1$ is some constant. From (1) and (3), we obtain

$$\frac{\nu RT_e(x)}{\nu - 1} + \phi(x) = \text{const} \quad (4)$$

E.g., pressure is

$$p_e(x) = C_1 \left[ C_2 - \phi(x) \right]^{\frac{\nu - 1}{\nu}}$$
Existing schemes

• Isothermal case: Xing and Shu [2], well-balanced WENO scheme
• If \( \nu = \gamma \) we are in isentropic case

\[
h(x) + \phi(x) = \text{const}
\]

has been considered by Kappeli and Mishra [1].
• Desveaux et al: Relaxation schemes, general hydrostatic states
Well-balanced scheme

• Scheme is well-balanced if it exactly preserves hydrostatic solution.

• General evolutionary PDE

\[ \frac{\partial q}{\partial t} = R(q) \]

• Stationary solution \( q_e \)

\[ R(q_e) = 0 \]

• We are interested in computing small perturbations

\[ q(x, 0) = q_e(x) + \varepsilon \tilde{q}(x, 0), \quad \varepsilon \ll 1 \]

• Perturbations are governed by linear equation

\[ \frac{\partial \tilde{q}}{\partial t} = R'(q_e)\tilde{q} \]
Well-balanced scheme

• Some numerical scheme

$$\frac{\partial q_h}{\partial t} = R_h(q_h)$$

• $q_{h,e} =$ interpolation of $q_e$ onto the mesh

• Scheme is well balanced if

$$R_h(q_{h,e}) = 0 \implies \frac{\partial q_h}{\partial t} = 0$$

• Suppose scheme is not well-balanced $R_h(q_{h,e}) \neq 0$. Solution

$$q_h(x, t) = q_{h,e}(x) + \varepsilon \tilde{q}_h(x, t)$$
Well-balanced scheme

- Linearize the scheme around $q_{h,e}$

\[ \frac{\partial}{\partial t}(q_{h,e} + \varepsilon \tilde{q}_h) = R_h(q_{h,e} + \varepsilon \tilde{q}_h) = R_h(q_{h,e}) + \varepsilon R'_h(q_{h,e}) \tilde{q}_h \]

or

\[ \frac{\partial \tilde{q}_h}{\partial t} = \frac{1}{\varepsilon} R_h(q_{h,e}) + R'_h(q_{h,e}) \tilde{q}_h \]

- Scheme is consistent of order $r$: $R_h(q_{h,e}) = Ch^r \| q_{h,e} \|

\[ \frac{\partial \tilde{q}_h}{\partial t} = \frac{1}{\varepsilon} Ch^r \| q_{h,e} \| + R'_h(q_{h,e}) \tilde{q}_h \]

- $\varepsilon \ll 1$ then first term may dominate the second term; need $h \ll 1$

- Canonical approach

\[ \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(p + \rho u^2) = -\rho \frac{\partial \phi}{\partial x} \]

\[ \frac{d}{dt}(\rho u)_i + \frac{1}{\Delta x} \left[ \hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}} \right] = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \]
Scope of present work

- Second order finite volume scheme
- Ideal gas model: well-balanced for both isothermal and polytropic solutions
- Most numerical fluxes can be used
Source term [2]

Define
\[
\psi(x) = -\int_{x_0}^{x} \frac{\phi'(s)}{RT(s)} ds, \quad x_0 \text{ is arbitrary}
\]

Then
\[
\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} \int_{x_0}^{x} \frac{\phi'(s)}{RT(s)} ds = -\frac{\phi'(x)}{RT(x)}
\]
and
\[
\frac{\partial}{\partial x} \exp(\psi(x)) = \exp(\psi(x)) \frac{\partial \psi}{\partial x} = -\exp(\psi(x)) \frac{\phi'(x)}{RT(x)}
\]
so that
\[
-\rho(x) \frac{\partial \phi}{\partial x} = p(x) \exp(-\psi(x)) \frac{\partial}{\partial x} \exp(\psi(x))
\]

Euler equations
\[
\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix} \exp(-\psi(x)) \frac{\partial}{\partial x} \exp(\psi(x))
\]
1-D finite volume scheme

- Divide domain into $N$ finite volumes each of size $\Delta x$
- $i'$th cell = $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$
- semi-discrete finite volume scheme for the $i'$th cell

$$\frac{dq_i}{dt} + \frac{\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}}{\Delta x} = e^{-\psi_i} \left( \frac{e^{\psi_{i+\frac{1}{2}}} - e^{\psi_{i-\frac{1}{2}}}}{\Delta x} \right) \begin{bmatrix} 0 \\ p_i \\ p_i u_i \end{bmatrix}$$

