Divergence-free DG method for ideal compressible MHD equations

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Hyperbolic eqns: Structure preserving methods & other problems
Univ. of Wuerzburg (via Zoom)
12 February, 2021

Maxwell Equations

Linear hyperbolic system

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}$$

$\boldsymbol{B}$ = magnetic flux density

$\boldsymbol{E}$ = electric field

$\boldsymbol{D}$ = electric flux density

$\boldsymbol{H}$ = magnetic field

$\boldsymbol{J}$ = electric current density

$$\boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{J} = \sigma \boldsymbol{E} \qquad \mu, \varepsilon \in \mathbb{R}^{3\times 3} \text{ symmetric}$$

$$\varepsilon = \text{permittivity tensor}$$
$$\mu = \text{magnetic permeability tensor}$$
$$\sigma = \text{conductivity}$$

Constraints

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{D}) + \nabla \cdot \boldsymbol{J} = 0$$

Two fluid MHD

Non-linear hyperbolic system

Conservation laws for each species: $\alpha = i, e$

$$
\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha) &= 0 \\
\frac{\partial (\rho_\alpha \boldsymbol{v}_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha + p_\alpha I) &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha (\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}) \\
\frac{\partial \mathcal{E}_\alpha}{\partial t} + \nabla \cdot [(\mathcal{E}_\alpha + p_\alpha) \boldsymbol{v}_\alpha] &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha \boldsymbol{E} \cdot \boldsymbol{v}_\alpha
\end{aligned}
$$

$$
\text{Total energy:} \qquad \mathcal{E}_\alpha = \frac{p_\alpha}{\gamma_\alpha - 1} + \frac{1}{2} \rho_\alpha |\boldsymbol{v}_\alpha|^2
$$

Coupled with Maxwell's equations

$$
\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu_0 (\rho_i q_i \boldsymbol{v}_i + \rho_e q_e \boldsymbol{v}_e)
$$

together with the <u>constraints</u>

$$
\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} (\rho_i q_i + \rho_e q_e)
$$

Compressible Euler equations with Lorentz force

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (P\mathrm{I} + \rho \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{B} \otimes \boldsymbol{B}) &= 0 \\
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot ((\mathcal{E} + P)\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}) &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) &= 0
\end{aligned}
$$

$$
P = p + \frac{1}{2}|\boldsymbol{B}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\boldsymbol{B}|^2
$$

Magnetic monopoles do not exist: $\implies \quad \nabla \cdot \boldsymbol{B} = 0$

Divergence constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\nabla \cdot \nabla \times}_{=0} \boldsymbol{E} = 0$$

$$\frac{\partial}{\partial t} \nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, 0) = 0 \quad \Longrightarrow \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0$$

Intrinsic property, not dynamical eqn

Lorentz force: $\boldsymbol{v} \times \boldsymbol{B} \perp \boldsymbol{B}$

$$\nabla \cdot \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2} |\boldsymbol{B}|^2 I \right)$$

$$= {\color{blue}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}} + {\color{red}(\nabla \cdot \boldsymbol{B})\boldsymbol{B}}$$

Divergence constraint

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$

$$\nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\nabla \cdot \nabla \times}_{=0} \boldsymbol{E} = 0$$

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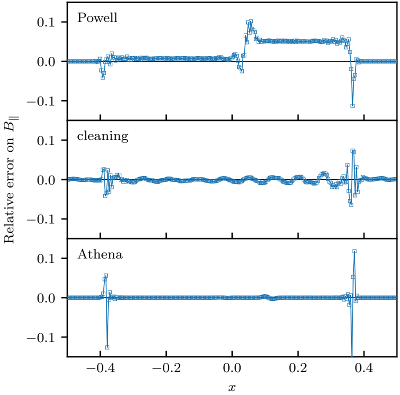
$$\nabla \cdot \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2} |\boldsymbol{B}|^2 I \right) = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + (\nabla \cdot \boldsymbol{B}) \boldsymbol{B}$$

Rotated shock tube



Guillet et al., MNRAS 2019

Discrete div-free $\Longrightarrow$ positivity (Kailiang Wu (2018))

Objectives

- Based on conservation form of the equations
- Upwind-type schemes using Riemann solvers (Godunov approach)
- High order accurate
  - discontinuous-Galerkin FEM
- Divergence-free schemes for Maxwell's and compressible MHD
  - Cartesian grids at present
  - Divergence preserving schemes (RT)
- Non-oscillatory schemes for MHD
  - using limiters
  - div-free reconstruction using BDM[1]
- Explicit time stepping
  - local mass matrices
- Based on
  - Induction eqn: J. Sci. Comp., Vol. 79, pp, 79-102, 2019
  - Compressible MHD: J. Sci. Comp., Vol. 84, 2020

[1] Hazra et al., JCP, Vol. 394, 2019

Some existing methods

Exactly divergence-free methods

- Yee scheme (Yee (1966))
- Projection methods (Brackbill & Barnes (1980))
- Constrained transport (Evans & Hawley (1989))
- Divergence-free reconstruction (Balsara (2001))
- Globally divergence-free scheme (Li et al. (2011), Fu et al, (2018))

Approximate methods

- Locally divergence-free schemes (Cockburn, Li & Shu (2005))
- Godunov's symmetrized version of MHD (Powell (1994), Winters/Gassner (2016), C/Klingenberg (2016))
- Divergence cleaning methods (Dedner et al. (2002))

MHD equations in 2-D

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} = 0$$

$$\mathcal{U} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \mathcal{E} \\ B_x \\ B_y \\ B_z \end{bmatrix}, \quad \mathcal{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x(\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ 0 \\ -E_z \\ v_x B_z - v_z B_x \end{bmatrix}, \quad \mathcal{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y(\boldsymbol{v} \cdot \boldsymbol{\mathfrak{B}}) \\ E_z \\ 0 \\ v_y B_z - v_z B_y \end{bmatrix}$$

where

$$\boldsymbol{\mathfrak{B}} = (B_x, B_y, B_z), \qquad P = p + \frac{1}{2}|\boldsymbol{\mathfrak{B}}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\boldsymbol{\mathfrak{B}}|^2$$

E_z is the electric field in the z direction

$$E_z = -(\boldsymbol{v} \times \boldsymbol{\mathfrak{B}})_z = v_y B_x - v_x B_y$$

Ideal MHD in one dimension

Divergence constraint $$\frac{\partial B_x}{\partial x} = 0 \quad \Longrightarrow \quad B_x = \text{constant}$$

Conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0 \qquad \boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \mathcal{E} \\ B_y \\ B_z \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \rho u \\ P + \rho u^2 - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ (\mathcal{E} + P)u - (\boldsymbol{v} \cdot \boldsymbol{B}) B_x \\ u B_y - v B_x \\ u B_z - w B_x \end{bmatrix}$$

Flux jacobian matrix

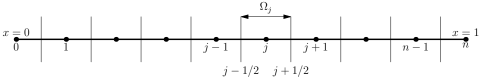
$$A = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$$ has seven real eigenvalues and eigenvectors

$$u - c_f \le u - c_a \le u - c_s \le u \le u + c_s \le u + c_a \le u + c_f$$

$$c_a = \frac{|B_x|}{\sqrt{\rho}} \qquad a = \sqrt{\frac{\gamma p}{\rho}} \qquad c_{f/s} = \sqrt{\frac{1}{2}\left[a^2 + |\boldsymbol{b}|^2 \pm \sqrt{(a^2 + |\boldsymbol{b}|^2)^2 - 4a^2 b_x^2}\right]} \qquad \boldsymbol{b} = \frac{\boldsymbol{B}}{\sqrt{\rho}}$$

Alfven speed — Sound speed — Fast/slow magnetosonic speeds

Finite volume method

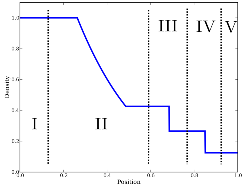


Weak solution: Satisfy conservation law on each finite volume

$$\Delta x \frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t} + \boldsymbol{F}_{j+\frac{1}{2}} - \boldsymbol{F}_{j-\frac{1}{2}} = 0$$

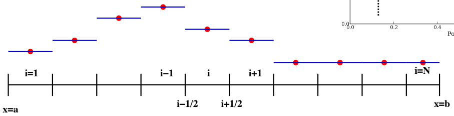
Basic unknown:
mean value in each cell

$$\boldsymbol{U}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \boldsymbol{U}(x, t_n)\mathrm{d}x$$



What is the flux ?

