Stable central schemes for compressible flows

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TOY Workshop, NCAR
18 May, 2017

Outline

1. Finite volume schemes for compressible Euler/NS equations
2. KE and entropy consistent flux functions
3. Numerical examples
4. Shallow water example

Conservation laws: Navier-Stokes equation

$\mathbf{u}$ = conserved variables

$\mathbf{f}$ = inviscid flux

$\mathbf{g}$ = viscous flux

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\partial \mathbf{g}}{\partial x}}$$

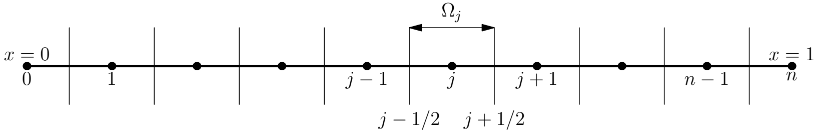
$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} f^\rho \\ f^m \\ f^e \end{bmatrix} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix} = \begin{bmatrix} f^\rho \\ p + u f^\rho \\ (E+p)u \end{bmatrix}$$

$$\mathbf{g} = \begin{bmatrix} 0 \\ \tau \\ u\tau - q \end{bmatrix}, \qquad \tau = \frac{4}{3}\mu \frac{\partial u}{\partial x}, \qquad q = -\kappa \frac{\partial T}{\partial x}$$

μ = coeff. of dynamic viscosity, $\quad \kappa$ = coeff. of heat conduction

$$p = \rho R T, \qquad E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2, \qquad \gamma = \frac{C_p}{C_v}$$

Finite volume method



$$\mathbf{u}_j = \text{Cell average value in } j\text{'th cell } \Omega_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

Semi-discrete FVM

$$\Delta x \frac{\mathrm{d}\mathbf{u}_j}{\mathrm{d}t} + \mathbf{f}_{j+\frac{1}{2}} - \mathbf{f}_{j-\frac{1}{2}} = \mathbf{g}_{j+\frac{1}{2}} - \mathbf{g}_{j-\frac{1}{2}}$$

Numerical flux function

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_j, \mathbf{u}_{j+1}), \qquad \mathbf{f}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u})$$

Centered approximation for $\mathbf{g}_{j+\frac{1}{2}}$

Locally and globally conserves mass, momentum and energy

- Conservation alone does not guarantee numerical stability
- Centered flux

$$\mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}[\mathbf{f}(\mathbf{u}_j) + \mathbf{f}(\mathbf{u}_{j+1})] \qquad \text{is not stable}$$

- Godunov/upwind schemes stable; dissipate kinetic energy
- Stable central schemes via
  - Consistent evolution of KE: KE preserving (KEP) schemes
  - Entropy condition: second law of thermodynamics
- Kinetic energy preserving: Incompressible flows
  - Harlow and Welch (1965): Staggered grids
    Ham (2002): Non-uniform grids
    Wesseling (1999): General structured grids
    Morinishi (1998): Fourth order scheme
    Verstappen et al. (2003): 2/4'th order symmetry preserving
    Mahesh et al. (2004): Unstructured grids
  - Sanderse (2012): Energy conserving RK for INS

- Kinetic energy preserving: Compressible flows
  - Jameson (2008): KEP scheme for compressible flow
  - Subbareddy et al. (2009): Fully discrete implicit KEP scheme
  - Shoeybi et al. (2010): KEP scheme, unstructured, IMEX-RK
  - Morinishi (2010): Skew symmetric, staggered grid schemes
- Entropy consistent/stable schemes: not fully conservative
  - Gerritsen et al. (1996): Entropy stable scheme for exponential entropy
  - Honein et al. (2004): Better entropy consistency using skew-symmetric form, internal energy equation
- Entropy consistent/stable schemes: fully conservative
  - Tadmor (1987): Entropy conservative flux
  - Lefloch et al. (2002): Higher order entropy conservative schemes
  - Roe (2006), PC (2013): Explicit entropy conservative flux for Euler equations
  - Fjordholm et al. (2011): Entropy stable ENO schemes

Burger's equation

Burger's equation $f(u) = \frac{1}{2}u^2$

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(1,t) = 0$$

$$\int_0^1 u \frac{\partial u}{\partial t} \mathrm{d}x + \int_0^1 u \frac{\partial f}{\partial x} \mathrm{d}x = \nu \int_0^1 u \frac{\partial^2 u}{\partial x^2} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \frac{1}{2} u^2 \mathrm{d}x + \frac{1}{3}[u^3(1,t) - u^3(0,t)] = -\nu \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \le 0$$

The energy cannot increase with time. Any increase of energy is due to flux through the boundary.

Burger's equation

$$\frac{\partial u}{\partial t} + \frac{2}{3}\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \frac{1}{3}u\frac{\partial u}{\partial x} = 0$$

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + \frac{2}{3}\frac{u_{j+1}^2 - u_{j-1}^2}{4\Delta x} + \frac{1}{3}u_j\frac{u_{j+1} - u_{j-1}}{2\Delta x} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u_j + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\Delta x} = 0, \qquad f_{j+\frac{1}{2}} = \frac{1}{6}(u_j^2 + u_{j+1}^2 + u_j u_{j+1})$$

At boundary points, use one-sided differences

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} + \frac{2}{\Delta x}(f_{\frac{1}{2}} - f_0) = 0, \qquad \frac{\mathrm{d}u_n}{\mathrm{d}t} + \frac{2}{\Delta x}(f_n - f_{n-\frac{1}{2}}) = 0$$

Define energy by trapezoidal rule of integration

$$E = \frac{\Delta x}{2}\left(\frac{u_0^2}{2} + \frac{u_n^2}{2}\right) + \Delta x\sum_{j=1}^{n-1}\frac{u_j^2}{2}$$

Burger's equation

Then it can be shown that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{u_0^3}{3} - \frac{u_n^3}{3}$$

Remark: The scheme has to be modified at boundaries to obtain stability

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le \frac{\mathrm{d}E_{\textsf{ex}}}{\mathrm{d}t}$$

see Jameson, SIAM JSC (2008) 34:152-187.

Remark: Similar ideas have been used for incompressible NS equations by writing the convective terms in **skew-symmetric** forms. This leads to stable central schemes.

