Numerical Methods for Ideal MHD (Plasma)

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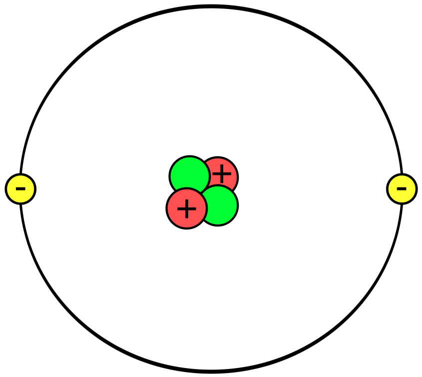
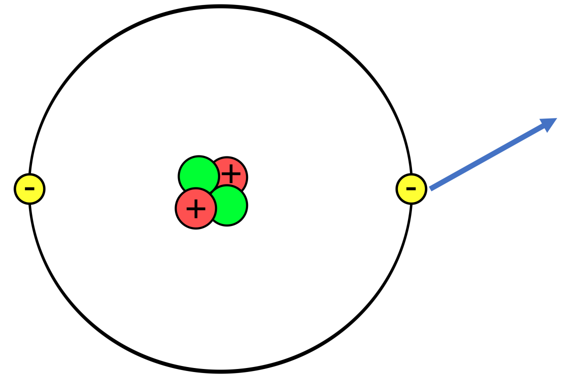
Plasma is everywhere

Neutral helium



High temperatures



Electron escapes

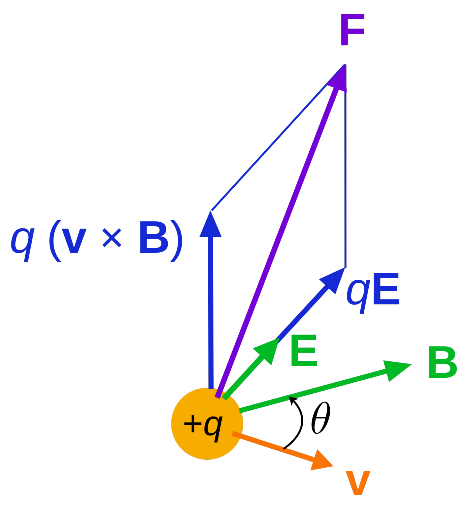
Plasma: completely ionized gas, consisting of freely moving positively charged ions, or nuclei, and negatively charged electrons.

90% of matter in the universe is in plasma state !!!

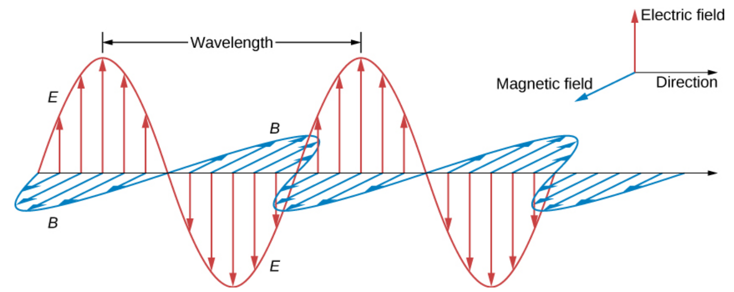
Many applications:

- Terrestrial fusion,
- interior of stars, solar wind,
- earth's magnetic core,
- magnetospheres of stars and planets,
- re-entry of space vehicles into earth's atmosphere,
- lasers

Forces on charges and Maxwell's equation



$\boldsymbol{E}$ = electric field
$\boldsymbol{B}$ = magnetic field



Lorentz force on moving charge q

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Maxwell's Equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = \frac{\tau}{\epsilon_0}, \qquad \tau = \text{charge density}$$

$\boldsymbol{J}$ = electric current, μ_0 = magnetic permeability, ϵ_0 = electrical permittivity

