Numerical solution of shear shallow water model

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Joint work with

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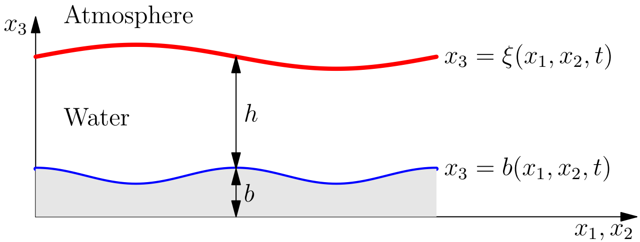
Shallow water (SW) model



$$h = h(x_1, x_2, t), \qquad \boldsymbol{v} = \boldsymbol{v}(x_1, x_2) = (v_1, v_2)$$

$$\begin{aligned} \frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{v}) &= 0 \\ \frac{\partial (h\boldsymbol{v})}{\partial t} + \nabla \cdot \left(h\boldsymbol{v} \otimes \boldsymbol{v} + \frac{gh^2}{2} I \right) &= -gh\nabla b - C_f |\boldsymbol{v}| \boldsymbol{v} \end{aligned}$$

Applications: Flow in channels, sea/oceans, tsunami modeling

Shear shallow water (SSW) model

Teshukov (2007), Richard & Gavrilyuk (2012), Gavrilyuk et al. (2018)

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{v}) &= 0 \\
\frac{\partial (h\boldsymbol{v})}{\partial t} + \nabla \cdot \left(h\boldsymbol{v} \otimes \boldsymbol{v} + \frac{gh^2}{2} I + h\mathcal{P} \right) &= -gh\nabla b - C_f |\boldsymbol{v}|\boldsymbol{v} \\
\frac{\partial \mathcal{P}}{\partial t} + \boldsymbol{v} \cdot \nabla \mathcal{P} + (\nabla \boldsymbol{v})\mathcal{P} + \mathcal{P}(\nabla \boldsymbol{v})^\top &= \mathcal{D}
\end{aligned}
$$

$\mathcal{R} = h\mathcal{P}$: stress tensor

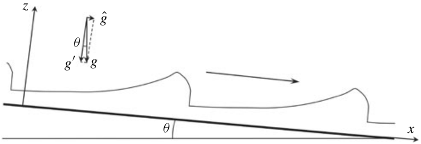
$$
\mathcal{P} = \mathcal{P}^\top > 0
$$

$\mathcal{D}$: dissipation tensor

Numerical methods: Gavrilyuk et al. (2018), Bhole et al. (2018)
(Based on splitting PDE into smaller sub-systems)

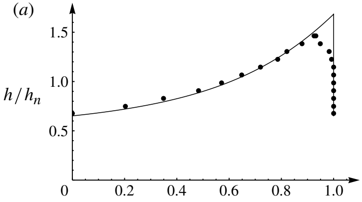
Brock's experimental results (1967)

Flow down an inclined plane due to gravity

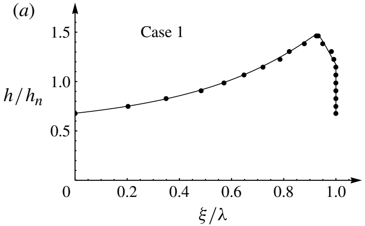


Formation of roll waves

$\tan\theta = 0.0502$ (Richard & Gavrilyuk, 2012)



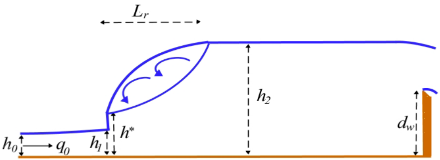
Shallow water



Shear shallow water

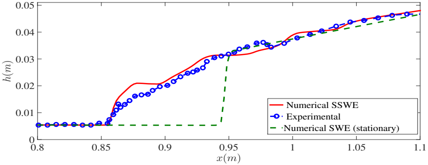
Flow over a weir

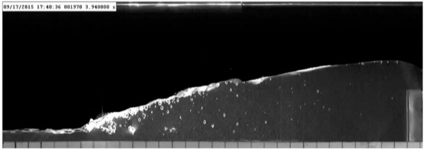
(Richard & Gavrilyuk, 2013)



Hydraulic jump and roll wave

Delis et al. (2018)