Kinetic energy

Kinetic energy per unit volume: $K = \frac{1}{2}\rho u^2$

$$\begin{aligned} \frac{\partial K}{\partial t} &= -\frac{1}{2}u^2\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial t} \\ &= -\frac{\partial}{\partial x}(p + \rho u^2/2 - \frac{4}{3}\mu\frac{\partial u}{\partial x})u + p\frac{\partial u}{\partial x} - \frac{4}{3}\mu\left(\frac{\partial u}{\partial x}\right)^2 \end{aligned}$$

Integrating, with periodic or wall bc

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega K \mathrm{d}x = \int_\Omega p\frac{\partial u}{\partial x}\mathrm{d}x - \frac{4}{3}\int_\Omega \mu\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \le \int_\Omega p\frac{\partial u}{\partial x}\mathrm{d}x$$

Work done by pressure forces, absent in incompressible flows

Irreversible destruction due to molecular diffusion

Note: Convection contributes to only flux of KE across $\partial\Omega$

KE preserving FVM

$$
\begin{aligned}
\sum_j \Delta x \frac{\mathrm{d}K_j}{\mathrm{d}t} &= \sum_j \left[-\frac{1}{2} u_j^2 \frac{\mathrm{d}\rho_j}{\mathrm{d}t} + u_j \frac{\mathrm{d}(\rho u)_j}{\mathrm{d}t} \right] \Delta x \\
&= \sum_j \left[\frac{1}{2} u_j^2 (f^\rho_{j+\frac{1}{2}} - f^\rho_{j-\frac{1}{2}}) - u_j (f^m_{j+\frac{1}{2}} - f^m_{j-\frac{1}{2}}) \right] \\
&= \sum_j \left[\frac{1}{2} (u_j^2 - u_{j+1}^2) f^\rho_{j+\frac{1}{2}} - (u_j - u_{j+1}) f^m_{j+\frac{1}{2}} \right] \\
&= \sum_j \frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} [\overline{u}_{j+\frac{1}{2}} f^\rho_{j+\frac{1}{2}} - f^m_{j+\frac{1}{2}}] \Delta x \\
&= \sum_j \frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} p_{j+\frac{1}{2}} \Delta x, \qquad \boxed{f^m_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^\rho_{j+\frac{1}{2}}}
\end{aligned}
$$

KE preserving FVM (Jameson)

Centered numerical flux

$$\mathbf{f}_{j+\frac{1}{2}} = \begin{bmatrix} f^\rho \\ f^m \\ f^e \end{bmatrix}_{j+\frac{1}{2}} = \begin{bmatrix} f^\rho \\ \tilde{p} + \overline{u} f^\rho \\ f^e \end{bmatrix}_{j+\frac{1}{2}}, \qquad g_{j+\frac{1}{2}} = \begin{bmatrix} 0 \\ \tau \\ \tilde{u}\tau - q \end{bmatrix}_{j+\frac{1}{2}}$$

where

$$\overline{u}_{j+\frac{1}{2}} = \frac{1}{2}(u_j + u_{j+1}), \quad \tau_{j+\frac{1}{2}} = \frac{4}{3}\mu \frac{u_{j+1} - u_j}{\Delta x}, \quad q_{j+\frac{1}{2}} = -\kappa \frac{T_{j+1} - T_j}{\Delta x}$$

Discrete KE equation

$$\sum_j \Delta x \frac{\mathrm{d}K_j}{\mathrm{d}t} = \sum_j \left[\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \tilde{p}_{j+\frac{1}{2}} - \frac{4}{3}\mu \left(\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \right)^2 \right] \Delta x$$

KE preserving FVM (Jameson)

Jameson's KEP flux

$$\mathbf{f}_{j+\frac{1}{2}} = \begin{bmatrix} \overline{\rho}\ \overline{u} \\ \overline{p} + \overline{u} f^\rho \\ \overline{H} f^\rho \end{bmatrix}_{j+\frac{1}{2}}, \qquad \text{compare with} \quad \mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}_j + \mathbf{f}_{j+1})$$

But there can be other choices, e.g.,

$$f^\rho = \overline{\rho u}, \qquad f^e = \overline{\rho H u}, \quad \text{etc.}$$

We are free to choose $\tilde{p}$, f^ρ, f^e in any consistent manner. We determine all flux components (uniquely) from entropy condition.

Entropy condition

Entropy-Entropy flux pair: $U(\mathbf{u}), F(\mathbf{u})$

$$U(\mathbf{u}) \text{ is strictly convex} \qquad \text{and} \qquad U'(\mathbf{u})\mathbf{f}'(\mathbf{u}) = F'(\mathbf{u})$$

Then, for hyperbolic problem (Euler equation)

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = 0 \quad \Longrightarrow \quad U'(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + U'(\mathbf{u})\mathbf{f}'(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = 0$$

$$\Downarrow$$

$$\frac{\partial U(\mathbf{u})}{\partial t} + \frac{\partial F(\mathbf{u})}{\partial x} = 0$$

For discontinuous solutions, only inequality

$$\frac{\partial U(\mathbf{u})}{\partial t} + \frac{\partial F(\mathbf{u})}{\partial x} \le 0$$

Second law of thermodynamics

$\int_\Omega U(\mathbf{u}) \mathrm{d}x$ for an isolated system decreases with time

Existence of entropy pair

For scalar problem, entropy exists (infinite)
Take any convex $U(u)$ and find

$$F(u) = \int^{u} U'(s) f'(s) \mathrm{d}s$$

For systems, there is no general result. We usually know there is an entropy function coming from second law of thermodynamics.