Riemann problem at each cell face

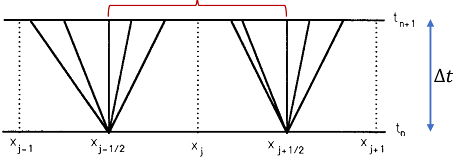


$$\boldsymbol{U}(x,t_n) = \begin{cases} \boldsymbol{U}_j^n & x < x_{j+\frac{1}{2}} \\ \boldsymbol{U}_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases} \Rightarrow \boldsymbol{U}(x,t) = \boldsymbol{U}_R\left(\frac{x - x_{i+\frac{1}{2}}}{t - t_n}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^{n+1}\right), \qquad t > t_n$$

Self-similar solution of RP

Finite volume method

Evolve waves for small time Δt (CFL condition)



Average solution at new time level

$$\boldsymbol{U}_j^{n+1} = \frac{1}{\Delta x}\left[\int_{x_{j-\frac{1}{2}}}^{x_j} \boldsymbol{U}_R\left(\frac{x - x_{j-\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right) \mathrm{d}x + \int_{x_j}^{x_{j+\frac{1}{2}}} \boldsymbol{U}_R\left(\frac{x - x_{j+\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n\right) \mathrm{d}x\right]$$

Finite volume form

$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}[\boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)) - \boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n))]$$

RP→Evolve→Average: Godunov finite volume scheme

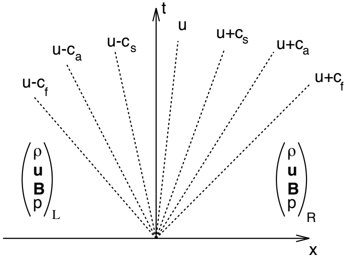
MHD Riemann problem

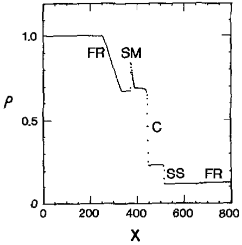
7 waves

$u \pm c_f$	fast shock/rarefaction wave
$u \pm c_a$	rotational discontinuity
$u \pm c_s$	slow shock/rarefaction wave
u	contact discontinuity

Neither genuinely non-linear nor linearly degenerate

Linearly degenerate





Brio-Wu

$c_s \leq c_a \leq c_f$: Wave speeds can coincide → non-strictly hyperbolic
Non-regular waves: compound waves, over-compressive intermediate shocks possible
Riemann solution is not always unique

MHD in multi-dimensions

x-direction Riemann problem

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} = 0, \quad \Longrightarrow \quad \frac{\partial \mathcal{U}}{\partial t} + \mathcal{A}_x \frac{\partial \mathcal{U}}{\partial x} = 0, \quad \mathcal{A}_x = \mathcal{F}_x'(\mathcal{U})$$

$\mathcal{A}_x$: 8 real eigenvalues, one zero, 7 lin. ind. eigenvectors only !!!

In the Riemann problem, $(B_x)_L \neq (B_x)_R$

Modify the MHD equations (Godunov, Powell)

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} + \Phi \nabla \cdot \boldsymbol{B} = 0$$

8 real eigenvalues and 8 lin. ind. eigenvectors

- Build approximate Riemann solver (Powell et al.)
- Build entropy stable schemes (Winters et al., C/Klingenberg)

BUT: not divergence-free, not conservative

MHD equations in 2-D

Split into two parts

$$\boldsymbol{U} = [\rho,\ \rho\boldsymbol{v},\ \mathcal{E},\ B_z]^\top, \qquad \boldsymbol{B} = (B_x, B_y)$$

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \boldsymbol{B}) = 0, \qquad \frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0}$$

Fluxes: $\boldsymbol{F} = (\boldsymbol{F}_x, \boldsymbol{F}_y)$

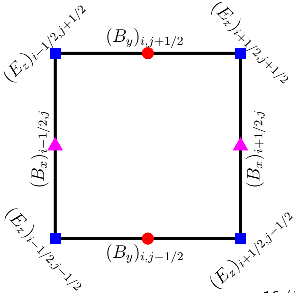
$$\boldsymbol{F}_x = \begin{bmatrix} \rho v_x \\ P + \rho v_x^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (\mathcal{E} + P) v_x - B_x(\boldsymbol{v} \cdot \mathfrak{B}) \\ v_x B_z - v_z B_x \end{bmatrix}, \qquad \boldsymbol{F}_y = \begin{bmatrix} \rho v_y \\ \rho v_x v_y - B_x B_y \\ P + \rho v_y^2 - B_y^2 \\ \rho v_y v_z - B_y B_z \\ (\mathcal{E} + P) v_y - B_y(\boldsymbol{v} \cdot \mathfrak{B}) \\ v_y B_z - v_z B_y \end{bmatrix}$$

Constraint preserving finite difference

Store magnetic field on the faces: $(B_x)_{i+\frac{1}{2},j}$, $(B_y)_{i,j+\frac{1}{2}}$

$$\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0 \quad \Longrightarrow \quad \frac{\mathsf{d}}{\mathsf{d}t}(B_x)_{i+\frac{1}{2},j} + \frac{(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} = 0$$

$$\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\mathsf{d}}{\mathsf{d}t}(B_y)_{i,j+\frac{1}{2}} - \frac{(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} = 0$$



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$$\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\mathsf{d}}{\mathsf{d}t}(B_y)_{i,j+\frac{1}{2}} - \frac{(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} = 0$$

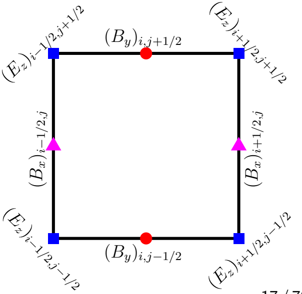
Measure divergence at cell center

$$\nabla_h \cdot \boldsymbol{B}_{i,j} = \frac{(B_x)_{i+\frac{1}{2},j} - (B_x)_{i-\frac{1}{2},j}}{\Delta x} + \frac{(B_y)_{i,j+\frac{1}{2}} - (B_y)_{i,j-\frac{1}{2}}}{\Delta y}$$

Then

$$\frac{\mathsf{d}}{\mathsf{d}t}\nabla_h \cdot \boldsymbol{B}_{i,j} = 0$$

The corner fluxes cancel one another !!!