- $\psi_i$, $\psi_{i+\frac{1}{2}}$ etc. are consistent approximations to the function $\psi(x)$
- consistent numerical flux $\hat{f}_{i+\frac{1}{2}} = \hat{f}(q_{i+\frac{1}{2}}^L, q_{i+\frac{1}{2}}^R)$
The numerical flux $\hat{f}$ is said to satisfy Property C if for any two states $q^L = [\rho^L, 0, p/(\gamma - 1)]$ and $q^R = [\rho^R, 0, p/(\gamma - 1)]$ we have

$$\hat{f}(q^L, q^R) = [0, p, 0]^\top$$

- states $q^L$, $q^R$ in the above definition correspond to a stationary contact discontinuity.
- Property C $\implies$ numerical flux exactly support a stationary contact discontinuity.
- Examples of such numerical flux: Roe, HLLC
1-D finite volume scheme

- First order scheme

\[ q_{i+\frac{1}{2}}^L = q_i, \quad q_{i+\frac{1}{2}}^R = q_{i+1} \]

- Higher order scheme: To obtain the states \( q_{i+\frac{1}{2}}^L, q_{i+\frac{1}{2}}^R \), reconstruct the following set of variables

\[ \mathbf{w} = \left[ \rho e^{-\psi}, u, pe^{-\psi} \right]^T \]

- Once \( \mathbf{w}_{i+\frac{1}{2}}^L \) etc. are computed, the primitive variables are obtained as

\[ \rho_{i+\frac{1}{2}}^L = e^{\psi_{i+\frac{1}{2}}^+} (w_1)_{i+\frac{1}{2}}^L, \quad u_{i+\frac{1}{2}}^L = (w_2)_{i+\frac{1}{2}}^L, \quad p_{i+\frac{1}{2}}^L = e^{\psi_{i+\frac{1}{2}}^+} (w_3)_{i+\frac{1}{2}}^L, \quad \text{etc.} \]
Well-balanced property

Theorem

The finite volume scheme (5) together with a numerical flux which satisfies property C and reconstruction of $w$ variables is well-balanced in the sense that the initial condition given by

$$u_i = 0, \quad p_i \exp(-\psi_i) = \text{const}, \quad \forall \ i$$

(6)

is preserved by the numerical scheme.

Proof: Start computation with an initial condition that satisfies (6). Since we reconstruct the variables $w$, at any interface $i + \frac{1}{2}$ we have

$$(w_2)_i^{L} = (w_2)_i^{R} = 0, \quad (w_3)_i^{L} = (w_3)_i^{R}$$

Hence

$$u_i^{L} = u_i^{R} = 0, \quad p_i^{L} = p_i^{R} = p_i \exp(\psi_{i+\frac{1}{2}} - \psi_i) =: p_{i+\frac{1}{2}}$$
Well-balanced property

and at $i - \frac{1}{2}$

$$u_{i-\frac{1}{2}}^L = u_{i-\frac{1}{2}}^R = 0, \quad p_{i-\frac{1}{2}}^L = p_{i-\frac{1}{2}}^R = p_i \exp(\psi_{i-\frac{1}{2}} - \psi_i) =: p_{i-\frac{1}{2}}$$

Since the numerical flux satisfies property C, we have

$$\hat{f}_{i-\frac{1}{2}} = [0, p_{i-\frac{1}{2}}, 0]^\top, \quad \hat{f}_{i+\frac{1}{2}} = [0, p_{i+\frac{1}{2}}, 0]^\top$$

Mass and energy equations are already well balanced, i.e.,

$$\frac{dq_i^{(1)}}{dt} = 0, \quad \frac{dq_i^{(3)}}{dt} = 0$$

Momentum equation: on the left we have

$$\frac{\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}}{\Delta x} = \frac{p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}}{\Delta x}$$
Well-balanced property

while on the right

\[ p_i e^{-\psi_i} \frac{e^{\psi_i + \frac{1}{2}} - e^{\psi_i - \frac{1}{2}}}{\Delta x} = p_i e^{\psi_i + \frac{1}{2} - \psi_i} - p_i e^{\psi_i - \frac{1}{2} - \psi_i} = \frac{p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}}{\Delta x} \]

and hence

\[ \frac{dq_i^{(2)}}{dt} = 0 \]

This proves that the initial condition is preserved under any time integration scheme. \(\square\)
Approximation of source term

- How to approximate $\psi_i$, $\psi_{i+\frac{1}{2}}$, etc.? Need some quadrature
- well-balanced property independent of quadrature rule to compute $\psi$.
- To preserve isothermal/polytropic solutions exactly, the quadrature rule has to be exact for these cases.
- To compute the source term in the $i$’th cell, we define the function $\psi(x)$ as follows

$$
\psi(x) = -\int_{x_i}^{x} \frac{\phi'(s)}{RT(s)} \, ds
$$

where we chose the reference position as $x_i$. 
Approximation of source term

- To approximate the integrals we define the piecewise constant temperature as follows

\[ T(x) = \hat{T}_{i+\frac{1}{2}}, \quad x_i < x < x_{i+1} \] (7)

where \( \hat{T}_{i+\frac{1}{2}} \) is the logarithmic average given by

\[ \hat{T}_{i+\frac{1}{2}} = \frac{T_{i+1} - T_i}{\log T_{i+1} - \log T_i} \]