Model derivation

3-D Inviscid, incompressible Euler eqns

$$\frac{\partial u_k}{\partial x_k} = 0$$
$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = -g\delta_{i3}, \qquad i = 1, 2, 3$$

Scales of motion

H : vertical length scale

L : horizontal length scale

U : horizontal velocity scale

Long wave limit

$$\varepsilon = \frac{H}{L} \ll 1$$

Non-dimensionalize

$$t^* = \frac{tU}{L}, \qquad (x_1^*, x_2^*) = \frac{1}{L}(x_1, x_2), \qquad x_3^* = \frac{x_3}{H}$$

Model derivation

$$(u_1^*, u_2^*) = \frac{1}{U}(u_1, u_2), \qquad u_3^* = \frac{u_3}{U_3}, \qquad p^* = \frac{p}{\rho U^2}$$

Continuity equation

$$\frac{U}{L}\frac{\partial u_1^*}{\partial x_1^*} + \frac{U}{L}\frac{\partial u_2^*}{\partial x_2^*} + \frac{U_3}{H}\frac{\partial u_3^*}{\partial x_3^*} = 0 \qquad \Longrightarrow \qquad U_3 = \varepsilon U$$

Vertical momentum equation

$$\varepsilon^2 \frac{D u_3^*}{D t^*} + \frac{\partial p^*}{\partial x_3^*} = -\frac{1}{\mathrm{Fr}^2}, \qquad \mathrm{Fr} = \frac{U}{\sqrt{gH}}$$

Hydrostatic approximation: for $\alpha = 1, 2$

$$\frac{\partial p_3}{\partial x_3} = -\rho g \quad \Longrightarrow \quad p - p_a = -\rho g(x_3 - \xi) \quad \Longrightarrow \quad \frac{\partial p}{\partial x_\alpha} = \rho g \frac{\partial \xi}{\partial x_\alpha}$$

Model derivation

Depth average

$$\overline{\phi}(x_1, x_2, t) = \frac{1}{h(x_1, x_2, t)} \int_{b(x_1,x_2,t)}^{\xi(x_1,x_2,t)} \phi(x_1, x_2, x_3, t) \mathsf{d}x_3$$

Fluctuations

$$\phi' = \phi - \overline{\phi}, \qquad \overline{\phi'} = 0$$

Boundary conditions

$$\begin{aligned} \frac{\partial \xi}{\partial t} + u_\alpha \frac{\partial \xi}{\partial x_\alpha} - u_3 &= 0 \qquad \text{on} \quad x_3 = \xi \\ \frac{\partial b}{\partial t} + u_\alpha \frac{\partial b}{\partial x_\alpha} - u_3 &= 0 \qquad \text{on} \quad x_3 = b \end{aligned}$$

Average continuity equation

$$\overline{\nabla \cdot u} = 0 \qquad \Longrightarrow \qquad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x_\alpha}(h \overline{u}_\alpha) = 0$$

Model derivation

Horizontal momentum eqn

$$\frac{\partial u_\alpha}{\partial t} + 2\frac{\partial K_{\alpha\beta}}{\partial x_\beta} + \frac{\partial}{\partial x_3}(u_\alpha u_3) + g\frac{\partial \xi}{\partial x_\alpha} = 0, \qquad K_{\alpha\beta} = \frac{1}{2}u_\alpha u_\beta$$

Average horizontal momentum eqn

$$\frac{\partial (h\overline{u}_\alpha)}{\partial t} + 2\frac{\partial \overline{K}_{\alpha\beta}}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}\left(\frac{1}{2}gh^2\right) = -\rho g h\frac{\partial b}{\partial x_\alpha}$$

$$\overline{K}_{\alpha\beta} = \frac{1}{2}\overline{u}_\alpha\overline{u}_\beta + \frac{1}{2}\mathcal{P}_{\alpha\beta}, \qquad \mathcal{P}_{\alpha\beta} = \overline{u'_\alpha u'_\beta}$$

Shallow water model	Shear shallow water model
$\mathcal{P}_{\alpha\beta} = 0$	$\mathcal{P}_{\alpha\beta} \neq 0$

Model derivation

$$2\frac{\partial K_{\alpha\beta}}{\partial t} + 2\frac{\partial}{\partial x_\gamma}(K_{\alpha\beta}u_\gamma) + 2\frac{\partial}{\partial x_3}(K_{\alpha\beta}u_3) + gu_\alpha\frac{\partial \xi}{\partial x_\beta} + gu_\beta\frac{\partial \xi}{\partial x_\alpha} = 0$$