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \text{speed of light}$$

Models for plasma

Kinetic models

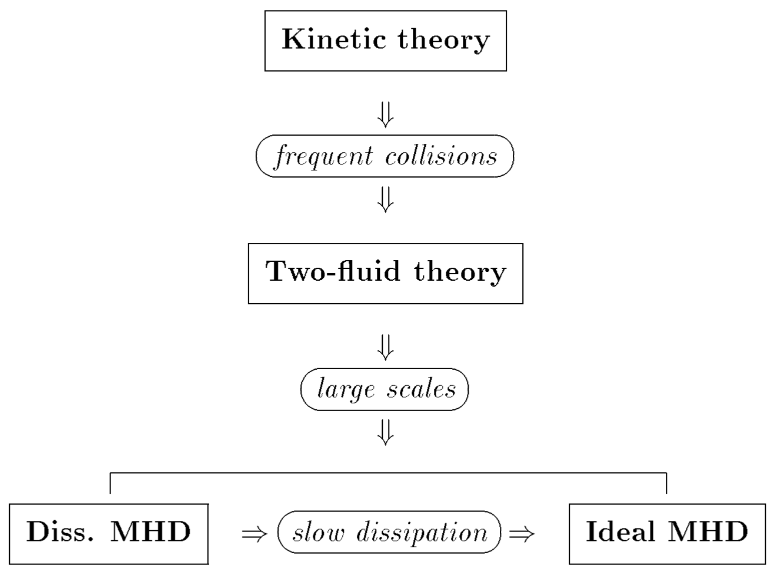
- distribution functions f_α for each species
- Boltzmann or Vlasov equation (collision-less)

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{x}} + \frac{q_\alpha}{m_\alpha}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{col}, \qquad \alpha = i, e$$

+ Maxwell's equations

Continuum models or fluid description

- based on conservation laws
  - Mass conservation
  - Momentum conservation (Newton)
  - Energy conservation
- Two fluid model
- Single fluid model



Two fluid model

Conservation laws for each species

$$
\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha) &= 0 \\
\frac{\partial (\rho_\alpha \boldsymbol{v}_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha + p_\alpha I) &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha (\boldsymbol{E} + \boldsymbol{v}_\alpha \times \boldsymbol{B}), \qquad \alpha = i, e \\
\frac{\partial \mathcal{E}_\alpha}{\partial t} + \nabla \cdot [(\mathcal{E}_\alpha + p_\alpha) \boldsymbol{v}_\alpha] &= \frac{1}{m_\alpha} \rho_\alpha q_\alpha \boldsymbol{E} \cdot \boldsymbol{v}_\alpha
\end{aligned}
$$

where the total energy is

$$
\mathcal{E}_\alpha = \frac{p_\alpha}{\gamma_\alpha - 1} + \frac{1}{2} \rho_\alpha |\boldsymbol{v}_\alpha|^2
$$

These equations are coupled with Maxwell's equations

$$
\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \qquad \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu_0 (\rho_i q_i \boldsymbol{v}_i + \rho_e q_e \boldsymbol{v}_e)
$$

together with the constraints

$$
\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} (\rho_i q_i + \rho_e q_e)
$$

Model contains waves in fluids and electromagnetic waves.

Ideal Magnetohydrodynamics

Single fluid conservation laws

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$
$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + pI) = \boldsymbol{J} \times \boldsymbol{B} + \tau \boldsymbol{E}$$
$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [(\mathcal{E} + p)\boldsymbol{v}] = \boldsymbol{J} \cdot \boldsymbol{E}$$

Perfectly conducting fluid ($\eta = 0$), $\boldsymbol{E}$ vanishes in co-moving frame

$$\eta \boldsymbol{J}' = \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0 \qquad \Longrightarrow \qquad \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$$

Ohm's Law

Non-relativistic velocities

$$|\boldsymbol{v}| \ll c$$

Displacement current is small

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J} \qquad \text{(Ampere's Law)}$$

Electrostatic force $\tau \boldsymbol{E}$ is also small, or quasi-neutral assumption $\tau = 0$.

Ideal MHD

Vector identity $$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla \cdot \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2}|\boldsymbol{B}|^2 I \right) - (\nabla \cdot \boldsymbol{B})\boldsymbol{B}$$

Constraint on magnetic field $$\nabla \cdot \boldsymbol{B} = 0$$

System of non-linear conservation laws ($\mu_0 = 1$)

Ideal MHD equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + PI - \boldsymbol{B} \otimes \boldsymbol{B}) &= 0 \\
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [(\mathcal{E} + P)\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}] &= 0 \\
\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{v}) &= 0
\end{aligned}$$

Conservation law form

$$\frac{\partial U}{\partial t} + div(F) = 0$$

$$P = p + \frac{1}{2}|\boldsymbol{B}|^2, \qquad \mathcal{E} = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{v}|^2 + \frac{1}{2}|\boldsymbol{B}|^2$$

Total pressure — Total energy

Divergence constraint

$$\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{B} = \underbrace{\nabla \cdot \nabla \times}_{=0} (\boldsymbol{v} \times \boldsymbol{B}) = 0$$

Property of the solution
Does not provide an equation

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, 0) = 0 \qquad \Longrightarrow \qquad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0$$

Standard numerical methods may not satisfy this constraint.