- The integrals are evaluated using the approximation of the temperature given in (7) leading to the following expressions for \( \psi \).

\[
\begin{align*}
\psi_i &= 0 \\
\psi_{i-\frac{1}{2}} &= -\frac{1}{RT\hat{T}_{i-\frac{1}{2}}} \int_{x_i}^{x_i-\frac{1}{2}} \phi'(s)ds = \frac{\phi_i - \phi_{i-\frac{1}{2}}}{RT\hat{T}_{i-\frac{1}{2}}} \\
\psi_{i+\frac{1}{2}} &= -\frac{1}{RT\hat{T}_{i+\frac{1}{2}}} \int_{x_i}^{x_i+\frac{1}{2}} \phi'(s)ds = \frac{\phi_i - \phi_{i+\frac{1}{2}}}{RT\hat{T}_{i+\frac{1}{2}}}
\end{align*}
\]
Approximation of source term

- Gravitational potential required at faces $\phi_{i+\frac{1}{2}}$
- $\phi$ is governed by Poisson equation and hence is a smooth function. We can interpolate

$$\phi_{i+\frac{1}{2}} = \frac{1}{2}(\phi_i + \phi_{i+1})$$

Sufficient to obtain second order accuracy. Then

$$\psi_{i-\frac{1}{2}} = \frac{\phi_i - \phi_{i-1}}{2R\hat{T}_{i-\frac{1}{2}}}, \quad \psi_i = 0, \quad \psi_{i+\frac{1}{2}} = \frac{\phi_i - \phi_{i+1}}{2R\hat{T}_{i+\frac{1}{2}}} \quad (8)$$
Approximation of source term

**Theorem**

*The source term discretization given by (8) is second order accurate.*

**Proof:** The source term in (5) has the factor

\[
e^{-\psi_i} \frac{e^{\psi_{i+\frac{1}{2}}} - e^{\psi_{i-\frac{1}{2}}}}{\Delta x} = \frac{\exp\left(\frac{\phi_i - \phi_{i+1}}{2R\hat{T}_{i+\frac{1}{2}}}\right) - \exp\left(\frac{\phi_i - \phi_{i-1}}{2R\hat{T}_{i-\frac{1}{2}}}\right)}{\Delta x}
\]

using (8)

Using a Taylor expansion around \(x_i\) we get

\[
\frac{1}{\hat{T}_{i-\frac{1}{2}}} = \frac{1}{T_i} [1 + O(\Delta x^2)], \quad \frac{1}{\hat{T}_{i+\frac{1}{2}}} = \frac{1}{T_i} [1 + O(\Delta x^2)]
\]
Approximation of source term

and

\[ e^{-\frac{\phi_i-\phi_{i+1}}{2RT_i} + \frac{\phi_i-\phi_{i-1}}{2RT_i} \frac{1}{2}} - e^{-\frac{\phi_i-\phi_{i+1}}{2RT_i} + \frac{\phi_i-\phi_{i-1}}{2RT_i} \frac{1}{2}} \]

\[ = e^{\frac{1}{2RT_i}}(-\phi'_i \Delta x - \phi''_i \Delta x^2 + O(\Delta x^3)) - e^{\frac{1}{2RT_i}}(+\phi'_i \Delta x - \phi''_i \Delta x^2 + O(\Delta x^3)) \]

\[ = \left[ 1 + \frac{1}{2RT_i}(-\phi'_i \Delta x - \phi''_i \Delta x^2) + \frac{1}{2(2RT_i)^2}(\phi'_i \Delta x)^2 + O(\Delta x^3) \right] - \left[ 1 + \frac{1}{2RT_i}(\phi'_i \Delta x - \phi''_i \Delta x^2) + \frac{1}{2(2RT_i)^2}(\phi'_i \Delta x)^2 + O(\Delta x^3) \right] \]

\[ = -\frac{1}{RT_i} \phi'(x_i) \Delta x + O(\Delta x^3) \]

Hence the source term discretization is second order accurate. \(\square\)
**Theorem**

Any hydrostatic solution which is isothermal or polytropic is exactly preserved by the finite volume scheme (5).

**Proof:** Take initial condition to be a hydrostatic solution. We have to verify that the initial condition satisfies equation (6).