Approximation of magnetic field

$$\boldsymbol{B}_h \in V_h = \text{FE polynomial space on mesh } \mathcal{T}_h$$

If $\nabla \cdot \boldsymbol{B}_h = 0$, then take

$$\boldsymbol{B}_h \in V_h \subset H(div, \Omega) = \{\boldsymbol{B} \in L^2(\Omega) : \text{div}(\boldsymbol{B}) \in L^2(\Omega)\}$$

Necessary condition

$$\boxed{\boldsymbol{B}_h \cdot \boldsymbol{n} \text{ continuous across element faces}}$$

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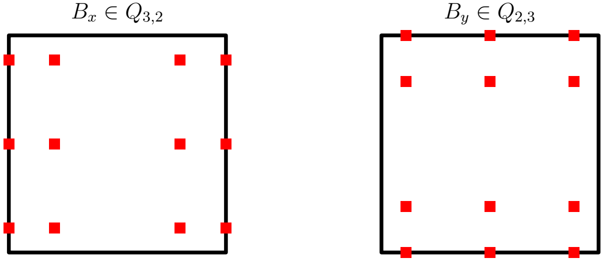
Possible options: Brezzi-Douglas-Marini, Raviart-Thomas, etc.



Approximation spaces: Degree $k \geq 0$

Map cell K to reference cell $\hat{K} = [-\frac{1}{2}, +\frac{1}{2}] \times [-\frac{1}{2}, +\frac{1}{2}]$

$$\mathbb{P}_r(\xi) = \text{span}\{1, \xi, \xi^2, \ldots, \xi^r\}, \quad \mathbb{Q}_{r,s}(\xi, \eta) = \mathbb{P}_r(\xi) \otimes \mathbb{P}_s(\eta)$$

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Hydrodynamic variables in each cell

$$\boldsymbol{U}(\xi,\eta) = \sum_{i=0}^{k}\sum_{j=0}^{k} \boldsymbol{U}_{ij}\phi_i(\xi)\phi_j(\eta) \in \mathbb{Q}_{k,k}$$

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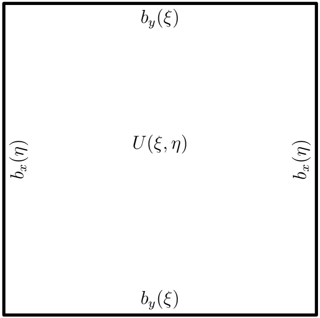

Normal component of $\boldsymbol{B}$ on faces

$$\text{on vertical faces} : b_x(\eta) = \sum_{j=0}^{k} a_j \phi_j(\eta) \in \mathbb{P}_k(\eta)$$

$$\text{on horizontal faces} : b_y(\xi) = \sum_{j=0}^{k} b_j \phi_j(\xi) \in \mathbb{P}_k(\xi)$$

$\{\phi_i(\xi)\}$ are **orthogonal polynomials** on $[-\frac{1}{2}, +\frac{1}{2}]$, with degree$(\phi_i) = i$.

Approximation spaces: Degree $k \geq 0$

For $k \geq 1$,define certain **cell moments**

$$\alpha_{ij} = \alpha_{ij}(B_x) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k-1,k}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k-1, \quad 0 \leq j \leq k$$

$$\beta_{ij} = \beta_{ij}(B_y) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k,k-1}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k, \quad 0 \leq j \leq k-1$$

$$m_{ij} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)\phi_j(\eta)]^2 \mathsf{d}\xi \mathsf{d}\eta = m_i m_j, \quad m_i = \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)]^2 \mathsf{d}\xi$$

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α_{00}, β_{00} are **cell averages** of B_x, B_y

Approximation spaces: Degree $k \geq 0$

For $k \geq 1$,define certain **cell moments**

$$\alpha_{ij} = \alpha_{ij}(B_x) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi, \eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k-1,k}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k-1, \quad 0 \leq j \leq k$$

$$\beta_{ij} = \beta_{ij}(B_y) := \frac{1}{m_{ij}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi, \eta) \underbrace{\phi_i(\xi)\phi_j(\eta)}_{\mathbb{Q}_{k,k-1}} \mathsf{d}\xi \mathsf{d}\eta, \quad 0 \leq i \leq k, \quad 0 \leq j \leq k-1$$

$$m_{ij} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)\phi_j(\eta)]^2 \mathsf{d}\xi \mathsf{d}\eta = m_i m_j, \quad m_i = \int_{-\frac{1}{2}}^{+\frac{1}{2}} [\phi_i(\xi)]^2 \mathsf{d}\xi$$

α_{00}, β_{00} are **cell averages** of B_x, B_y

Solution variables

$$\{\boldsymbol{U}(\xi, \eta)\}, \qquad \{b_x(\eta)\}, \qquad \{b_y(\xi)\}, \qquad \{\alpha, \beta\}$$

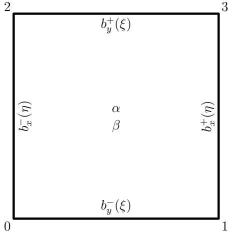
The set $\{b_x, b_y, \alpha, \beta\}$ are the dofs for the **Raviart-Thomas** space.

RT reconstruction: $\{b_x^\pm(\eta), b_y^\pm(\xi), \alpha, \beta\} \to \boldsymbol{B}(\xi, \eta)$

Given $b_x^\pm(\eta) \in \mathbb{P}_k$ and $b_y^\pm(\xi) \in \mathbb{P}_k$,
and set of cell moments

$$\{\alpha_{ij},\ 0 \le i \le k-1,\ 0 \le j \le k\}$$

$$\{\beta_{ij},\ 0 \le i \le k,\ 0 \le j \le k-1\}$$



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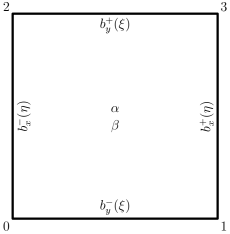

Find $B_x \in \mathbb{Q}_{k+1,k}$ and $B_y \in \mathbb{Q}_{k,k+1}$ such that

$$B_x(\pm\tfrac{1}{2},\eta) = b_x^\pm(\eta), \quad \eta \in [-\tfrac{1}{2},\tfrac{1}{2}], \qquad B_y(\xi,\pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \xi \in [-\tfrac{1}{2},\tfrac{1}{2}]$$

$$\frac{1}{m_{ij}}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} B_x(\xi,\eta)\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta = \alpha_{ij}, \qquad 0 \le i \le k-1, \quad 0 \le j \le k$$

$$\frac{1}{m_{ij}}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} B_y(\xi,\eta)\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta = \beta_{ij}, \qquad 0 \le i \le k, \quad 0 \le j \le k-1$$

RT reconstruction: $\{b_x^\pm(\eta), b_y^\pm(\xi), \alpha, \beta\} \to \boldsymbol{B}(\xi,\eta)$

Given $b_x^\pm(\eta) \in \mathbb{P}_k$ and $b_y^\pm(\xi) \in \mathbb{P}_k$,
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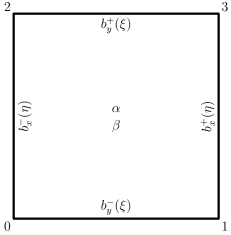

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(1) $\exists$ unique solution. (2) $\boldsymbol{B} \cdot \boldsymbol{n}$ continuous.
(3) Data div-free $\Longrightarrow$ reconstructed $\boldsymbol{B}$ is div-free.