Averaging the above equation over the depth

$$\frac{\partial(h\overline{K}_{\alpha\beta})}{\partial t} + \frac{\partial}{\partial x_\gamma}(h\overline{K_{\alpha\beta}u_\gamma}) + \frac{1}{2}gh\overline{u}_\alpha\frac{\partial h}{\partial x_\beta} + \frac{1}{2}gh\overline{u}_\beta\frac{\partial h}{\partial x_\alpha} = - \frac{1}{2}gh\overline{u}_\alpha\frac{\partial b}{\partial x_\beta} - \frac{1}{2}gh\overline{u}_\beta\frac{\partial b}{\partial x_\alpha}$$

$$\overline{K_{\alpha\beta}u_\gamma} = \frac{1}{2}\overline{u_\alpha u_\beta u_\gamma} = \overline{K}_{\alpha\beta}\overline{u}_\gamma + \frac{1}{2}\overline{u}_\alpha\mathcal{P}_{\beta\gamma} + \frac{1}{2}\overline{u}_\beta\mathcal{P}_{\alpha\gamma} + \frac{1}{2}\overline{u'_\alpha u'_\beta u'_\gamma}$$

Small horizontal shear case: $\overline{u'_\alpha u'_\beta u'_\gamma} = 0$

Closed system of 6 equations

SSW model

$$v_\alpha = \overline{u}_\alpha, \qquad \mathcal{R}_{\alpha\beta} = h\mathcal{P}_{\alpha\beta}, \qquad E_{\alpha\beta} = h\overline{K}_{\alpha\beta} = \frac{1}{2}\mathcal{R}_{\alpha\beta} + \frac{1}{2}\overline{u}_\alpha\overline{u}_\beta$$

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}_1(\boldsymbol{U})}{\partial x_1} + \frac{\partial \boldsymbol{F}_2(\boldsymbol{U})}{\partial x_2} + \boldsymbol{B}_1(\boldsymbol{U})\frac{\partial h}{\partial x_1} + \boldsymbol{B}_2(\boldsymbol{U})\frac{\partial h}{\partial x_2} = \boldsymbol{S}(\boldsymbol{U})}$$

$$\boldsymbol{U} = \begin{bmatrix} h \\ hv_1 \\ hv_2 \\ E_{11} \\ E_{12} \\ E_{22} \end{bmatrix}, \quad \boldsymbol{F}_1 = \begin{bmatrix} hv_1 \\ \mathcal{R}_{11} + hv_1^2 + \frac{1}{2}gh^2 \\ \mathcal{R}_{12} + hv_1v_2 \\ (E_{11} + \mathcal{R}_{11})v_1 \\ E_{12}v_1 + \frac{1}{2}(\mathcal{R}_{11}v_2 + \mathcal{R}_{12}v_1) \\ E_{22}v_1 + \mathcal{R}_{12}v_2 \end{bmatrix}, \quad \boldsymbol{F}_2 = \begin{bmatrix} hv_2 \\ \mathcal{R}_{12} + hv_1v_2 \\ \mathcal{R}_{22} + hv_2^2 + \frac{1}{2}gh^2 \\ E_{11}v_2 + \mathcal{R}_{12}v_1 \\ E_{12}v_2 + \frac{1}{2}(\mathcal{R}_{12}v_2 + \mathcal{R}_{22}v_1) \\ (E_{22} + \mathcal{R}_{22})v_2 \end{bmatrix}$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ghv_1 \\ \frac{1}{2}ghv_2 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}ghv_1 \\ ghv_2 \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 0 \\ -gh\frac{\partial b}{\partial x_1} - C_f|\boldsymbol{v}|v_1 \\ -gh\frac{\partial b}{\partial x_2} - C_f|\boldsymbol{v}|v_2 \\ -ghv_1\frac{\partial b}{\partial x_1} + \frac{1}{2}h\mathcal{D}_{11} - C_f|\boldsymbol{v}|v_1^2 \\ -\frac{1}{2}ghv_2\frac{\partial b}{\partial x_1} - \frac{1}{2}ghv_1\frac{\partial b}{\partial x_2} + \frac{1}{2}h\mathcal{D}_{12} - C_f|\boldsymbol{v}|v_1v_2 \\ -ghv_2\frac{\partial b}{\partial x_2} + \frac{1}{2}h\mathcal{D}_{22} - C_f|\boldsymbol{v}|v_2^2 \end{bmatrix}$$