**Isothermal case:**  
\[
\hat{T}_{i+\frac{1}{2}} = T_e = \text{const}, \quad \text{and using (2) we obtain}
\]

\[
\frac{p_{i+1}e^{-\psi_{i+1}}}{p_i e^{-\psi_i}} = \frac{p_{i+1}}{p_i} e^{\psi_i - \psi_{i+1}} = \frac{p_{i+1}}{p_i} \exp \left( \frac{\phi_{i+1} - \phi_i}{RT_e} \right) = \frac{p_{i+1} \exp(\phi_{i+1}/RT_e)}{p_i \exp(\phi_i/RT_e)}
\]

**Polytropic case:**

\[
\frac{p_{i+1}e^{-\psi_{i+1}}}{p_i e^{-\psi_i}} = \frac{p_{i+1}}{p_i} e^{\psi_i - \psi_{i+1}} = \frac{p_{i+1}}{p_i} \exp \left( \frac{\phi_{i+1} - \phi_i}{R\hat{T}_{i+\frac{1}{2}}} \right)
\]
But from (3), (4) we have

$$\frac{\phi_{i+1} - \phi_i}{R\hat{T}_{i+\frac{1}{2}}} = -\frac{\nu R}{\nu-1} \frac{(T_{i+1} - T_i)}{R \frac{T_{i+1} - T_i}{\log(T_{i+1}) - \log(T_i)}} = \log \left( \frac{T_i}{T_{i+1}} \right)^{\frac{\nu}{\nu-1}}$$

and hence

$$\frac{p_{i+1}e^{-\psi_{i+1}}}{p_i e^{-\psi_i}} = \frac{p_{i+1}T_{i+1}^{-\nu/(\nu-1)}}{p_i T_i^{-\nu/(\nu-1)}} = 1$$

Hence in both cases, the initial condition is preserved by the finite volume scheme.
Summary of the scheme

Using the approximations given by (8), the semi-discrete finite volume scheme is given by

\[
\frac{dq_i}{dt} + \frac{\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}}{\Delta x} = e^{\hat{\beta}_{i+\frac{1}{2}}(\phi_i - \phi_{i+1})} - e^{\hat{\beta}_{i-\frac{1}{2}}(\phi_i - \phi_{i-1})} \left[ \begin{array}{c} 0 \\ p_i \\ p_i u_i \end{array} \right]
\]

where we have introduced the quantity

\[
\hat{\beta}_{i+\frac{1}{2}} = \frac{1}{2RT_{i+\frac{1}{2}}}
\]

As an example of reconstruction, we discuss the minmod-type scheme for the interface \(i + \frac{1}{2}\) which is given by

\[
\omega_{i+\frac{1}{2}}^L = \omega_i + \frac{1}{2}m(\theta(\omega_i - \omega_{i-1}), (\omega_{i+1} - \omega_{i-1})/2, \theta(\omega_{i+1} - \omega_i))
\]
Summary of the scheme

\[ w_{i+\frac{1}{2}}^R = w_{i+1} - \frac{1}{2} m(\theta(w_{i+1} - w_i), (w_{i+2} - w_{i+1})/2, \theta(w_{i+2} - w_{i+1})) \]

where \( \theta \in [1, 2] \) and \( m(\cdot, \cdot, \cdot) \) is the minmod limiter function given by

\[
m(a, b, c) = \begin{cases} 
s \min(|a|, |b|, |c|) & \text{if } s = \text{sign}(a) = \text{sign}(b) = \text{sign}(c) \\
0 & \text{otherwise}
\end{cases}
\]

The variables \( w \) are defined using the potential relative to \( x_{i+\frac{1}{2}} \)

\[
\psi(x) = -\int_{x_{i+\frac{1}{2}}}^{x} \frac{\phi'(s)}{RT(s)} \, ds
\]
Summary of the scheme

Then

\[ \psi_{i-1} = \frac{\phi_i - \phi_{i-1}}{RT_i^{-1/2}} + \frac{\phi_{i+1/2} - \phi_i}{RT_i^{+1/2}} = 2\beta_i^{-1/2} (\phi_i - \phi_{i-1}) + \beta_i^{+1/2} (\phi_{i+1} - \phi_i) \]

\[ \psi_i = \frac{\phi_{i+1/2} - \phi_i}{RT_i^{+1/2}} = \beta_i^{+1/2} (\phi_{i+1} - \phi_i) \]

\[ \psi_{i+1} = -\frac{\phi_{i+1} - \phi_{i+1/2}}{RT_i^{+1/2}} = -\beta_i^{+1/2} (\phi_{i+1} - \phi_i) \]

\[ \psi_{i+2} = -\frac{\phi_{i+1} - \phi_{i+1/2}}{RT_i^{+1/2}} - \frac{\phi_{i+2} - \phi_{i+1}}{RT_i^{+3/2}} = -\beta_i^{+1/2} (\phi_{i+1} - \phi_i) - 2\beta_i^{+3/2} (\phi_{i+2} - \phi_{i+1}) \]