Parallel component if $\nabla \cdot B \neq 0$

Lorentz force is perpendicular to $\boldsymbol{B}$

$$\nabla \cdot \left(\boldsymbol{B} \otimes \boldsymbol{B} - \frac{1}{2}|\boldsymbol{B}|^2 I \right) = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + (\nabla \cdot \boldsymbol{B})\boldsymbol{B}$$

Affects accuracy and stability of the scheme.

Solutions must remain positive

$$\rho > 0, \qquad p > 0$$

Positivity of density and pressure requires some discrete div-free condition to be satisfied (Kailiang Wu)

Ideal MHD in one dimension

Divergence constraint $$\frac{\partial B_x}{\partial x} = 0 \quad \Longrightarrow \quad B_x = \text{constant}$$

Conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0$$

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \mathcal{E} \\ B_y \\ B_z \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} \rho u \\ P + \rho u^2 - B_x^2 \\ \rho u v - B_x B_y \\ \rho u w - B_x B_z \\ (\mathcal{E} + P)u - (\boldsymbol{v} \cdot \boldsymbol{B}) B_x \\ u B_y - v B_x \\ u B_z - w B_x \end{bmatrix}$$

Flux jacobian matrix

$$A = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$$ has seven real eigenvalues and eigenvectors

$$u - c_f \le u - c_a \le u - c_s \le u \le u + c_s \le u + c_a \le u + c_f$$

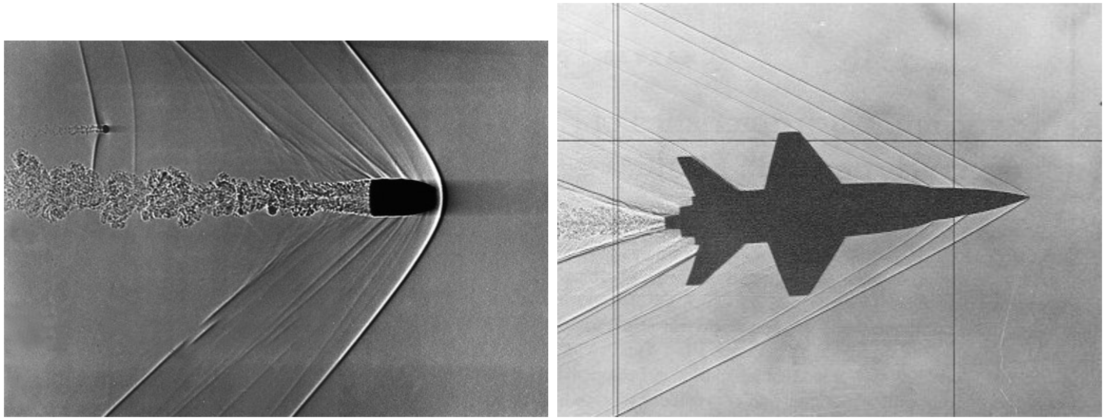
$$c_a = \frac{|B_x|}{\sqrt{\rho}} \qquad a = \sqrt{\frac{\gamma p}{\rho}} \qquad c_{f/s} = \sqrt{\frac{1}{2}\left[a^2 + |\boldsymbol{b}|^2 \pm \sqrt{(a^2 + |\boldsymbol{b}|^2)^2 - 4a^2 b_x^2}\right]} \qquad \boldsymbol{b} = \frac{\boldsymbol{B}}{\sqrt{\rho}}$$

Alfven speed — Sound speed — Fast/slow magnetosonic speeds

Shocks, etc. and weak solutions

Non-linear hyperbolic PDE: Cannot expect smooth solutions even for very smooth initial data.

Shock waves, contact waves, rarefactions can develop which are not smooth solutions.