DG scheme for $\boldsymbol{B}$ on faces

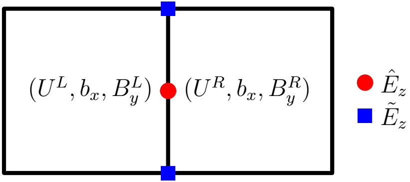
<u>Vertical face</u> of the mesh: $\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_x}{\partial t} \phi_i \mathsf{d}\eta - \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\eta} \mathsf{d}\eta + \frac{1}{\Delta y} [\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

Numerical fluxes

$\hat{E}_z$: on face, 1-D Riemann solver

$\tilde{E}_z$: at vertex, 2-D Riemann solver



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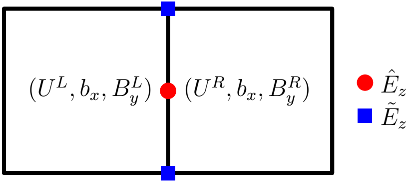

Horizontal face of the mesh: $\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial b_y}{\partial t} \phi_i \mathsf{d}\xi + \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{E}_z \frac{\mathsf{d}\phi_i}{\mathsf{d}\xi} \mathsf{d}\xi - \frac{1}{\Delta x} [\tilde{E}_z \phi_i] = 0, \qquad 0 \le i \le k$$

Unique vertex flux $\tilde{E}_z$ used in all equations

DG scheme for $\boldsymbol{B}$ on cells: $\frac{\partial B_x}{\partial t} + \frac{1}{\Delta y}\frac{\partial E_z}{\partial \eta} = 0$

$$
\begin{aligned}
m_{ij}\frac{\mathsf{d}\alpha_{ij}}{\mathsf{d}t} &= \int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial B_x}{\partial t}\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta, \quad 0 \le i \le k-1, \quad 0 \le j \le k \\
&= -\frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial E_z}{\partial \eta}\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta \\
&= -\frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}} [\hat{E}_z(\xi, \tfrac{1}{2})\phi_i(\xi)\phi_j(\tfrac{1}{2}) - \hat{E}_z(\xi, -\tfrac{1}{2})\phi_i(\xi)\phi_j(-\tfrac{1}{2})]\mathsf{d}\xi \\
&\quad + \frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} E_z(\xi,\eta)\phi_i(\xi)\phi_j'(\eta)\mathsf{d}\xi\mathsf{d}\eta
\end{aligned}
$$

Numerical fluxes

$$\hat{E}_z : \text{on face, 1-D Riemann solver}$$

DG scheme for $\boldsymbol{B}$ on cells: $\frac{\partial B_x}{\partial t} + \frac{1}{\Delta y}\frac{\partial E_z}{\partial \eta} = 0$

$$
\begin{aligned}
m_{ij}\frac{\mathsf{d}\alpha_{ij}}{\mathsf{d}t} &= \int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial B_x}{\partial t}\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta, \quad 0 \le i \le k-1, \quad 0 \le j \le k \\
&= -\frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial E_z}{\partial \eta}\phi_i(\xi)\phi_j(\eta)\mathsf{d}\xi\mathsf{d}\eta \\
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&\quad +\frac{1}{\Delta y}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} E_z(\xi,\eta)\phi_i(\xi)\phi_j'(\eta)\mathsf{d}\xi\mathsf{d}\eta
\end{aligned}
$$

Numerical fluxes

$$\hat{E}_z : \text{on face, 1-D Riemann solver}$$

Not a Galerkin method, test functions ($\mathbb{Q}_{k-1,k}$) different from trial functions ($\mathbb{Q}_{k+1,k}$)

DG scheme for $\boldsymbol{U}$ on cells

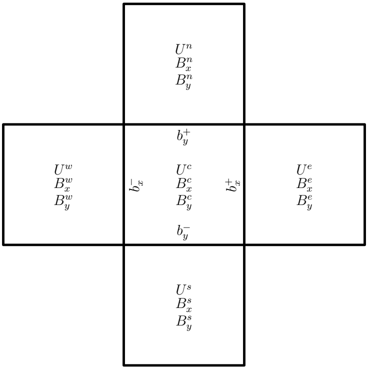
For each test function $\Phi(\xi,\eta) = \phi_i(\xi)\phi_j(\eta) \in \mathbb{Q}_{k,k}$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial \boldsymbol{U}^c}{\partial t} \Phi(\xi,\eta) \mathsf{d}\xi \mathsf{d}\eta - \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left[\frac{1}{\Delta x} \boldsymbol{F}_x \frac{\partial \Phi}{\partial \xi} + \frac{1}{\Delta y} \boldsymbol{F}_y \frac{\partial \Phi}{\partial \eta} \right] \mathsf{d}\xi \mathsf{d}\eta$$
$$+ \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_x^+ \Phi(\tfrac{1}{2},\eta) \mathsf{d}\eta - \frac{1}{\Delta x} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_x^- \Phi(-\tfrac{1}{2},\eta) \mathsf{d}\eta$$
$$+ \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_y^+ \Phi(\xi,\tfrac{1}{2}) \mathsf{d}\xi - \frac{1}{\Delta y} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \hat{\boldsymbol{F}}_y^- \Phi(\xi,-\tfrac{1}{2}) \mathsf{d}\xi = 0$$

Numerical fluxes

$$\hat{\boldsymbol{F}}_x^\pm, \hat{\boldsymbol{F}}_y^\pm : \text{on face, 1-D Riemann solver}$$

DG scheme for $\boldsymbol{U}$ on cells



$$\boldsymbol{F}_x = \boldsymbol{F}_x(\boldsymbol{U}^c, B_x^c, B_y^c), \qquad \boldsymbol{F}_y = \boldsymbol{F}_y(\boldsymbol{U}^c, B_x^c, B_y^c)$$

$$\hat{\boldsymbol{F}}_x^+ = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^c, b_x^+, B_y^c), (\boldsymbol{U}^e, b_x^+, B_y^e)), \qquad \hat{\boldsymbol{F}}_x^- = \hat{\boldsymbol{F}}_x((\boldsymbol{U}^w, b_x^-, B_y^w), (\boldsymbol{U}^c, b_x^-, B_y^c))$$

$$\hat{\boldsymbol{F}}_y^+ = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^c, B_x^c, b_y^+), (\boldsymbol{U}^n, B_x^n, b_y^+)), \qquad \hat{\boldsymbol{F}}_y^- = \hat{\boldsymbol{F}}_y((\boldsymbol{U}^s, B_x^s, b_y^-), (\boldsymbol{U}^c, B_x^c, b_y^-))$$

Constraints on $\boldsymbol{B}$

Definition (Globally divergence-free)

A vector field $\boldsymbol{B}$ defined on a mesh is globally divergence-free if

1. $\nabla \cdot \boldsymbol{B} = 0$ in each cell $K \in \mathcal{T}_h$
2. $\boldsymbol{B} \cdot \boldsymbol{n}$ is continuous at each face $F \in \mathcal{T}_h$

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Theorem

(1) The DG scheme satisfies

$$\frac{d}{dt} \int_K (\nabla \cdot \boldsymbol{B}) \phi \, dx dy = 0, \qquad \forall \phi \in \mathbb{Q}_{k,k}$$

and since $\nabla \cdot \boldsymbol{B} \in \mathbb{Q}_{k,k} \implies \nabla \cdot \boldsymbol{B} =$ *constant wrt time.*

(2) If $\nabla \cdot \boldsymbol{B} \equiv 0$ *at* $t = 0 \implies \nabla \cdot \boldsymbol{B} \equiv 0$ *for* $t > 0$

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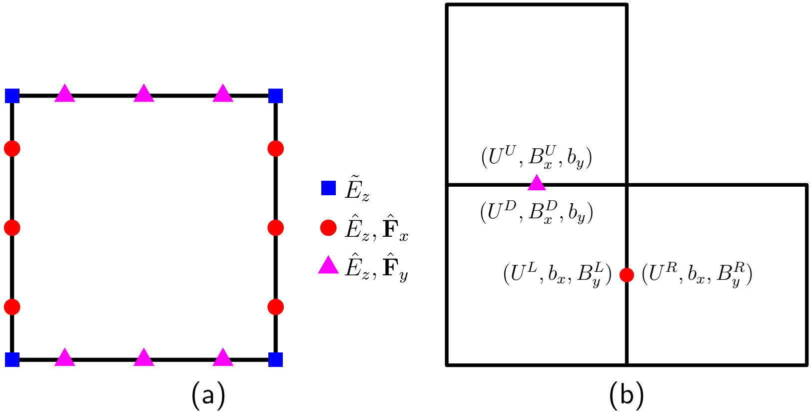
But: Applying a limiter in a post-processing step destroys div-free property !!!