In terms of the above \( \psi_i \)'s, the variables \( w \) are defined as follows

\[
\mathbf{w}_j = \begin{bmatrix} \rho_j e^{-\psi_j} \\ u_j \\ p_j e^{-\psi_j} \end{bmatrix}, \quad j = i - 1, i, i + 1, i + 2
\]
Summary of the scheme

Since $\psi_{i+\frac{1}{2}} = 0$ we obtain the reconstructed values as

$$
\begin{bmatrix}
\rho \\
u \\
p
\end{bmatrix}^{L}_{i+\frac{1}{2}} = w^L_{i+\frac{1}{2}},
\quad
\begin{bmatrix}
\rho \\
u \\
p
\end{bmatrix}^{R}_{i+\frac{1}{2}} = w^R_{i+\frac{1}{2}}
$$

For the first and last cells, we extrapolate the potential from inside the domain to the faces located on the domain boundary

$$
\phi_{\frac{1}{2}} = \frac{3}{2} \phi_1 - \frac{1}{2} \phi_2, \quad
\phi_{N+\frac{1}{2}} = \frac{3}{2} \phi_N - \frac{1}{2} \phi_{N-1}
$$
Isothermal examples: well-balanced test

Density and pressure are given by

$$\rho_e(x) = p_e(x) = \exp(-\phi(x))$$

$$N = 100, 1000, \text{ final time } = 2$$

<table>
<thead>
<tr>
<th>Potential 1</th>
<th>Potential 2</th>
<th>Potential 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(x)$</td>
<td>$x$</td>
<td>$\frac{1}{2}x^2$</td>
</tr>
</tbody>
</table>

**Table:** Potential functions used for well-balanced tests
### Isothermal examples: well-balanced test

<table>
<thead>
<tr>
<th>Potential</th>
<th>Cells</th>
<th>Density</th>
<th>Velocity</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
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**Table:** Error in density, velocity and pressure for isothermal example
Isentropic examples: well-balanced test

Isentropic hydrostatic solution

\[ T_e(x) = 1 - \frac{\gamma - 1}{\gamma} \phi(x), \quad \rho_e = T_e^{\frac{1}{\gamma - 1}}, \quad p_e = \rho_e^\gamma \]

\[ N = 100, 1000, \quad \text{final time} = 2 \]

<table>
<thead>
<tr>
<th>Potential</th>
<th>Cells</th>
<th>Density</th>
<th>Velocity</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>( \sin(2\pi x) )</td>
<td>100</td>
<td>1.27570e-14</td>
<td>5.18212e-16</td>
<td>1.65185e-14</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>1.29020e-13</td>
<td>1.12837e-15</td>
<td>1.66566e-13</td>
</tr>
</tbody>
</table>

**Table:** Error in density, velocity and pressure for isentropic example
Polytropic examples: well-balanced test

Polytropic hydrostatic solutions

\[ T_e(x) = 1 - \frac{\nu - 1}{\nu} \phi(x), \quad \rho_e = T_e^{\nu - 1}, \quad p_e = \rho_e^\nu \]

\[ \nu = 1.2, \quad N = 100, 1000, \quad \text{final time} = 2 \]

<table>
<thead>
<tr>
<th>Potential</th>
<th>Cells</th>
<th>Density</th>
<th>Velocity</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>100</td>
<td>6.86395e-15</td>
<td>2.65535e-16</td>
<td>7.88869e-15</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>7.03820e-14</td>
<td>7.79350e-16</td>
<td>8.03623e-14</td>
</tr>
<tr>
<td>\frac{1}{2}x^2</td>
<td>100</td>
<td>1.06604e-14</td>
<td>2.27512e-16</td>
<td>1.04128e-14</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>1.10726e-13</td>
<td>1.15415e-15</td>
<td>1.09185e-13</td>
</tr>
<tr>
<td>sin(2\pi x)</td>
<td>100</td>
<td>1.27570e-14</td>
<td>5.18212e-16</td>
<td>1.65185e-14</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>1.29020e-13</td>
<td>1.12837e-15</td>
<td>1.66566e-13</td>
</tr>
</tbody>
</table>

Table: Error in density, velocity and pressure for polytropic example
Non-isothermal example

- Stationary solution
  \[
  \phi(x) = \frac{1}{2} x^2, \quad \rho_e(x) = \exp(-x), \quad p_e(x) = (1 + x) \exp(-x)
  \]

- corresponds to a non-uniform temperature profile
  \[
  T_e(x) = 1 + x
  \]

- Neither isothermal nor polytropic; present scheme will not be able to preserve the exact hydrostatic solution

- Instead, we construct an approximation to the above hydrostatic solution by numerically integrating the hydrostatic equations (1)

\[
p_1 = p_e(x_1), \quad \rho_1 = \frac{p_1}{RT_e(x_1)}
\]

\[
p_i = p_{i-1} \exp(-2\hat{\beta}_{i-1/2} (\phi_i - \phi_{i-1})), \quad \rho_i = \frac{p_i}{RT_e(x_i)}, \quad i = 2, 3, \ldots
\]
Non-isothermal example

- The above solution satisfies equation (6) and hence is preserved by the numerical scheme.
- Solution converges at second order; velocity is zero up to machine precision indicating that we have a stationary solution.