Similar to 10 moment Gaussian model for gases: Levermore, Berthon

SSW model: entropy

Solution space

$$\mathcal{U}_{\mathrm{ad}} := \{\boldsymbol{U} \in \mathbb{R}^6 : h > 0,\ \mathcal{R} = \mathcal{R}^\top > 0\}$$

Convex entropy function (Levermore)

$$\eta = \eta(\boldsymbol{U}) := -h \log\left(\frac{\det \mathcal{R}}{h^4}\right)$$

Entropy equation

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \boldsymbol{v}) = -\frac{h}{\det(\mathcal{P})}[\mathrm{trace}(\mathcal{P})\,\mathrm{trace}(\mathcal{D}) - \mathrm{trace}(\mathcal{P}\mathcal{D})]$$

Entropy condition

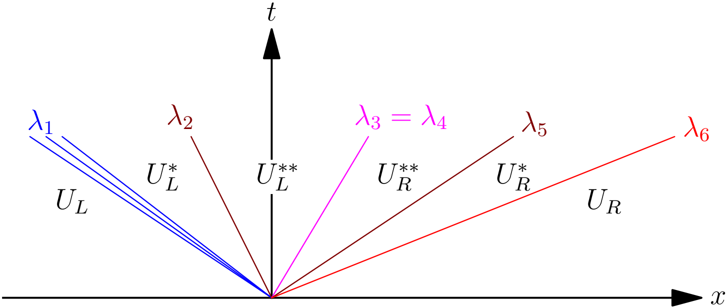
$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \boldsymbol{v}) \leq 0$$

Hyperbolicity

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x_1} + \boldsymbol{B}(\boldsymbol{U})\frac{\partial h}{\partial x_1} = 0$$

Six real eigenvalues, full set of eigenvectors

λ_1	λ_2	$\lambda_3 = \lambda_4$	λ_5	λ_6
$v_1 - \sqrt{gh + 3\mathcal{P}_{11}}$	$v_1 - \sqrt{\mathcal{P}_{11}}$	v_1	$v_1 + \sqrt{\mathcal{P}_{11}}$	$v_1 + \sqrt{gh + 3\mathcal{P}_{11}}$
Gen non-lin	Lin deg	Lin deg	Lin deg	Gen non-lin
Shock/rarefaction	Shear	Contact	Shear	Shock/rarefaction



Dissipation model: Richard & Gavrilyuk

Stokes hypothesis: isotropic tensor function of $\mathcal{P}$

$$\mathcal{D} = -\frac{2\alpha|\boldsymbol{v}|^3}{h}\mathcal{P}, \qquad \alpha = \max\left(0, C_r \frac{T - \phi h^2}{T^2}\right), \qquad T = \mathrm{trace}(\mathcal{P})$$

ϕ : enstrophy of small vortices near bottom

C_r : dissipation coefficient associated to roller formation

Entropy equation

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \boldsymbol{v}) = 4\alpha|\boldsymbol{v}|^3 \geq 0 \quad \text{Wrong entropy inequality !!!}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \frac{\det(\mathcal{P})}{h^2} = -\frac{4\alpha}{h^3}\det(\mathcal{P}) < 0 \quad \det(\mathcal{P}) \downarrow \text{along particle path}$$

Path conservative solution

Conservative system: $\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = 0$

$$\int_0^\infty \int_{\mathbb{R}} (\boldsymbol{U}\phi_t + \boldsymbol{F}(\boldsymbol{U})\phi_x)\mathsf{d}x\mathsf{d}t + \int_{\mathbb{R}} \boldsymbol{U}_0(x)\phi(x,0)\mathsf{d}x = 0, \qquad \phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$$

Non-conservative system: $\boldsymbol{U}_t + \boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x = 0$

$$\boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x = ? \quad \text{when } \boldsymbol{U} \text{ is discontinuous at } x = x_0$$

Dal Maso, LeFloch, Murat (1995)

- Choose a nice path: $\Psi : [0,1] \times \mathcal{U}_{\text{ad}} \times \mathcal{U}_{\text{ad}} \to \mathcal{U}_{\text{ad}}$

$$\Psi(0; \boldsymbol{U}_L, \boldsymbol{U}_R) = \boldsymbol{U}_L, \qquad \Psi(1; \boldsymbol{U}_L, \boldsymbol{U}_R) = \boldsymbol{U}_R$$