Numerical fluxes

$(k+1)$-point Gauss-Legendre quadrature on faces



(a) Face quadrature points & numerical fluxes, $k = 2$. (b) 1-D Riemann problems at a vertical and horizontal face of a cell

Numerical fluxes

$\hat{\boldsymbol{F}}_x$, $\hat{E}_z$: solve 1-D Riemann problem at each face quadrature point

$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}_x}{\partial x} = 0, \qquad \mathcal{U}(x,0) = \begin{cases} \mathcal{U}^L = \mathcal{U}(\boldsymbol{U}^L, b_x, B_y^L) & x < 0 \\ \mathcal{U}^R = \mathcal{U}(\boldsymbol{U}^R, b_x, B_y^R) & x > 0 \end{cases}$$

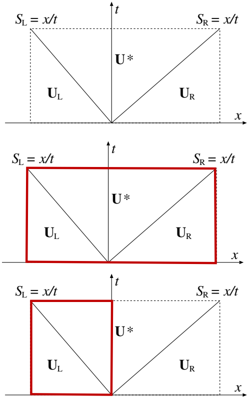
$$\hat{\boldsymbol{F}}_x = \begin{bmatrix} (\hat{\mathcal{F}}_x)_1 \\ (\hat{\mathcal{F}}_x)_2 \\ (\hat{\mathcal{F}}_x)_3 \\ (\hat{\mathcal{F}}_x)_4 \\ (\hat{\mathcal{F}}_x)_5 \\ (\hat{\mathcal{F}}_x)_8 \end{bmatrix}, \qquad \hat{E}_z = -(\hat{\mathcal{F}}_x)_7$$

Riemann problem can lead to 7 waves !!! Solve approximately.

HLL solver in 1-D: slowest and fastest waves: $S_L < S_R$

- Intermediate state from conservation law

$$\mathcal{U}^* = \frac{S_R\mathcal{U}^R - S_L\mathcal{U}^L - (\mathcal{F}_x^R - \mathcal{F}_x^L)}{S_R - S_L}$$



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- Satisfying conservation law over half Riemann fan

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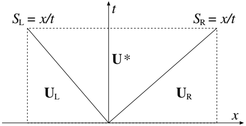

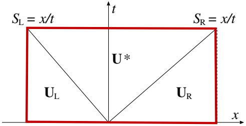

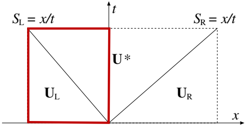

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- Satisfying conservation law over half Riemann fan

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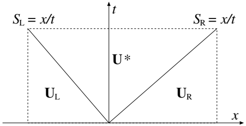
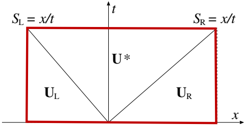
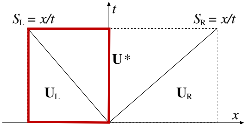
- Numerical flux is given by

$$\hat{\mathcal{F}}_x = \begin{cases} \mathcal{F}_x^L & S_L > 0 \\ \mathcal{F}_x^R & S_R < 0 \\ \mathcal{F}_x^* & \text{otherwise} \end{cases}$$







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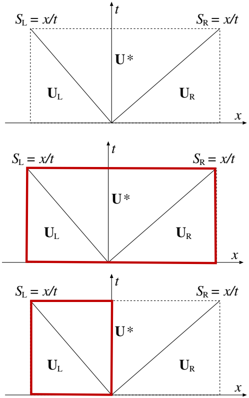

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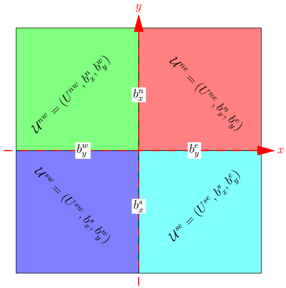
- Numerical flux is given by

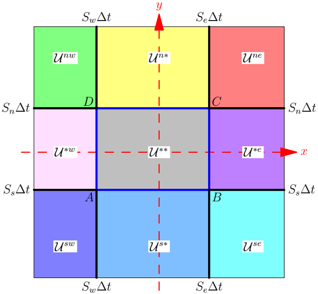
$$\hat{\mathcal{F}}_x = \begin{cases} \mathcal{F}_x^L & S_L > 0 \\ \mathcal{F}_x^R & S_R < 0 \\ \mathcal{F}_x^* & \text{otherwise} \end{cases}$$

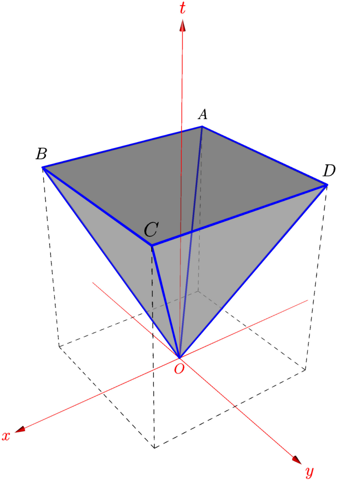
- Electric field from 7'th component

$$\hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = -(\hat{\mathcal{F}}_x)_7 = \begin{cases} E_z^L & S_L > 0 \\ E_z^R & S_R < 0 \\ \frac{S_R E_z^L - S_L E_z^R - S_L S_R (B_y^R - B_y^L)}{S_R - S_L} & \text{otherwise} \end{cases}$$

2-D Riemann problem







2-D Riemann problem

Strongly interacting state

$$B_x^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_x^{ne} - 2S_n S_w B_x^{nw} + 2S_s S_w B_x^{sw} - 2S_s S_e B_x^{se} - S_e(E_z^{ne} - E_z^{se}) + S_w(E_z^{nw} - E_z^{sw}) - (S_e - S_w)(E_z^{n*} - E_z^{s*}) \Big]$$

$$B_y^{**} = \frac{1}{2(S_e - S_w)(S_n - S_s)} \Big[2S_e S_n B_y^{ne} - 2S_n S_w B_y^{nw} + 2S_s S_w B_y^{sw} - 2S_s S_e B_y^{se} + S_n(E_z^{ne} - E_z^{nw}) - S_s(E_z^{se} - E_z^{sw}) + (S_n - S_s)(E_z^{*e} - E_z^{*w}) \Big]$$

Jump conditions b/w $**$ and $\{n*, s*, *e, *w\}$

$$E_z^{**} = E_z^{n*} - S_n(B_x^{n*} - B_x^{**})$$
$$E_z^{**} = E_z^{s*} - S_s(B_x^{s*} - B_x^{**}) \qquad \text{4 equations}$$
$$E_z^{**} = E_z^{*e} + S_e(B_y^{*e} - B_y^{**}) \qquad \text{1 unknown}$$
$$E_z^{**} = E_z^{*w} + S_w(B_y^{*w} - B_y^{**})$$

2-D Riemann problem

Over-determined, least-squares solution (Vides et al.)