<table>
<thead>
<tr>
<th>Cells</th>
<th>( \rho ) error</th>
<th>( \rho ) rate</th>
<th>Velocity</th>
<th>( p ) error</th>
<th>( p ) rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>5.41510e-06</td>
<td>-</td>
<td>3.90665e-16</td>
<td>8.51248e-06</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>1.37964e-06</td>
<td>1.97</td>
<td>1.06754e-15</td>
<td>2.16486e-06</td>
<td>1.97</td>
</tr>
<tr>
<td>200</td>
<td>3.48173e-07</td>
<td>1.98</td>
<td>4.82755e-16</td>
<td>5.45846e-07</td>
<td>1.98</td>
</tr>
<tr>
<td>400</td>
<td>8.74530e-08</td>
<td>1.99</td>
<td>1.94554e-15</td>
<td>1.37043e-07</td>
<td>1.99</td>
</tr>
<tr>
<td>800</td>
<td>2.19146e-08</td>
<td>1.99</td>
<td>2.62298e-15</td>
<td>3.43336e-08</td>
<td>1.99</td>
</tr>
<tr>
<td>1600</td>
<td>5.48521e-09</td>
<td>1.99</td>
<td>6.56911e-15</td>
<td>8.59273e-09</td>
<td>1.99</td>
</tr>
</tbody>
</table>

Table: Convergence of error for hydrostatic solution of section (34).
Evolution of small perturbations

The initial condition is taken to be the following

\[ \phi = \frac{1}{2} x^2, \quad u = 0, \quad \rho(x) = \exp(-\phi(x)) \]

Add small perturbation to equilibrium pressure

\[ p(x) = \exp(-\phi(x)) + \varepsilon \exp(-100(x - 1/2)^2), \quad 0 < \varepsilon \ll 1 \]

Non-well-balanced scheme

\[ \frac{\partial \phi}{\partial x}(x_i) \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}, \quad \text{reconstruct } \rho, u, p \]

Using exact derivative of potential does not improve results. In practice, \( \phi \) is only available at grid points.
Evolution of small perturbations

\[ \varepsilon = 10^{-3}, \quad N = 100 \text{ cells} \]

\[ \varepsilon = 10^{-5}, \quad N = 100 \text{ cells} \]
Evolution of small perturbations

\[ \varepsilon = 10^{-5}, \; N = 500 \text{ cells} \]
Shock tube under gravitational field

Gravitational field

\[ \phi(x) = x \]

The domain is \([0, 1]\) and the initial conditions are given by

\[
(\rho, u, p) = \begin{cases} 
(1, 0, 1) & x < \frac{1}{2} \\
(0.125, 0, 0.1) & x > \frac{1}{2}
\end{cases}
\]

Solid wall boundary conditions. Final time \(t = 0.2, \ N = 100, 2000\) cells
Shock tube under gravitational field
2-D Euler equations with gravity

2-D Euler equations in Cartesian coordinates

\[
\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = s
\]

Here the conserved variables \( q \), fluxes \((f, g)\) and source terms \( s \) are given by

\[
q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ (E + p)u \end{bmatrix}, \quad g = \begin{bmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ (E + p)v \end{bmatrix}, \quad s = \begin{bmatrix} 0 \\ -\rho \frac{\partial \phi}{\partial x} \\ -\rho \frac{\partial \phi}{\partial y} \\ -\rho(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y}) \end{bmatrix}
\]

In the above equations

\[
\rho = \text{density}, \quad (u, v) = \text{Cartesian components of the velocity}
\]

\[
p = \text{pressure}, \quad E = \text{total energy per unit volume}
\]

\[
\phi = \phi(x, y) = \text{gravitational potential}
\]
Hydrostatic solution

The hydrostatic equilibrium is characterized by the following set of equations

\[ u_e = v_e = 0, \quad \frac{\partial p_e}{\partial x} = -\rho_e \frac{\partial \phi}{\partial x}, \quad \frac{\partial p_e}{\partial y} = -\rho_e \frac{\partial \phi}{\partial y} \]

These equations can be integrated along \( y = \text{const} \) lines

\[ p_e(x, y) = a(y) \exp \left( -\int_{x_0}^{x} \frac{\phi_x(s, y)}{RT(s, y)} ds \right) \]

and \( x = \text{const} \) lines

\[ p_e(x, y) = b(x) \exp \left( -\int_{y_0}^{y} \frac{\phi_y(x, s)}{RT(x, s)} ds \right) \]