Entropy conserving FVM

Entropy variables

$$\mathbf{v}(\mathbf{u}) = U'(\mathbf{u})$$

$$U(\mathbf{u}) \text{ is strictly convex} \quad \Longrightarrow \quad \mathbf{u} = \mathbf{u}(\mathbf{v})$$

Define dual $\psi(\mathbf{v})$ of the entropy flux $F(\mathbf{u})$

$$\psi(\mathbf{v}) = \mathbf{v} \cdot \mathbf{f}(\mathbf{u}(\mathbf{v})) - F(\mathbf{u}(\mathbf{v}))$$

Entropy conservative flux (Tadmor)

$$\boxed{(\mathbf{v}_{j+1} - \mathbf{v}_j) \cdot \mathbf{f}_{j+\frac{1}{2}} = \psi_{j+1} - \psi_j}$$

$$\mathbf{v}_j \cdot \left(\Delta x \frac{\mathrm{d}\mathbf{u}_j}{\mathrm{d}t} + \mathbf{f}_{j+\frac{1}{2}} - \mathbf{f}_{j-\frac{1}{2}} = 0 \right) \quad \Longrightarrow \quad \Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = 0$$

Entropy conserving FVM

Consistent entropy flux

$$F_{j+\frac{1}{2}} = \overline{\mathbf{v}}_{j+\frac{1}{2}} \cdot \mathbf{f}_{j+\frac{1}{2}} - \overline{\psi}_{j+\frac{1}{2}}$$

In the scalar case

$$(v_{j+1} - v_j) \cdot f_{j+\frac{1}{2}} = \psi_{j+1} - \psi_j \quad \Longrightarrow \quad f_{j+\frac{1}{2}} = \frac{\psi_{j+1} - \psi_j}{v_{j+1} - v_j}$$

Example: Burger's equation $f(u) = u^2/2$, $U(u) = \frac{1}{2}u^2$, $F(u) = \frac{1}{3}u^3$

$$v = U'(u) = u, \qquad \psi = vf(u) - F(u) = \frac{1}{6}u^3$$

and hence

$$f_{j+\frac{1}{2}} = \frac{\psi_{j+1} - \psi_j}{v_{j+1} - v_j} = \frac{1}{6}(u_j^2 + u_j u_{j+1} + u_{j+1}^2)$$

Entropy conserving FVM

For systems, we have an under-determined problem.

Entropy conservative flux of Tadmor (1987)

$$\mathbf{f}_{j+\frac{1}{2}} = \int_0^1 \mathbf{f}(\mathbf{v}_{j+\frac{1}{2}}(\theta))\mathrm{d}\theta, \qquad \mathbf{v}_{j+\frac{1}{2}}(\theta) = \mathbf{v}_j + \theta(\mathbf{v}_{j+1} - \mathbf{v}_j)$$

Cannot be explicitly evaluated, requires numerical quadrature

Entropy condition for Euler equation

Entropy-Entropy flux pair

$$U = -\frac{\rho s}{\gamma - 1}, \qquad F = -\frac{\rho u s}{\gamma - 1}, \qquad s = \ln(p/\rho^\gamma)$$

Entropy variables

$$\mathbf{v} = \begin{bmatrix} \frac{\gamma - s}{\gamma - 1} - \beta u^2 \\ 2\beta u \\ -2\beta \end{bmatrix}, \qquad \beta = \frac{1}{2RT}, \qquad \psi = \rho u$$

Entropy conservative numerical flux for the Euler equations

$$(\mathbf{v}_{j+1} - \mathbf{v}_j) \cdot \mathbf{f}_{j+\frac{1}{2}} = (\rho u)_{j+1} - (\rho u)_j$$

Remark: There are other entropy functions U but this is the only one which is consistent with NS equations in presence of heat conduction.

Roe's entropy conservative flux: Euler equation

Parameter vector and logarithmic average

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \sqrt{\frac{\rho}{p}} \begin{bmatrix} 1 \\ u \\ p \end{bmatrix}, \qquad \hat{\varphi}(\varphi_l, \varphi_r) = \frac{\varphi_r - \varphi_l}{\ln \varphi_r - \ln \varphi_l} = \frac{\Delta \varphi}{\Delta \ln \varphi}$$

Entropy conserving numerical flux

$$\mathbf{f}^* = \begin{bmatrix} \tilde{\rho}\tilde{u} \\ \tilde{p}_1 + \tilde{u} f^\rho \\ \tilde{H} f^\rho \end{bmatrix}$$

where

$$\tilde{\rho} = \overline{z}_1 \hat{z}_3, \quad \tilde{u} = \frac{\overline{z}_2}{\overline{z}_1}, \quad \tilde{p}_1 = \frac{\overline{z}_3}{\overline{z}_1}, \quad \tilde{p}_2 = \frac{\gamma + 1}{2\gamma} \frac{\hat{z}_3}{\hat{z}_1} + \frac{\gamma - 1}{2\gamma} \frac{\overline{z}_3}{\overline{z}_1}$$

$$\tilde{a} = \left(\frac{\gamma \tilde{p}_2}{\tilde{\rho}} \right)^{\frac{1}{2}}, \quad \tilde{H} = \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2}\tilde{u}^2$$

KEP and entropy conserving flux (CICP, 2013)

Condition for entropy conservative flux: $\Delta \mathbf{v} \cdot \mathbf{f} = \Delta(\rho u)$

$$f^\rho \Delta v_1 + f^m \Delta v_2 + f^e \Delta v_3 = \Delta(\rho u) = \overline{\rho} \Delta u + \overline{u} \Delta \rho$$

Jump in entropy variables in terms of (ρ, u, β)

$$\begin{aligned}
\Delta v_1 &= \frac{\Delta \rho}{\hat{\rho}} + \left[\frac{1}{(\gamma - 1)\hat{\beta}} - \overline{u^2} \right] \Delta \beta - 2 \overline{u}\,\overline{\beta} \Delta u \\
\Delta v_2 &= 2 \overline{\beta} \Delta u + 2 \overline{u} \Delta \beta \\
\Delta v_3 &= -2 \Delta \beta
\end{aligned}$$

KEP and Entropy conserving numerical flux

$$\mathbf{f}^* = \begin{bmatrix} \hat{\rho} \overline{u} \\ \tilde{p} + \overline{u} f^\rho \\ \left\{ \frac{1}{2(\gamma-1)\hat{\beta}} - \frac{1}{2} \overline{u^2} \right\} f^\rho + \overline{u} f^m \end{bmatrix}, \qquad \tilde{p} = \frac{\overline{\rho}}{2 \overline{\beta}}$$

FVM for NS equation

The semi-discrete finite volume method for NS equations using the centered KEP and entropy conservative flux is stable for the kinetic energy and entropy, i.e.,

$$
\begin{aligned}
\sum_j \Delta x \frac{\mathrm{d}K_j}{\mathrm{d}t} &= \sum_j \left[\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \tilde{p}_{j+\frac{1}{2}} - \frac{4}{3}\mu \left(\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \right)^2 \right] \Delta x \\
&\le \sum_j \left[\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \tilde{p}_{j+\frac{1}{2}} \right] \Delta x
\end{aligned}
$$

and

$$
\sum_j \Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} = -\sum_j \left[\frac{8\mu\overline{\beta}_{j+\frac{1}{2}}}{3} \left(\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \right)^2 + \frac{\kappa}{RT_j T_{j+1}} \left(\frac{\Delta T_{j+\frac{1}{2}}}{\Delta x} \right)^2 \right] \Delta x \le 0
$$