- Non-conservative product is

$$\left\langle \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} \right\rangle_\Psi = \int_0^1 \boldsymbol{A}(\Psi(\xi; \boldsymbol{U}_L, \boldsymbol{U}_R))\frac{\mathsf{d}\Psi}{\mathsf{d}\xi}\mathsf{d}\xi$$

Path conservative solution

Generalized RH jump condition: discontinuity moving with speed S

$$\int_0^1 \left[\boldsymbol{A}(\Psi(\xi; \boldsymbol{U}_L, \boldsymbol{U}_R)) - SI\right] \frac{\mathsf{d}\Psi}{\mathsf{d}\xi} \mathsf{d}\xi = 0$$

Linear path

$$\Psi(\xi; \boldsymbol{U}_L, \boldsymbol{U}_R) = \boldsymbol{U}_L + \xi(\boldsymbol{U}_R - \boldsymbol{U}_L)$$

RH condition for SSW model

$$\boldsymbol{F}_R - \boldsymbol{F}_L + \boldsymbol{B}(\boldsymbol{m}_L, \boldsymbol{m}_R)(h_R - h_L) = S(\boldsymbol{U}_R - \boldsymbol{U}_L)$$

$$\boldsymbol{B}(\boldsymbol{m}_L, \boldsymbol{m}_R) = \boldsymbol{B}\left(\frac{\boldsymbol{m}_L + \boldsymbol{m}_R}{2}\right)$$

Riemann problem: exact solution

Assume that $\boldsymbol{U}_L, \boldsymbol{U}_R \in \mathcal{U}_{\text{ad}}$ and we use the linear path.

$$\boldsymbol{U}(x,0) = \begin{cases} \boldsymbol{U}_L & x < 0 \\ \boldsymbol{U}_R & x > 0 \end{cases}$$

- The RP has a unique positive and entropy satisfying solution if

$$(v_1)_R - (v_1)_L \leq A(h_L, c_L) + A(h_R, c_R)$$

where

$$A(h,c) = \sqrt{gh + 3ch^2} + \frac{g}{\sqrt{3c}} \sinh^{-1} \sqrt{\frac{3ch}{g}}, \qquad c = \frac{\mathcal{P}_{11}}{h^2}$$

- Across a shock

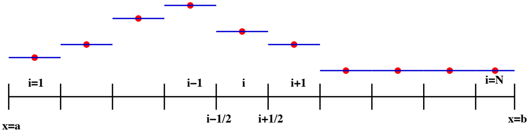
$$\frac{1}{2} < \frac{h_R}{h_L} < 2$$

Path conservative schemes (Pares, 2006)

Partition domain Ω into cells: $\Omega_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $\Omega = \cup_j \Omega_j$

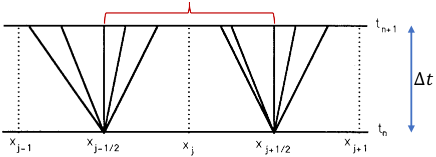
Approximate solution by piecewise constant functions, the **cell averages**

$$\boldsymbol{U}_j^n \approx \frac{1}{\Delta x} \int_{I_j} \boldsymbol{U}(x, t_n) \mathrm{d}x$$



Godunov's idea:

1. Solve RP for small time-step Δt
2. Average solution onto piecewise constant

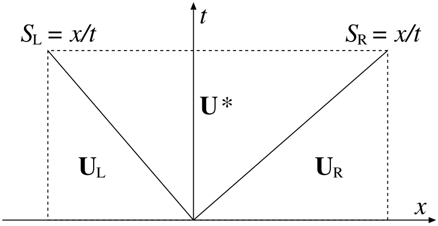


$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x} (\boldsymbol{D}^{+,n}_{j-\frac{1}{2}} + \boldsymbol{D}^{-,n}_{j+\frac{1}{2}}) + \Delta t \boldsymbol{S}(\boldsymbol{U}_j^{n+\theta}), \quad \boldsymbol{D}^{\pm,n}_{j+\frac{1}{2}} = \boldsymbol{D}^{\pm}(\boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)$$

$$\boldsymbol{D}^{-}(\boldsymbol{U}_L, \boldsymbol{U}_R) + \boldsymbol{D}^{+}(\boldsymbol{U}_L, \boldsymbol{U}_R) = \int_0^1 \boldsymbol{A}(\Psi(s; \boldsymbol{U}_L, \boldsymbol{U}_R)) \frac{\mathrm{d}}{\mathrm{d}s} \Psi(s; \boldsymbol{U}_L, \boldsymbol{U}_R) \mathrm{d}s$$