As in the 1-D case, we will exploit the structure of these solutions to construct the well-balanced scheme.
Source term

Define

\[ \psi(x, y) = -\int_{x_0}^{x} \frac{\phi_x(s, y)}{RT(s, y)} ds, \quad \chi(x, y) = -\int_{y_0}^{y} \frac{\phi_y(x, s)}{RT(x, s)} ds \]

Then the gravitational force can be written as

\[ -\rho \phi_x = pe^{-\psi} \frac{\partial}{\partial x} e^\psi, \quad -\rho \phi_y = pe^{-\chi} \frac{\partial}{\partial y} e^\chi \] (9)
2-d finite volume scheme on Cartesian meshes

- Partition computational domain into rectangular cells

\[ \Omega_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \]

with

\[ x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \Delta x \quad \text{and} \quad y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = \Delta y \]

- Semi-discrete finite volume scheme for the cell \((i, j)\)

\[ \Omega_{i,j} \frac{d}{dt} q_{i,j} + \hat{f}_{i+\frac{1}{2},j} - \hat{f}_{i-\frac{1}{2},j} + \hat{g}_{i,j+\frac{1}{2}} - \hat{g}_{i,j-\frac{1}{2}} = \hat{s}_{i,j} \quad (10) \]
2-d finite volume scheme on Cartesian meshes

- The gravitational source term is discretized as

\[
\hat{s}^{(1)}_{i,j} = 0
\]

\[
\hat{s}^{(2)}_{i,j} = p_{i,j}e^{-\psi_{i,j}} \left[ e^{\psi_{i+1/2,j} - e^{\psi_{i-1/2,j}}} \right]
\]

\[
\hat{s}^{(3)}_{i,j} = p_{i,j}e^{-\chi_{i,j}} \left[ e^{\chi_{i,j+1/2} - e^{\chi_{i,j-1/2}}} \right]
\]

\[
\hat{s}^{(4)}_{i,j} = u_{i,j}\hat{s}^{(2)}_{i,j} + v_{i,j}\hat{s}^{(3)}_{i,j}
\]

- Following the steps in the 1-D case, we can write the source terms as

\[
\hat{s}^{(2)}_{i,j} = p_{i,j} e^{\hat{\beta}_{i+1/2,j} (\phi_{i+1,j} - \phi_{i,j})} - e^{\hat{\beta}_{i-1/2,j} (\phi_{i-1,j} - \phi_{i,j})}
\]

\[
\hat{s}^{(3)}_{i,j} = p_{i,j} e^{\hat{\beta}_{i,j+1/2} (\phi_{i,j+1} - \phi_{i,j})} - e^{\hat{\beta}_{i,j-1/2} (\phi_{i,j-1} - \phi_{i,j})}
\]
To obtain the values at the face $q^L_{i+\frac{1}{2},j}$, $q^R_{i+\frac{1}{2},j}$ we reconstruct the following set of variables

$$w = [\rho e^{-\psi}, u, v, pe^{-\psi}]^T$$

and to obtain $q^L_{i,j+\frac{1}{2}}$, $q^R_{i,j+\frac{1}{2}}$, we reconstruct the following set of variables

$$w = [\rho e^{-\chi}, u, v, pe^{-\chi}]^T$$
Theorem

The finite volume scheme (10) together with a numerical flux which satisfies property C and reconstruction of $w$ variables is well-balanced in the sense that the initial condition given by

$$u_{i,j} = v_{i,j} = 0, \quad p_{i,j} \exp(-\psi_{i,j}) = a_j, \quad p_{i,j} \exp(-\chi_{i,j}) = b_i, \quad \forall \ i, j$$

is preserved by the numerical scheme.

Theorem

Any hydrostatic solution which is isothermal or polytropic is exactly preserved by the finite volume scheme (10).
Isothermal hydrostatic solution

unit square, potential

\[ \phi(x, y) = x + y \]

\[ \rho_e(x, y) = \rho_0 \exp(-\rho_0 g(x+y)/p_0), \quad p_e(x, y) = p_0 \exp(-\rho_0 g(x+y)/p_0) \]

\[ \rho_0 = 1.21, \quad p_0 = 1, \quad g = 1, \quad \text{final time} = 1 \]

<table>
<thead>
<tr>
<th>Grid</th>
<th>( \rho )</th>
<th>( u )</th>
<th>( v )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 ( \times ) 50</td>
<td>0.19050E-14</td>
<td>0.14660E-15</td>
<td>0.14439E-15</td>
<td>0.20428E-14</td>
</tr>
<tr>
<td>200 ( \times ) 200</td>
<td>0.75677E-14</td>
<td>0.12908E-14</td>
<td>0.12853E-14</td>
<td>0.83936E-14</td>
</tr>
</tbody>
</table>