Flow around a bullet and X-15 model at mach = 3.5

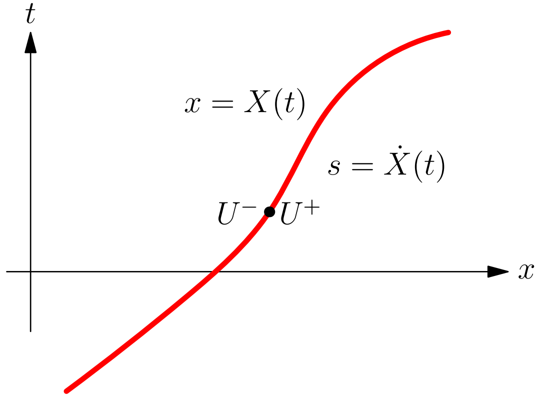
We relax the notion of solution and look for weak solutions

$$\int_0^\infty \int_{\mathbb{R}} \left(\boldsymbol{U} \frac{\partial \phi}{\partial t} + \boldsymbol{F}(\boldsymbol{U}) \frac{\partial \phi}{\partial x} \right) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}} \boldsymbol{U}(x,0)\phi(x,0)\mathrm{d}x = 0, \qquad \forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$$

No derivatives required in this notion of solution !!!

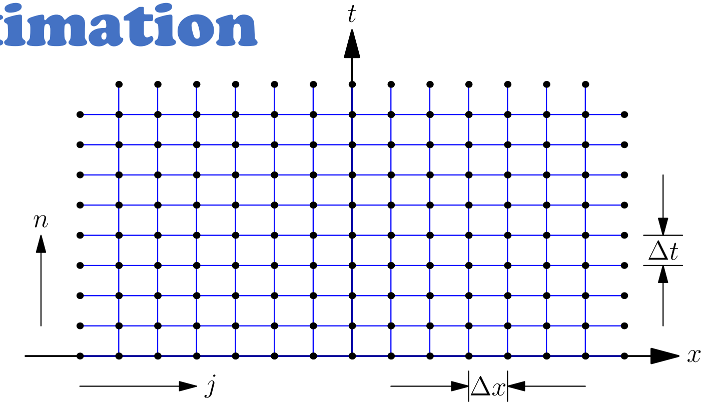
Weak solution: piecewise smooth solution satisfying RH jump conditions

$$\boldsymbol{F}^+ - \boldsymbol{F}^- = s(\boldsymbol{U}^+ - \boldsymbol{U}^-)$$



Numerical approximation

Partition space and time into intervals;
finite dimensional approximation of
unknown solution



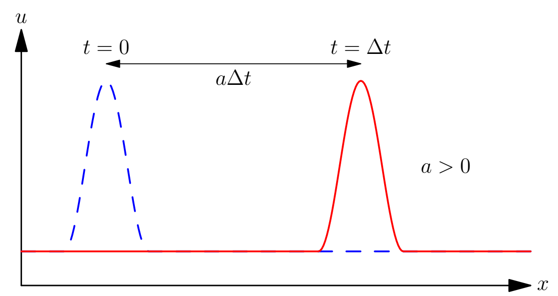
Central difference approximation

$$\frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t} + \frac{\boldsymbol{F}_{j+1} - \boldsymbol{F}_{j-1}}{\Delta x} = 0$$

Fails due to oscillatory solutions, loss of positivity, etc. !!!

Linear advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

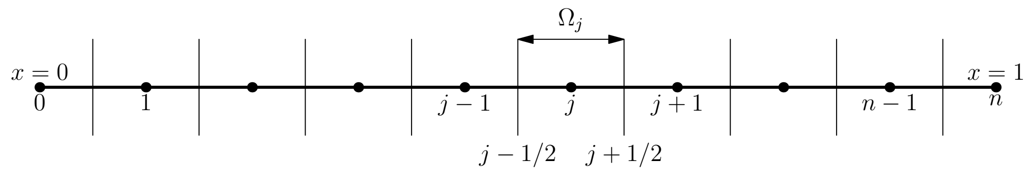


$$a > 0: \qquad \frac{\mathrm{d}u_j}{\mathrm{d}t} + a\frac{u_j - u_{j-1}}{\Delta x} = 0 \qquad \text{(backward difference)}$$

$$a < 0: \qquad \frac{\mathrm{d}u_j}{\mathrm{d}t} + a\frac{u_{j+1} - u_j}{\Delta x} = 0 \qquad \text{(forward difference)}$$

Stencil must be tailored to the waves in the problem !!!