Approximate Riemann solver: HLL[1]

* Include only the slowest (S_L) and fastest wave (S_R)



* RH condition across the two waves

$$\boldsymbol{F}_* - \boldsymbol{F}_L + \boldsymbol{B}(\boldsymbol{m}_L, \boldsymbol{m}_*)(h_* - h_L) = S_L(\boldsymbol{U}_* - \boldsymbol{U}_L)$$

$$\boldsymbol{F}_R - \boldsymbol{F}_* + \boldsymbol{B}(\boldsymbol{m}_*, \boldsymbol{m}_R)(h_R - h_*) = S_R(\boldsymbol{U}_R - \boldsymbol{U}_*)$$

* Solve for $\boldsymbol{U}_*$ and $\boldsymbol{F}_*$
* Fluctuations

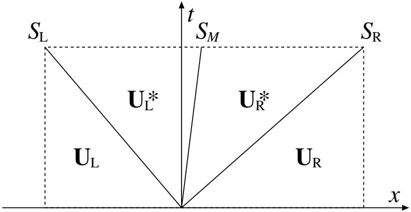
$$\boldsymbol{D}^{\pm}(\boldsymbol{U}_L, \boldsymbol{U}_R) = S_L^{\pm}(\boldsymbol{U}_* - \boldsymbol{U}_L) + S_R^{\pm}(\boldsymbol{U}_R - \boldsymbol{U}_*)$$

[1] Harten, Lax, van Leer

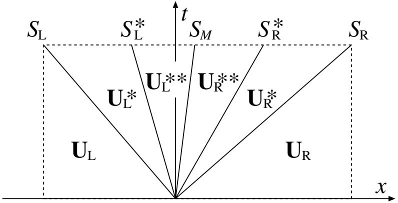
Multi-state approximate Riemann solvers

Include more waves in the model

More accurate resolution of linear waves



3 wave model

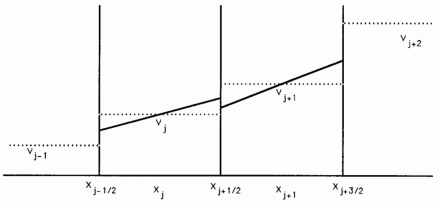


5 wave model

See `10.1016/j.jcp.2020.109457`

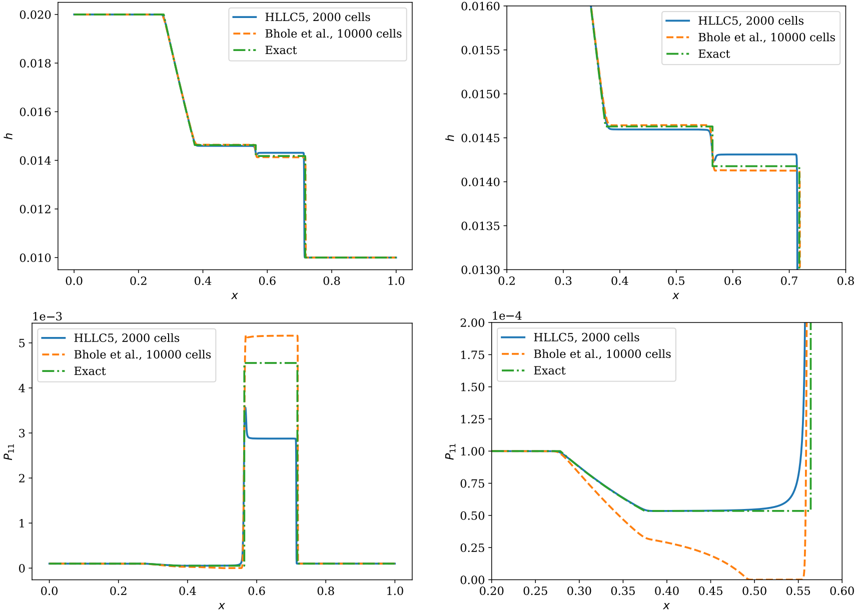
High order schemes

- Piecewise linear reconstruction
- TVD-type limiters (minmod)
- MUSCL-Hancock in time
- Source term is implicit
    - essential for positivity