- Expect good stability property
- No control of density/pressure

Higher order schemes (LeFloch)

Two point fluxes $\mathbf{f}^*$ lead to second order schemes (Tadmor)
For any integer $p \geq 1$, let $\alpha_1^p, \ldots, \alpha_p^p$ solve the linear equations

$$2 \sum_{r=1}^{p} r \alpha_r^p = 1, \qquad \sum_{r=1}^{p} r^{2s-1} \alpha_r^p = 0, \qquad s = 2, \ldots, p$$

and define the numerical flux

$$\mathbf{f}_{j+\frac{1}{2}}^{*,2p} = \mathbf{f}^{*,2p}(\mathbf{u}_{j-p+1}, \ldots, \mathbf{u}_{j+p}) = \sum_{r=1}^{p} \alpha_r^p \sum_{s=0}^{r-1} \mathbf{f}^*(\mathbf{u}_{j-s}, \mathbf{u}_{j-s+r})$$

- $2p$'th order accurate

$$\frac{\mathbf{f}_{j+\frac{1}{2}}^{*,2p} - \mathbf{f}_{j-\frac{1}{2}}^{*,2p}}{\Delta x} = \frac{\partial \mathbf{f}}{\partial x}(\mathbf{u}_j) + O(h^{2p})$$

Higher order schemes (LeFloch)

- Entropy conservative

$$\frac{\mathrm{d}U_j}{\mathrm{d}t} + \frac{F^{*,2p}_{j+\frac{1}{2}} - F^{*,2p}_{j-\frac{1}{2}}}{\Delta x} = 0$$

where

$$F^{*,2p}_{j+\frac{1}{2}} = \sum_{r=1}^{p} \alpha_r^p \sum_{s=0}^{r-1} F^*(\mathbf{u}_{j-s}, \mathbf{u}_{j-s+r})$$

Example: For $p = 2$, the fourth order flux is given by

$$\mathbf{f}^{*,4}_{j+\frac{1}{2}} = \frac{4}{3}\mathbf{f}^*(\mathbf{u}_j, \mathbf{u}_{j+1}) - \frac{1}{6}[\mathbf{f}^*(\mathbf{u}_{j-1}, \mathbf{u}_{j+1}) + \mathbf{f}^*(\mathbf{u}_j, \mathbf{u}_{j+2})]$$

Entropy stable schemes

Roe flux

$$\mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}_j + \mathbf{f}_{j+1}) - \frac{1}{2}R_{j+\frac{1}{2}}|\Lambda_{j+\frac{1}{2}}|R^{-1}_{j+\frac{1}{2}}\Delta\mathbf{u}_{j+\frac{1}{2}}$$

Eigenvectors and eigenvalues

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{bmatrix}, \quad |\Lambda| = |\Lambda|^{Roe} = \operatorname{diag}\{|u-a|,\ |u|,\ |u+a|\}$$

Write $\Delta\mathbf{u}$ in terms of $\Delta\mathbf{v}$: $\mathrm{d}\mathbf{u} = \mathbf{u}'(\mathbf{v})\mathrm{d}\mathbf{v}$
Barth: Rescale eigenvectors $\tilde{R} = RS^{\frac{1}{2}}$ such that $\mathbf{u}'(\mathbf{v}) = \tilde{R}\tilde{R}^\top$

$$R^{-1}\mathrm{d}\mathbf{u} = SR^\top\mathrm{d}\mathbf{v}, \qquad S = \operatorname{diag}\left[\frac{\rho}{2\gamma},\ \frac{(\gamma-1)\rho}{\gamma},\ \frac{\rho}{2\gamma}\right]$$

Entropy stable schemes

Entropy-variable numerical flux (Tadmor)

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}^*_{j+\frac{1}{2}} - \frac{1}{2} \underbrace{R_{j+\frac{1}{2}} |\Lambda_{j+\frac{1}{2}}| S_{j+\frac{1}{2}} R^\top_{j+\frac{1}{2}}}_{Q_{j+\frac{1}{2}} \ge 0} \Delta \mathbf{v}_{j+\frac{1}{2}}$$

Entropy equation

$$\Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = -\frac{1}{4}\left[\Delta \mathbf{v}^\top_{j-\frac{1}{2}} Q_{j-\frac{1}{2}} \Delta \mathbf{v}_{j-\frac{1}{2}} + \Delta \mathbf{v}^\top_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta \mathbf{v}_{j+\frac{1}{2}}\right] \le 0$$

with consistent entropy flux

$$F_{j+\frac{1}{2}} = \overline{\mathbf{v}}_{j+\frac{1}{2}} \cdot \mathbf{f}^*_{j+\frac{1}{2}} - \overline{\psi}_{j+\frac{1}{2}} + \frac{1}{2} \bar{\mathbf{v}}^\top_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta \mathbf{v}_{j+\frac{1}{2}}$$

Entropy stable schemes

Stationary contact waves: Exactly resolved if

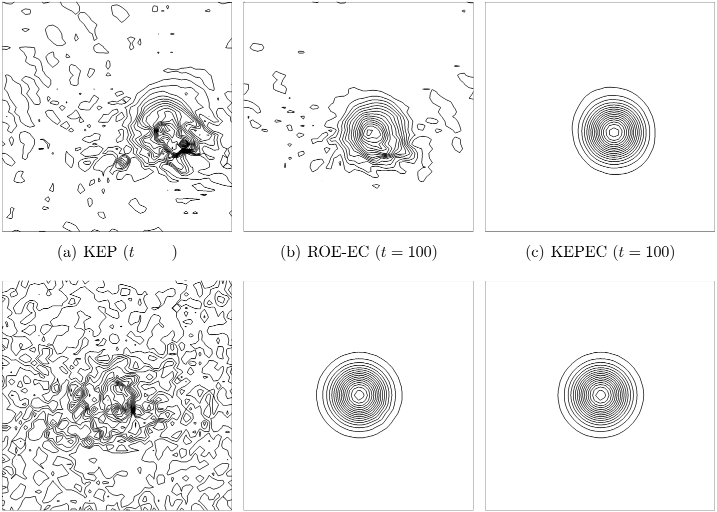
$$a_{j+\frac{1}{2}} = \sqrt{\frac{\gamma}{2\hat{\beta}_{j+\frac{1}{2}}}}, \quad H_{j+\frac{1}{2}} = \frac{a^2_{j+\frac{1}{2}}}{\gamma - 1} + \frac{1}{2}\overline{u}^2_{j+\frac{1}{2}}$$