$$E_z^{**} = \frac{1}{4}(E_z^{n*} + E_z^{s*} + E_z^{*e} + E_z^{*w}) - \frac{1}{4}S_n(B_x^{n*} - B_x^{**}) - \frac{1}{4}S_s(B_x^{s*} - B_x^{**}) + \frac{1}{4}S_e(B_y^{*e} - B_y^{**}) + \frac{1}{4}S_w(B_y^{*w} - B_y^{**})$$

Consistency with 1-D solver

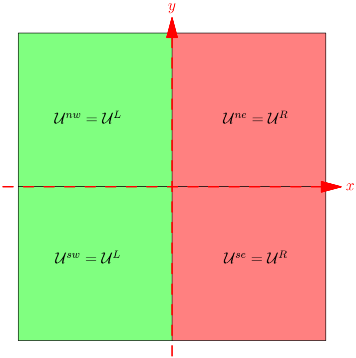
If

$$\mathcal{U}^{nw} = \mathcal{U}^{sw} = \mathcal{U}^L$$

$$\mathcal{U}^{ne} = \mathcal{U}^{se} = \mathcal{U}^R$$

then

$$E_z^{**} = \hat{E}_z(\mathcal{U}^L, \mathcal{U}^R) = \text{1-D HLL}$$



HLLC Riemann solver*

1-D solver

- Slowest and fastest waves S_L, S_R, and contact wave $S_M = u_*$
- Two intermediate states: $\mathcal{U}^{*L}$, $\mathcal{U}^{*R}$
- No unique way to satisfy all jump conditions: Gurski (2004), Li (2005)
- Common value of magnetic field $\boldsymbol{B}^{*L} = \boldsymbol{B}^{*R}$
- Common electric field $E_z^{*L} = E_z^{*R}$, same as in HLL

2-D solver

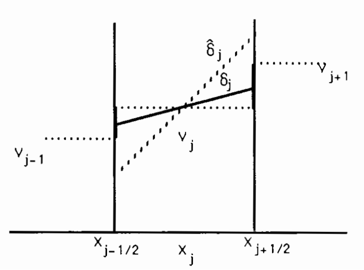
- Electric field estimate E_z^{**} same as HLL
- Consistent with 1-D solver

Limiting procedure: Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

1. Perform RT reconstruction $\implies \boldsymbol{B}(\xi, \eta)$.

Limiting procedure: Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

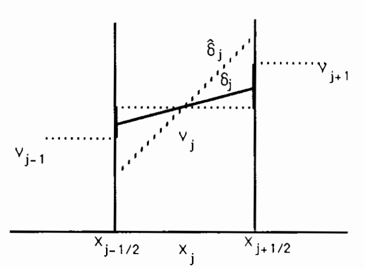
1. Perform RT reconstruction $\implies \boldsymbol{B}(\xi,\eta)$.
2. Apply TVD limiter in characteristic variables to $\{\boldsymbol{U}(\xi,\eta), \boldsymbol{B}(\xi,\eta)\}$.



Limiting procedure: Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

1. Perform RT reconstruction $\Longrightarrow \boldsymbol{B}(\xi,\eta)$.
2. Apply TVD limiter in characteristic variables to $\{\boldsymbol{U}(\xi,\eta), \boldsymbol{B}(\xi,\eta)\}$.
3. Reset cell moments

$$\alpha_{ij} = a_{ij}, \qquad \beta_{ij} = b_{ij}$$



Limiting procedure: Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

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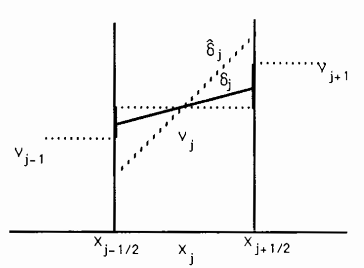
3. Reset cell moments

$$\alpha_{ij} = a_{ij}, \qquad \beta_{ij} = b_{ij}$$

4. On each face, use limited left/right $\boldsymbol{B}(\xi, \eta)$ to limit b_x, b_y

$$b_x(\eta) \leftarrow \mathrm{minmod}\left(b_x(\eta), B_x^L(\tfrac{1}{2}, \eta), B_x^R(-\tfrac{1}{2}, \eta)\right)$$

Do not change mean value on faces: $\oint_{\partial C} \boldsymbol{B} \cdot \boldsymbol{n} = 0$



Limiting procedure: Given $\boldsymbol{U}^{n+1}, b_x^{n+1}, b_y^{n+1}, \alpha^{n+1}, \beta^{n+1}$

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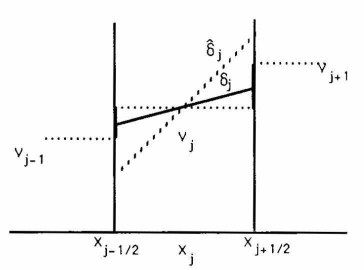
   Do not change mean value on faces: $\oint_{\partial C} \boldsymbol{B} \cdot \boldsymbol{n} = 0$
5. Restore divergence-free property using divergence-free-reconstruction



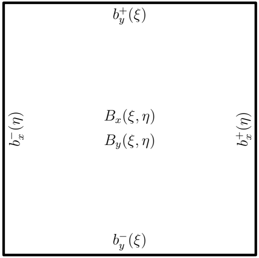
Divergence-free reconstruction[2]: $(b_x^\pm, b_y^\pm, *) \to (B_x, B_y)$

For each cell, find $\boldsymbol{B}(\xi, \eta)$ such that

$$B_x(\pm\tfrac{1}{2}, \eta) = b_x^\pm(\eta), \quad \forall \eta \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

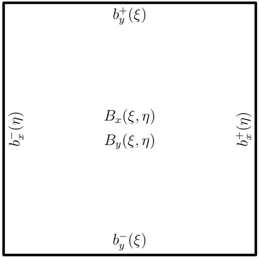
$$B_y(\xi, \pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \forall \xi \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

$$\nabla\cdot\boldsymbol{B}(\xi, \eta) = 0, \qquad \forall (\xi, \eta) \in [-\tfrac{1}{2}, \tfrac{1}{2}]\times[-\tfrac{1}{2}, \tfrac{1}{2}]$$



[2] Hazra et al., JCP, Vol. 394, 2019

Divergence-free reconstruction[2]: $(b_x^\pm, b_y^\pm, *) \to (B_x, B_y)$



For each cell, find $\boldsymbol{B}(\xi,\eta)$ such that

$$B_x(\pm\tfrac{1}{2},\eta) = b_x^\pm(\eta), \quad \forall \eta \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

$$B_y(\xi,\pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \forall \xi \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

$$\nabla\cdot\boldsymbol{B}(\xi,\eta) = 0, \qquad \forall (\xi,\eta) \in [-\tfrac{1}{2},\tfrac{1}{2}]\times[-\tfrac{1}{2},\tfrac{1}{2}]$$

We look for $\boldsymbol{B}$ in (Brezzi & Fortin, Section III.3.2)

$$\mathrm{BDM}(k) = \mathbb{P}_k^2 \oplus \nabla\times(x^{k+1}y) \oplus \nabla\times(xy^{k+1}) \supset \mathbb{P}_k^2$$