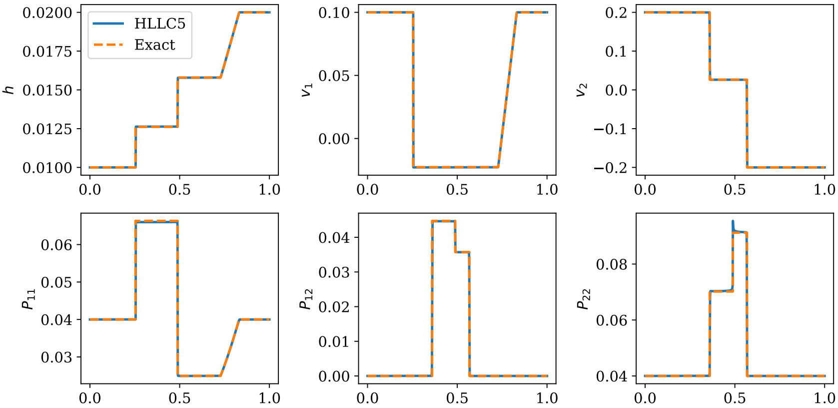


Numerical Results

1-D dam-break: I

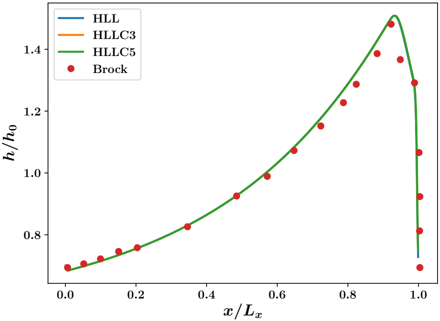


1-D dam-break: II

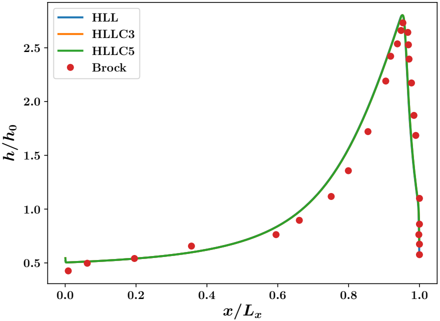


1-D roll wave

Compared to Brock's experiments

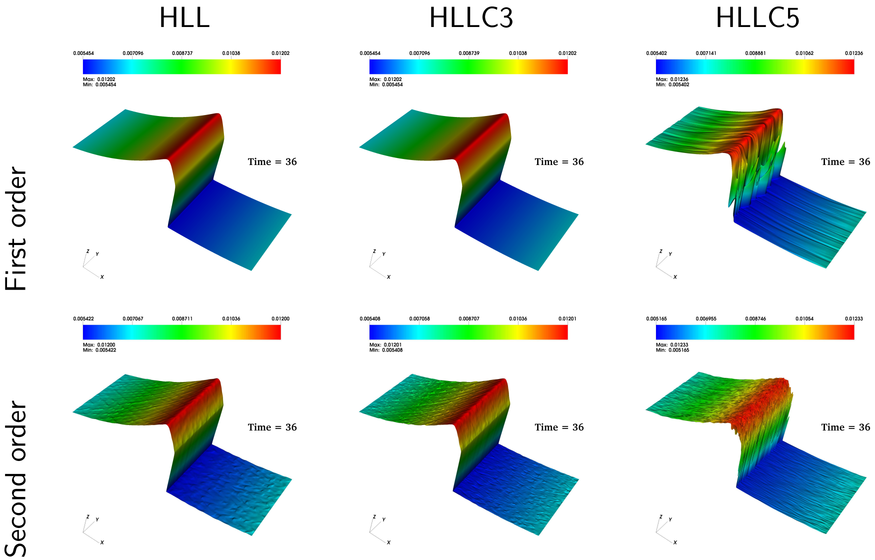


(a) Case 1

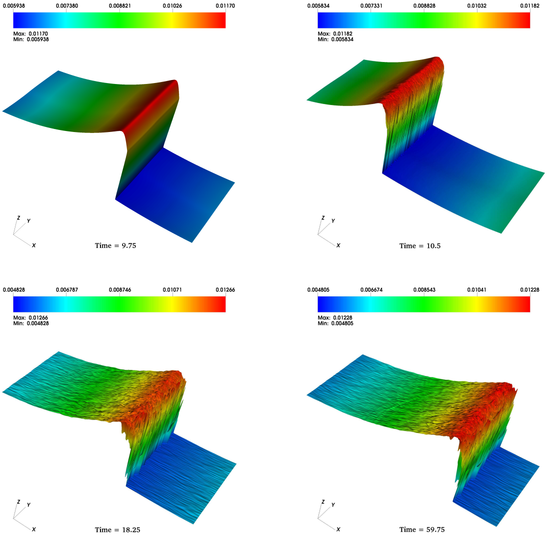


(b) Case 2

2-D roll wave: 2080×800 mesh



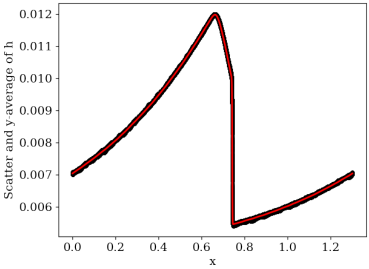
2-D roll wave: HLLC5, 2080×800 mesh

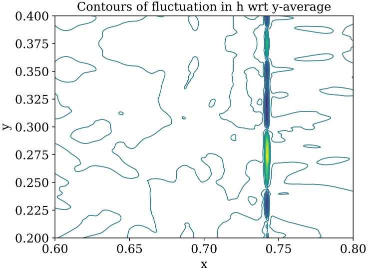