**Table:** Error in density, velocity and pressure for isothermal hydrostatic example
Isothermal hydrostatic solution

To study the accuracy of the scheme, we add an initial perturbation to the pressure and take the following initial condition

\[ p(x, y, 0) = p_0 \exp(-\rho_0 g(x+y)/p_0) + \eta \exp(-100\rho_0 g((x-0.3)^2+(y-0.3)^2)/p_0) \]

mesh $= 50 \times 50$, transmissive bc, final time $= 0.15$

pressure perturbation with $\eta = 0.1$
Isothermal hydrostatic solution

well-balanced
20 equally spaced contours between -0.03 and +0.03 are shown

non well-balanced
Isothermal hydrostatic solution

pressure perturbation with $\eta = 0.001$

well-balanced

20 contours in $[-0.00026, +0.00026]$

non well-balanced

20 contours in $[-0.02, +0.00026]$
Polytropic hydrostatic solution

Unit square, potential \( \phi(x, y) = x + y \)

\[
T_e = 1 - \frac{\nu - 1}{\nu} (x + y), \quad p_e = T_e^{\frac{\nu}{\nu - 1}}, \quad \rho_e = T_e^{\frac{1}{\nu - 1}}
\]

\( \nu = 1.2, \) \hspace{1cm} \text{final time} = 1

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Grid} & \rho & u & \nu & p \\
\hline
50 \times 50 & 0.20449E-14 & 0.41148E-15 & 0.39802E-15 & 0.24637E-14 \\
200 \times 200 & 0.83747E-14 & 0.18037E-14 & 0.17986E-14 & 0.10107E-13 \\
\hline
\end{array}
\]

Table: Error in density, velocity and pressure
Polytropic hydrostatic solution

Perturbation of the initial pressure from the above polytropic solution

\[ p(x, y, 0) = p_e(x, y) + \eta \exp(-100 \rho_0 g ((x - 0.3)^2 + (y - 0.3)^2) / p_0) \]

mesh = 50 \times 50, transmissive bc, final time = 0.15

pressure perturbation with \( \eta = 0.1 \)
Polytropic hydrostatic solution

well-balanced
20 equally spaced contours between -0.03 and +0.03

non well-balanced
Polytropic hydrostatic solution

Pressure perturbation with $\eta = 0.001$

well-balanced
20 contours in $[-0.00025, +0.00025]$  

non well-balanced
20 contours in $[-0.015, +0.0003]$
Rayleigh-Taylor instability

- isothermal radial solution with potential $\phi = r$: $\rho = p = \exp(-r)$
- Add perturbation: initial pressure and density are given by

\[
p = \begin{cases} 
  e^{-r} & r \leq r_0 \\
  e^{-\frac{r}{\alpha} + r_0 \frac{(1-\alpha)}{\alpha}} & r > r_0 
\end{cases}, \quad \rho = \begin{cases} 
  e^{-r} & r \leq r_i \\
  \frac{1}{\alpha} e^{-\frac{r}{\alpha} + r_0 \frac{(1-\alpha)}{\alpha}} & r > r_i 
\end{cases}
\]

\[
r_i = r_0 (1 + \eta \cos(k\theta)), \quad \alpha = \exp(-r_0)/(\exp(-r_0) + \Delta_{\rho})
\]

- density jumps by an amount $\Delta_{\rho} > 0$ at interface $r = r_i$, pressure is continuous.

\[
\Delta_{\rho} = 0.1, \quad \eta = 0.02, \quad k = 20, \quad \text{mesh} = 240 \times 240 \text{ cells}
\]

\[
\text{domain} = [-1, +1] \times [-1, +1].
\]
Rayleigh-Taylor instability

- In the regions $r < r_0(1 - \eta)$ and $r > r_0(1 + \eta)$ the initial condition is in stable equilibrium
- but due to the discontinuous density, a Rayleigh-Taylor instability develops near interface defined by $r = r_i$.
- Due to well-balanced scheme, instability is concentrated only around the discontinuous interface
Rayleigh-Taylor instability

$t = 0$

$t = 2.9$

$t = 3.8$

$t = 5.0$
Extensions, ongoing work

- 2/3-D curvilinear meshes
- General equation of state, e.g., ideal gas with radiation pressure
  \[ p = \rho RT + \frac{1}{3}aT^4 \]

  No exact hydrostatic solutions known, preserve an approximate hydrostatic solution
- Weak formulation
  \[
  \text{find } u \in V \text{ such that } a(u, v) = \ell(v) \quad \forall v \in V
  \]
- Galerkin method
  \[
  \text{find } u_h \in V_h \text{ such that } a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h
  \]

  In practice
  \[
  \text{find } u_h \in V_h \text{ such that } a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h
  \]

  Exact solution \( u \) is not a solution of above problem.
- Discontinuous Galerkin method: well-balanced for isothermal hydrostatic solution
Thank You