[2] Hazra et al., JCP, Vol. 394, 2019

Divergence-free reconstruction[2]: $(b_x^\pm, b_y^\pm, *) \to (B_x, B_y)$

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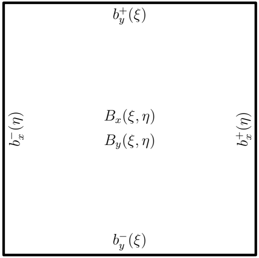
$$B_y(\xi, \pm\tfrac{1}{2}) = b_y^\pm(\xi), \quad \forall \xi \in [-\tfrac{1}{2}, +\tfrac{1}{2}]$$

$$\nabla \cdot \boldsymbol{B}(\xi, \eta) = 0, \qquad \forall (\xi, \eta) \in [-\tfrac{1}{2}, \tfrac{1}{2}] \times [-\tfrac{1}{2}, \tfrac{1}{2}]$$



We look for $\boldsymbol{B}$ in (Brezzi & Fortin, Section III.3.2)

$$\mathrm{BDM}(k) = \mathbb{P}_k^2 \oplus \nabla \times (x^{k+1} y) \oplus \nabla \times (x y^{k+1}) \supset \mathbb{P}_k^2$$

- For $k = 0, 1, 2$, we can solve the above problem
- For $k \geq 3$, we need additional information
  - $k = 3$: $b_{10} - a_{01} = \omega_1 = \nabla \times \boldsymbol{B}(0, 0)$
  - $k = 4$: ω_1, $b_{20} - a_{11} = \omega_2 \approx \frac{\partial}{\partial x} \nabla \times \boldsymbol{B}$, $b_{11} - a_{02} = \omega_3 \approx \frac{\partial}{\partial y} \nabla \times \boldsymbol{B}$
  - ω_1, etc. are known from α, β

[2] Hazra et al., JCP, Vol. 394, 2019

Divergence-free reconstruction

- Preserves mean value of $\boldsymbol{B} \cdot \boldsymbol{n}$ on the faces

$$\implies \quad \oint_{\partial C} \boldsymbol{B} \cdot \boldsymbol{n} = 0$$

- Does not preserve mean value of $\boldsymbol{B}$ in the cells
- Fundamental principle: magnetic flux conservation across surfaces

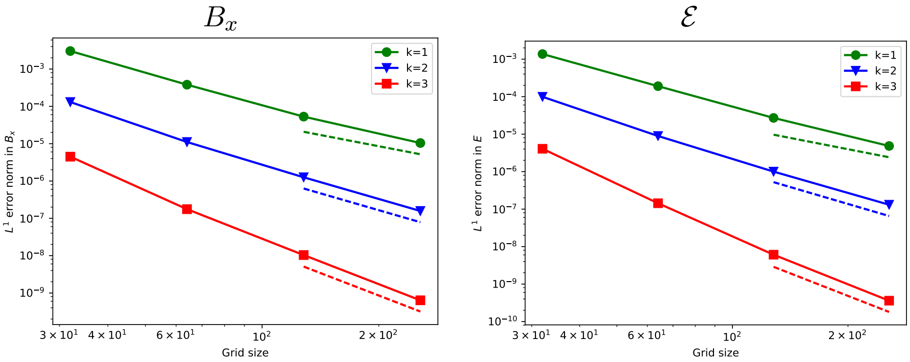
$$\frac{\mathsf{d}}{\mathsf{d}t} \int_S \boldsymbol{B} \cdot \boldsymbol{n} \mathsf{d}s = - \oint_{\partial S} \boldsymbol{E} \cdot \mathsf{d}\boldsymbol{l}$$

Algorithm 1: Constraint preserving scheme for ideal compressible MHD

Allocate memory for all variables;
Set initial condition for $(\boldsymbol{U}, b_x, b_y, \alpha, \beta)$;
Loop over cells and reconstruct (B_x, B_y);
Set time counter $t = 0$;
while $t < T$ **do**
    Copy current solution into old solution;
    Compute time step Δt;
    for *each RK stage* **do**
        Loop over vertices and compute vertex flux;
        Loop over faces and compute all face integrals;
        Loop over cells and compute all cell integrals;
        Update solution $(b_x, b_y, \alpha, \beta)$ to next stage;
        Loop over cells and do RT reconstruction $(b_x, b_y, \alpha, \beta) \to \boldsymbol{B}$;
        Loop over cells and apply limiter on $\boldsymbol{U}, \boldsymbol{B}$;
        Loop over faces and limit solution b_x, b_y;
        Loop over faces and perform div-free reconstruction;
        Apply positivity limiter;
    end
    $t = t + \Delta t$;
end

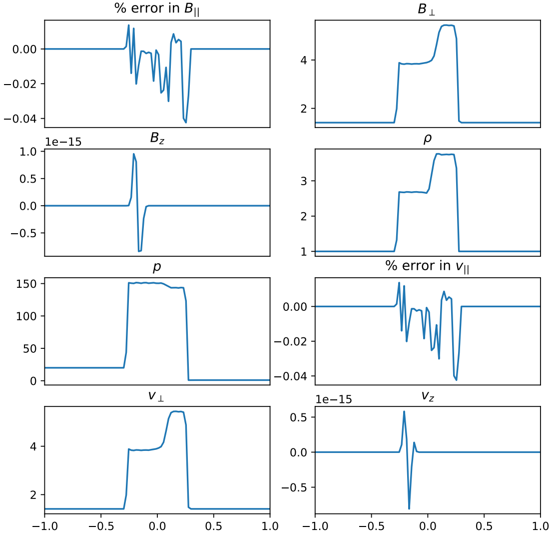
Numerical Results

Smooth vortex



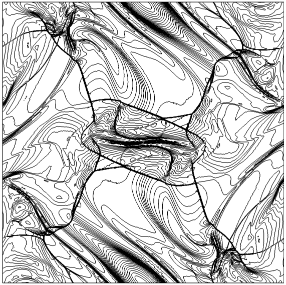
$$\text{Error} = O(h^{k+1})$$

Rotated shock tube: $k = 1$, 128 cells, HLL

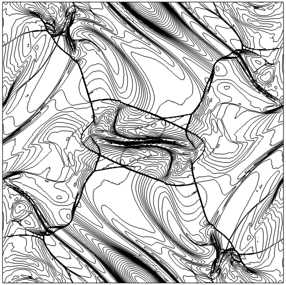


Orszag-Tang test

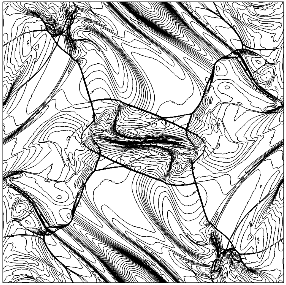
Density, $t = 0.5$, $k = 3$, 512×512 cells, TVD limiter



LxF



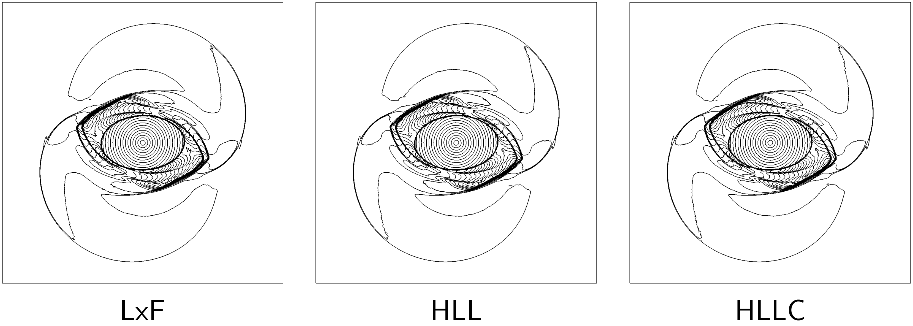
HLL



HLLC

Rotor test

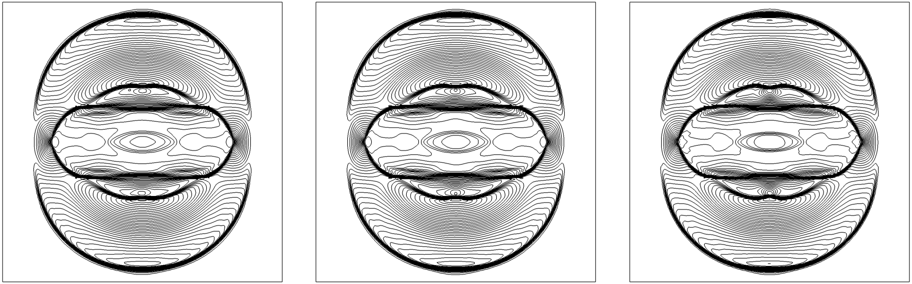
Mach, $t = 0.15$, $k = 3$, 512×512 cells, TVD limiter