$\Longrightarrow$ Accurate computation of boundary layers and shear layers

Higher order extension: ENO/WENO-type reconstruction (Fjordholm et al, Ray)

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}^{*,2p}_{j+\frac{1}{2}} - \frac{1}{2} R_{j+\frac{1}{2}} |\Lambda_{j+\frac{1}{2}}| S_{j+\frac{1}{2}} R^\top_{j+\frac{1}{2}} (\mathbf{v}^R_{j+\frac{1}{2}} - \mathbf{v}^L_{j+\frac{1}{2}})$$

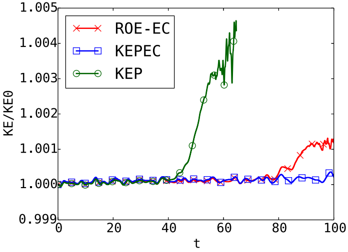
Isentropic vortex



(a) KEP (t)
(b) ROE-EC ($t = 100$)
(c) KEPEC ($t = 100$)
(d) KEP4 ($t = 100$)
(e) ROE-EC4 ($t = 100$)
(f) KEPEC4 ($t = 100$)

Figure 5.9: Isentropic vortex with $= 0$, 0×0 cells: density contours.

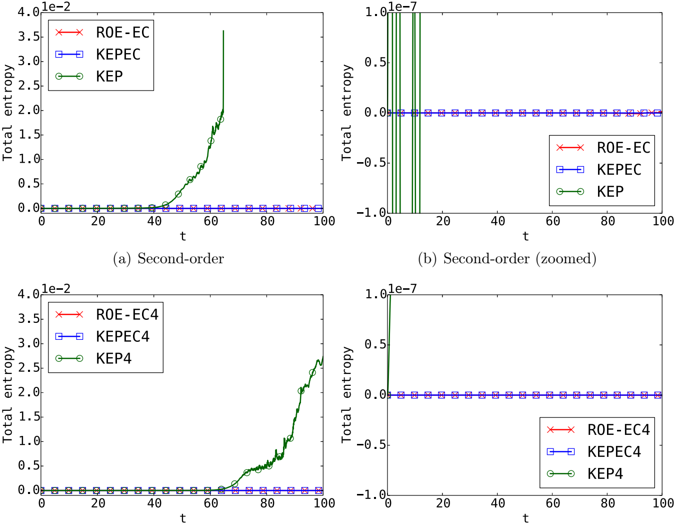
Isentropic vortex



(a) Second-order (b) Fourth-order

Figure 5.10: Evolution of relative total kinetic energy for isentropic vortex with $= 0$ **,** $0 \times \ 0$ **cells.**

Isentropic vortex



(a) Second-order (b) Second-order (zoomed)

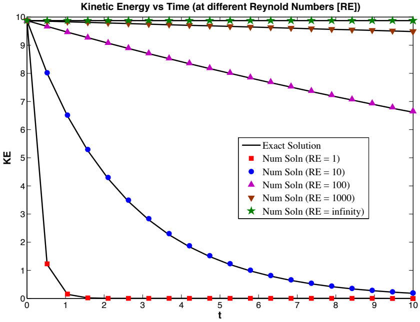
(c) Fourth-order (d) Fourth-order (zoomed)

Figure 5.11: Evolution of total entropy for isentropic vortex with = 0 , 0 × 0 cells.

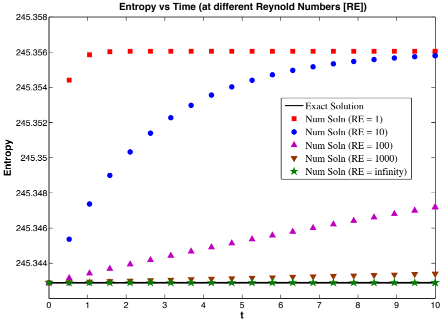
2-D Taylor-Green vortex: $[0, 2\pi]^2$, 32^2 grid

$$
\begin{aligned}
u &= -\cos(x)\sin(y)\exp(-2\mu t) \\
v &= \sin(x)\cos(y)\exp(-2\mu t) \\
p &= 500 - \frac{1}{4}(\cos(2x) + \cos(2y))\exp(-4\mu t)
\end{aligned}
$$