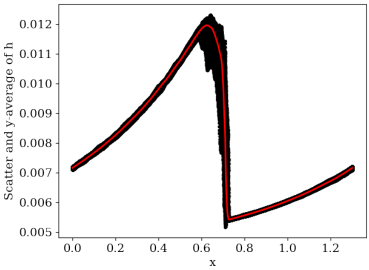
2-D roll wave: 2080×800 mesh, $t = 36$

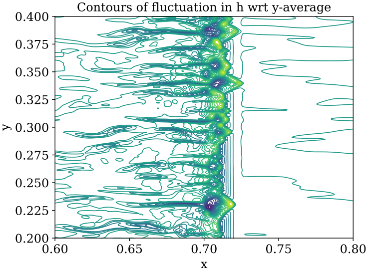
HLLC3

HLLC5

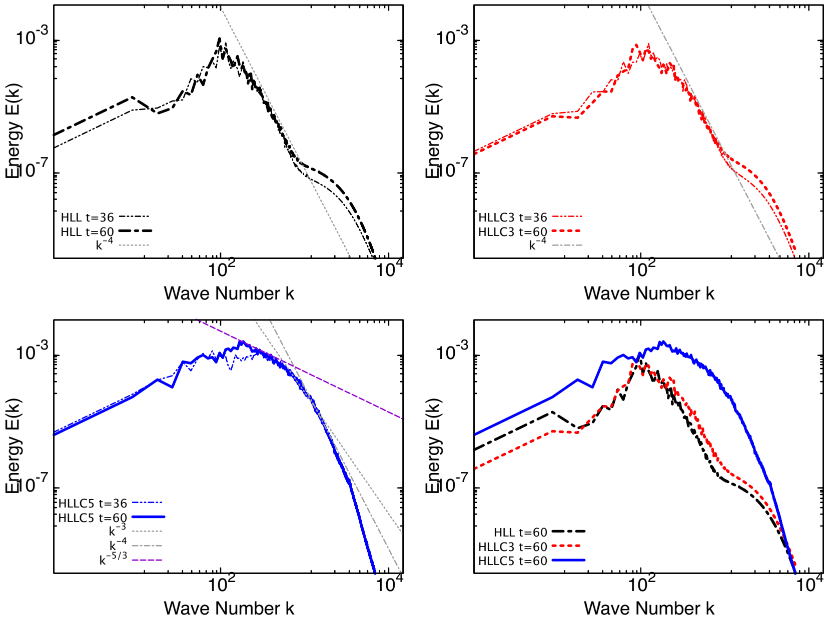








2-D roll wave: spectrum of fluctuation KE



2-D roll wave

Movie

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- Multi-wave Riemann solvers
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$\boxed{\texttt{Thank You}}$

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