Blast wave: 200×200 cells, TVD limiter

$$\rho = 1, \quad \boldsymbol{v} = (\ 0,\ 0,\ 0), \quad \mathfrak{B} = \frac{1}{\sqrt{4\pi}}(100,\ 0,\ 0), \quad p = \begin{cases} 1000 & r < 0.1 \\ 0.1 & r > 0.1 \end{cases}$$

Contours of $\frac{1}{2}(B_x^2 + B_y^2)$



LxF HLL HLLC

Summary

- Solve the conservation form of PDE
- Div-free DG scheme using RT basis for $\boldsymbol{B}$
- Multi-D Riemann solvers essential
  - consistency with 1-d solver is not automatic; ok for HLL (2-wave) and HLLC (3-wave); what about HLLD (5-wave) ?
- Div-free limiting needs to ensure strong div-free condition
  - Reconstruction of $\boldsymbol{B}$ using div$=0$ and curl$=$given
- Limiters are still major obstacle for high order
  - WENO-type ideas
  - sub-cell limiter
  - Machine learning ideas (Ray & Hesthaven)
- No proof of positivity for div-free scheme
  - Not a fully discontinuous solution
  - Variables are not co-located
  - No proof of positivity of first order div-free scheme
  - Div-free limiter is non-conservative

Thank You

Divergence-free reconstruction: $\mathrm{RT}(1)$

On faces

$$b_x^{\pm}(\eta) = a_0^{\pm} + a_1^{\pm}\eta, \qquad b_y^{\pm}(\eta) = b_0^{\pm} + b_1^{\pm}\xi$$

with

$$\oint_{\partial C} \boldsymbol{B} \cdot \boldsymbol{n} = (a_0^+ - a_0^-)\Delta y + (b_0^+ - b_0^-)\Delta x = 0$$

In the cell

$$B_x(\xi, \eta) = \sum_{i=0}^{2} \sum_{j=0}^{1} a_{ij}\phi_i(\xi)\phi_j(\eta), \qquad B_y(\xi, \eta) = \sum_{i=0}^{1} \sum_{j=0}^{2} b_{ij}\phi_i(\xi)\phi_j(\eta)$$

Divergence-free reconstruction: $\mathrm{RT}(1)$

Matching the cell solution with the face solution

$$
\begin{aligned}
a_{00} - a_{10}/2 + a_{20}/6 &= a_0^- && (1)\\
a_{01} - a_{11}/2 + a_{21}/6 &= a_1^- && (2)\\
a_{00} + a_{10}/2 + a_{20}/6 &= a_0^+ && (3)\\
a_{01} + a_{11}/2 + a_{21}/6 &= a_1^+ && (4)\\
b_{00} - b_{01}/2 + b_{02}/6 &= b_0^- && (5)\\
b_{10} - b_{11}/2 + b_{12}/6 &= b_1^- && (6)\\
b_{00} + b_{01}/2 + b_{02}/6 &= b_0^+ && (7)\\
b_{10} + b_{11}/2 + b_{12}/6 &= b_1^+ && (8)
\end{aligned}
$$

$\nabla \cdot \boldsymbol{B}$ is in $\mathbb{Q}_1$

$$
\begin{aligned}
\mathrm{div}(\boldsymbol{B})\Delta x \Delta y = {} & (a_{10}\Delta y + b_{01}\Delta y) + (2a_{20}\Delta y + b_{11}\Delta x)\xi \\
& + (a_{11}\Delta y + 2b_{02}\Delta x)\eta + (2a_{21}\Delta y + 2b_{12}\Delta x)\xi\eta
\end{aligned}
$$

Divergence-free reconstruction: $\mathrm{RT}(1)$

$\nabla \cdot \boldsymbol{B}(\xi, \eta) = 0$ yields four equations

$$\begin{aligned} a_{10}\Delta y + b_{01}\Delta y &= 0 \qquad (\text{constant}) \quad (9) \\ 2a_{20}\Delta y + b_{11}\Delta x &= 0 \qquad (\xi) \quad (10) \\ a_{11}\Delta y + 2b_{02}\Delta x &= 0 \qquad (\eta) \quad (11) \\ 2a_{21}\Delta y + 2b_{12}\Delta x &= 0 \qquad (\xi\eta) \quad (12) \end{aligned}$$

12 coefficients and 12 equations, but not linearly independent.
To show this, combine the equations in the form
$[(3) - (1)]\Delta y + [(7) - (5)]\Delta x$ which yields

$$a_{10}\Delta y + b_{01}\Delta x = (a_0^+ - a_0^-)\Delta y + (b_0^+ - b_0^-)\Delta x = 0 \quad (13)$$

Equation (9) is contained in the remaining equations.

Only **11 equations** (i.e., (1)-(12) but excluding (9)) for the **12 unknown coefficients**.

Divergence-free reconstruction: $\mathrm{RT}(1)$

We can solve for some of the variables from the above equations

$$\begin{aligned} a_{00} &= \frac{1}{2}(a_0^- + a_0^+) + \tfrac{1}{12}(b_1^+ - b_1^-)\frac{\Delta x}{\Delta y} & b_{00} &= \frac{1}{2}(b_0^- + b_0^+) + \tfrac{1}{12}(a_1^+ - a_1^-)\frac{\Delta y}{\Delta x} \\ a_{10} &= a_0^+ - a_0^- & b_{01} &= b_0^+ - b_0^- \\ a_{11} &= a_1^+ - a_1^- & b_{11} &= b_1^+ - b_1^- \\ a_{20} &= -\frac{1}{2}(b_1^+ - b_1^-)\frac{\Delta x}{\Delta y} & b_{02} &= -\frac{1}{2}(a_1^+ - a_1^-)\frac{\Delta y}{\Delta x} \end{aligned}$$

The remaining unknowns are $a_{01}, a_{21}, b_{10}, b_{21}$ which satisfy the equations

$$a_{01} + \frac{1}{6}a_{21} = \frac{1}{2}(a_1^- + a_1^+) \tag{14}$$

$$b_{10} + \frac{1}{6}b_{12} = \frac{1}{2}(b_1^- + b_1^+) \tag{15}$$

$$a_{21}\Delta y + b_{12}\Delta x = 0 \tag{16}$$

Divergence-free reconstruction: $\mathrm{RT}(1)$

We have four remaining unknowns but only three equations. Hence we have to make additional assumptions or simplificiations in order to solve the problem.

For second order accuracy, it is enough to include $\mathbb{P}_1$ in our approximation space and hence we can set

$$a_{21} = b_{12} = 0$$

The remaining two coefficients are given by

$$a_{01} = \frac{1}{2}(a_1^- + a_1^+), \qquad b_{10} = \frac{1}{2}(b_1^- + b_1^+)$$

This approach can be used at higher degrees; the resulting solution is same as BDM polynomial.