Central KEP-EC flux

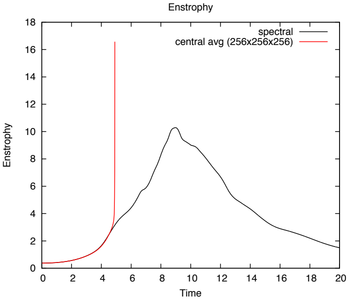


Total KE

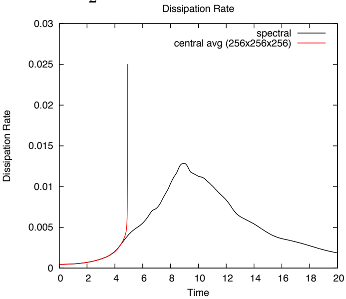


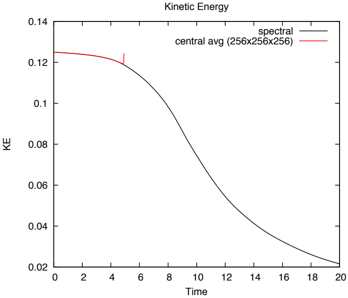
Total Entropy

DNS of 3-D Taylor-Green vortex: central scheme

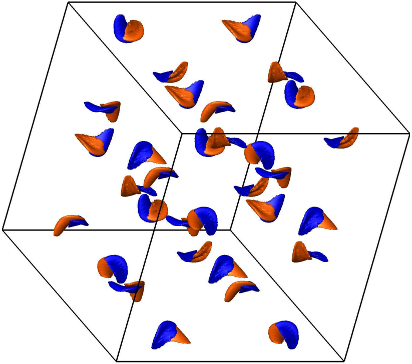


$$\mathbf{f}_{j+\frac{1}{2}} = \frac{1}{2}(\mathbf{f}_j + \mathbf{f}_{j+1})$$

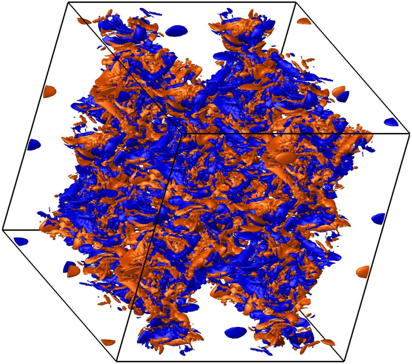




DNS of 3-D Taylor-Green vortex

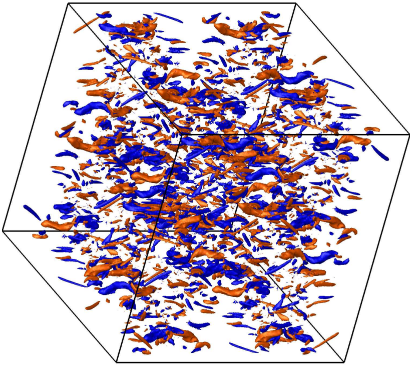


$t = 5$



$t = 10$

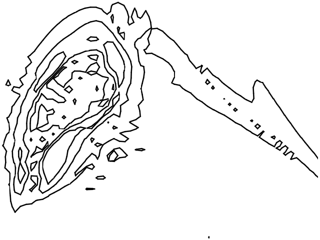
DNS of 3-D Taylor-Green vortex



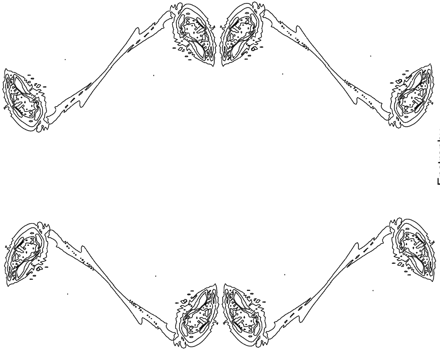
$t = 15$ $\qquad$ $t = 20$

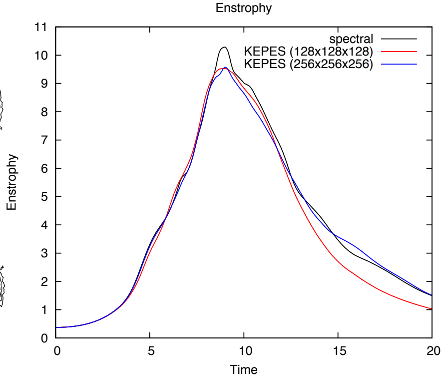
DNS of 3-D Taylor-Green vortex

igure 3: so contours of the dimensionless vorticity norm $-|\boldsymbol{\omega}| = 1, , 10, 20, 30$ on a subset of the periodic face $- = -$ at time $\frac{}{c} = 8$. ompar ison bet een the results obtained using the pseudo spectral code (blac) and those obtained using a code ith $p = 3$ and on a 6 mesh (red).

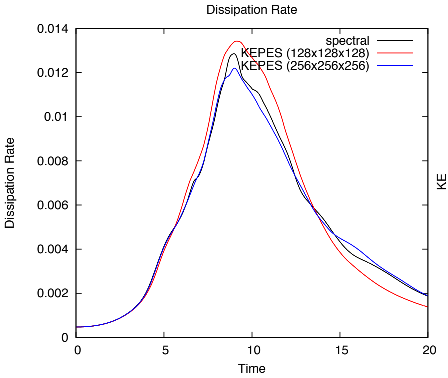


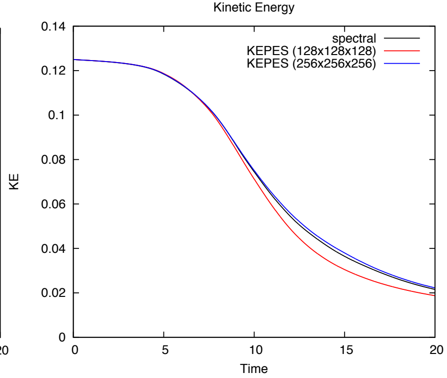
DNS of 3-D Taylor-Green vortex





DNS of 3-D Taylor-Green vortex





Summary

KE preserving + Entropy conservative scheme = non-linearly stable

- We can do DNS with such schemes: need small Δx, Δt
- Under-resolved case: mesh is coarse and/or large gradients
  - need additional stabilization or filtering
  - goal is to do this adaptively
  - Ducros sensor

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}^{*,2p}_{j+\frac{1}{2}} - \alpha_{j+\frac{1}{2}} \boldsymbol{d}_{j+\frac{1}{2}}$$

$$\alpha = \max\left(\frac{-div(\boldsymbol{v})}{\sqrt{|div(\boldsymbol{v})|^2 + |curl(\boldsymbol{v})|^2 + \omega_{ref}^2}}, 0 \right)$$

- SBP scheme: Fisher & Carpenter, JCP, 2013
  Gassner, IJNMF, 2014

Rotating shallow water model

Vector invariant form

$$
\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \Phi + (\omega + f)\boldsymbol{v}^{\perp} &= 0 \\
\frac{\partial D}{\partial t} + \nabla \cdot (\boldsymbol{v} D) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{v} &= \text{velocity} \\
D &= \text{depth} \\
H_s &= \text{bottom} \\
H &= \text{height of free surface} \\
&= D + H_s \\
K &= \frac{1}{2}|\boldsymbol{v}|^2 \\
\Phi &= gH + K \\
\omega &= \mathbf{k} \cdot \nabla \times \boldsymbol{v} \\
\boldsymbol{v}^{\perp} &= \mathbf{k} \times \boldsymbol{v} \\
f &= 2\Omega \sin \theta
\end{aligned}
$$

Additional properties

Total energy is conserved

$$\frac{\partial}{\partial t}\left(\frac{1}{2}D|\boldsymbol{v}|^2 + \frac{1}{2}gH^2\right) + \nabla\cdot\left[\left(gH + \frac{1}{2}|\boldsymbol{v}|^2\right)\boldsymbol{v}D\right] = 0$$

Vorticity equation: $\eta := \omega + f$

$$\frac{\partial\eta}{\partial t} + \nabla\cdot(\boldsymbol{v}\eta) = 0 \qquad \Longrightarrow \qquad \int_S \eta \mathrm{d}s = \text{const.}$$

Potential enstrophy is conserved

$$\frac{\partial}{\partial t}\left(\frac{\eta^2}{D}\right) + \nabla\cdot\left(\frac{\eta^2}{D}\boldsymbol{v}\right) = 0 \qquad \Longrightarrow \qquad \int_S \frac{\eta^2}{D}\mathrm{d}s = \text{const.}$$

Potential vorticity $q := \frac{\eta}{D}$ is advected by the flow

$$\frac{\partial q}{\partial t} + \boldsymbol{v}\cdot\nabla q = 0 \qquad \Longrightarrow \qquad q_{min} \le q(x,y,z,t) \le q_{max}$$

Additional properties

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Update vorticity in addition to $\boldsymbol{v}, D$

INS: Olshanskii et al. (JCP 2010), Benzi et al. (CMAME 2012), Palha & Gerritsma (2016)

Finite difference scheme in the plane

- Vector-invariant form
- Central fourth order FD for $\boldsymbol{v}$, D

$$\begin{bmatrix} \Phi \\ vD \end{bmatrix}_{j+1/2} = \frac{1}{2}\left\{ \begin{bmatrix} \Phi \\ vD \end{bmatrix}_{j} + \begin{bmatrix} \Phi \\ vD \end{bmatrix}_{j+1} \right\}$$

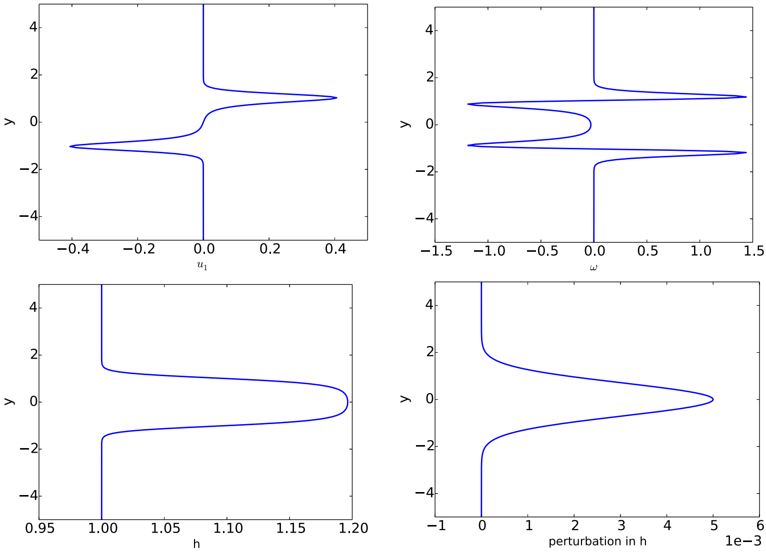
  - Semi-discrete scheme conserves energy
- 5'th order FD-LF-WENO for η
- Co-located variables
- Refered to as: VI-EP4

CM-EP4 (Fjordholm et al.)	Conservative model $(D\boldsymbol{v}, D)$ central scheme conserves energy
CM-WENO5	Conservative model $(D\boldsymbol{v}, D)$ WENO5 for all equations

Joint work with Deep Ray, TIFR-CAM

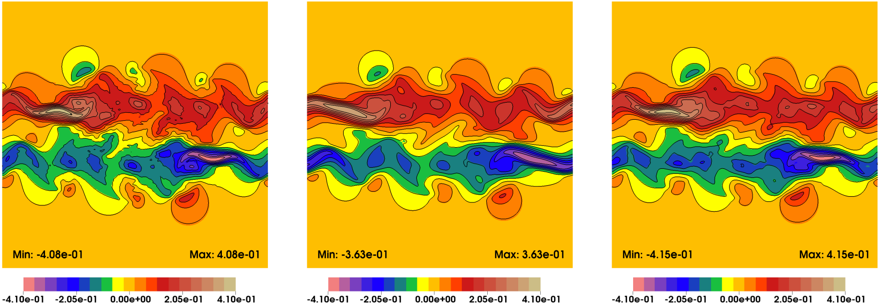
Perturbed geostrophic balance ($g = 1$, $f = 1$)

Initial condition



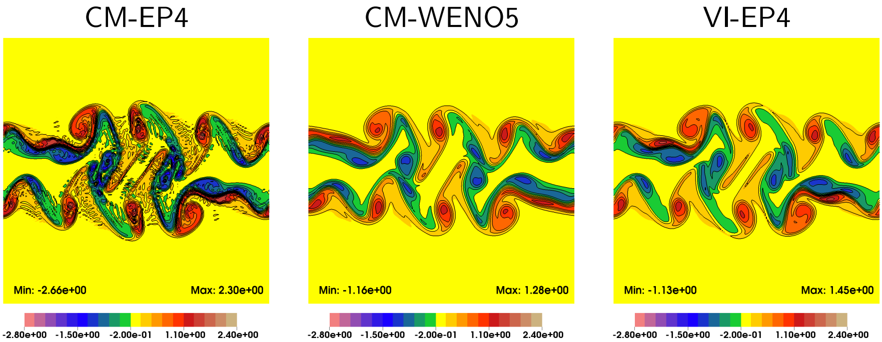
Perturbed geostrophic balance: v_1 at $t = 50$

CM-EP4 | CM-WENO5 | VI-EP4



200×200 mesh

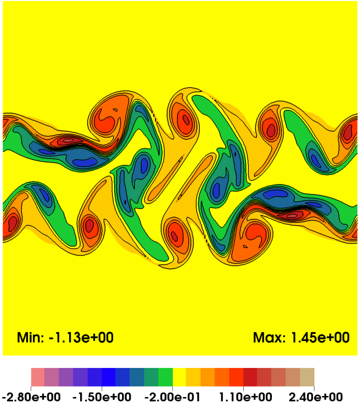
Perturbed geostrophic balance: ω at $t = 50$



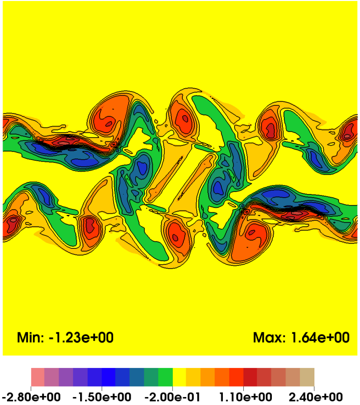
200×200 mesh

Perturbed geostrophic balance: VI-EP4 at $t = 50$

ω from WENO5

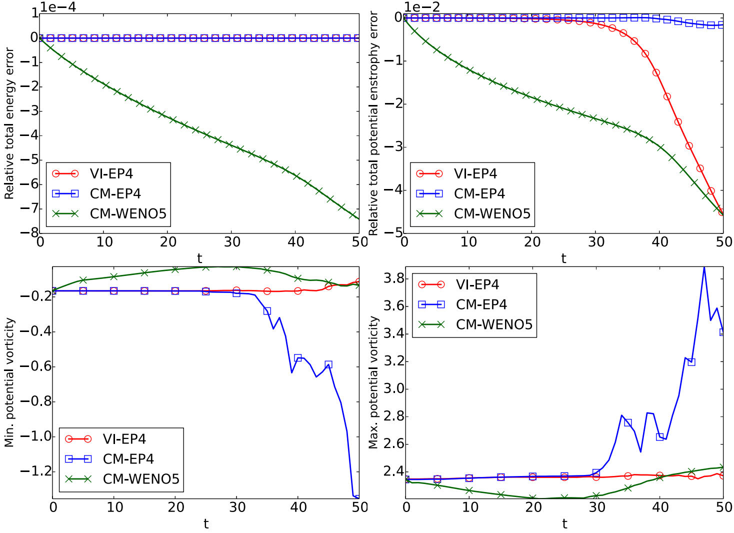


$\nabla \times \boldsymbol{v}$ using 4'th order FD



200×200 mesh

Perturbed geostrophic balance

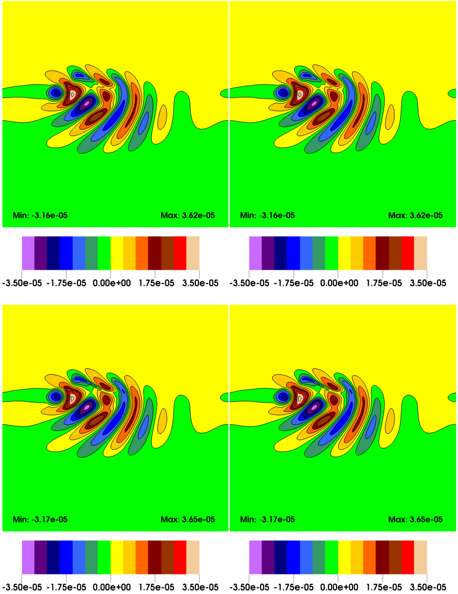


Flow over isolated mountain (Toy & Nair, 2017)

Vorticity from VI-EP4 scheme

200×200
$\Delta t = 6s$

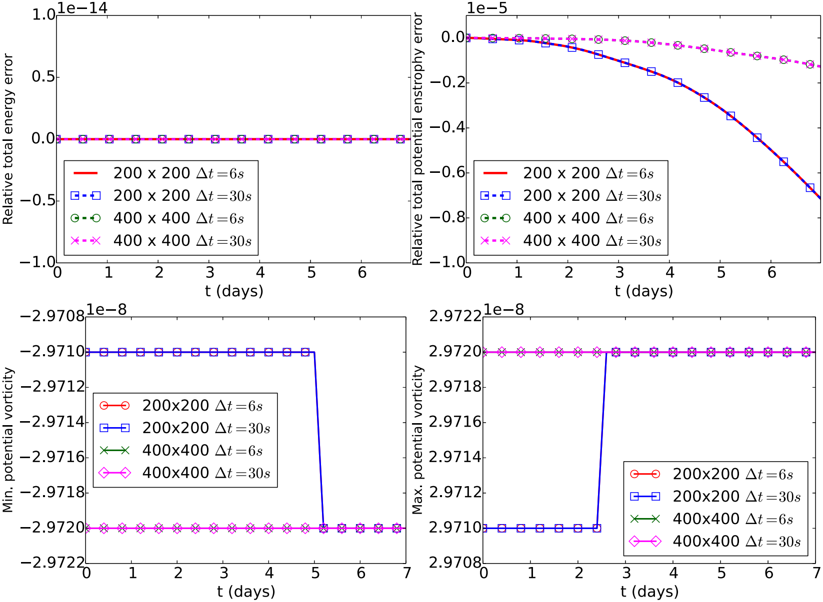
200×200
$\Delta t = 30s$



400×400
$\Delta t = 6s$

400×400
$\Delta t = 30s$

Flow over isolated mountain: VI-EP4 scheme

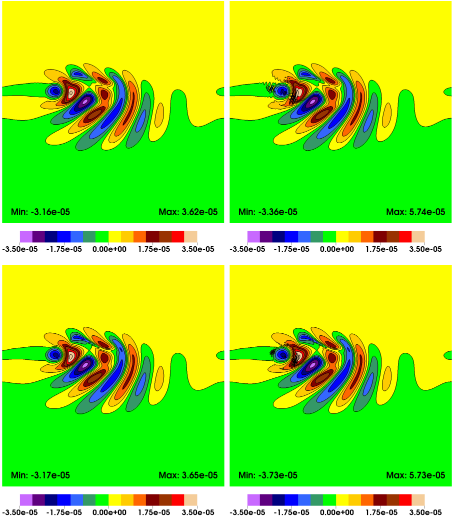


Flow over isolated mountain

VI-EP4 scheme, $\Delta t = 30s$

200×200
ω from WENO

200×200
$\nabla \times \boldsymbol{v}$ using FD



400×400
ω from WENO

400×400
$\nabla \times \boldsymbol{v}$ using FD

Summary

- Finite difference scheme in the plane
  - Central scheme for $\boldsymbol{v} - D$
  - WENO for vorticity equation
  - conserves mass, total vorticity, energy*
  - very high orders possible
  - extension to cube-sphere grid using SBP/SAT (Todo)