Finite volume method

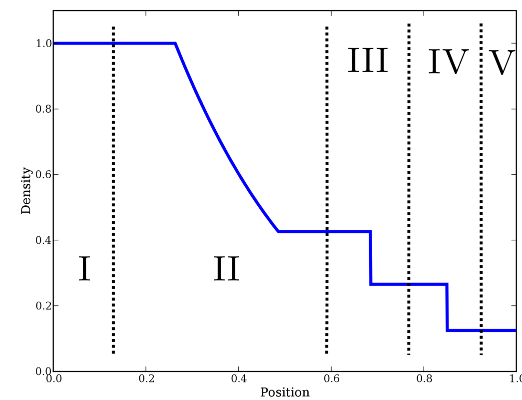


Weak solution: Satisfy conservation law on each finite volume

$$\Delta x \frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t} + \boldsymbol{F}_{j+\frac{1}{2}} - \boldsymbol{F}_{j-\frac{1}{2}} = 0$$

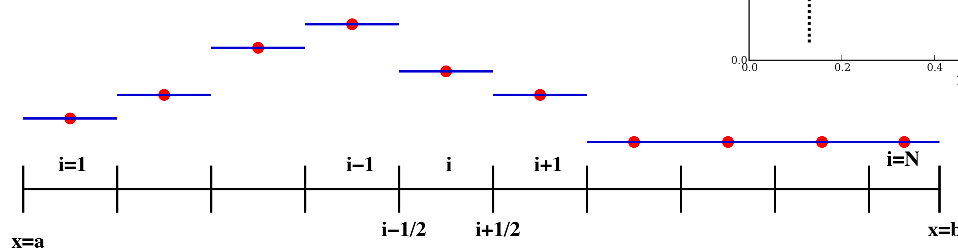
Basic unknown:
mean value in each cell

$$\boldsymbol{U}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \boldsymbol{U}(x, t_n) \mathrm{d}x$$



What is the flux ?

Riemann problem at each cell face

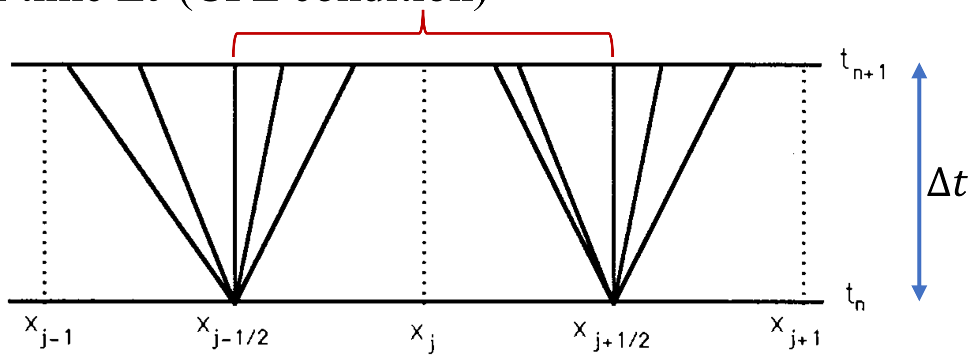


$$\boldsymbol{U}(x, t_n) = \begin{cases} \boldsymbol{U}_j^n & x < x_{j+\frac{1}{2}} \\ \boldsymbol{U}_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases} \Rightarrow \boldsymbol{U}(x, t) = \boldsymbol{U}_R\left(\frac{x - x_{i+\frac{1}{2}}}{t - t_n}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^{n+1}\right), \qquad t > t_n$$

Self-similar solution of RP

Finite volume method

Evolve waves for small time Δt (CFL condition)



Average solution at new time level

$$\boldsymbol{U}_j^{n+1} = \frac{1}{\Delta x}\left[\int_{x_{j-\frac{1}{2}}}^{x_j} \boldsymbol{U}_R\left(\frac{x - x_{j-\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n\right) \mathrm{d}x + \int_{x_j}^{x_{j+\frac{1}{2}}} \boldsymbol{U}_R\left(\frac{x - x_{j+\frac{1}{2}}}{\Delta t}; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n\right) \mathrm{d}x\right]$$

Finite volume form

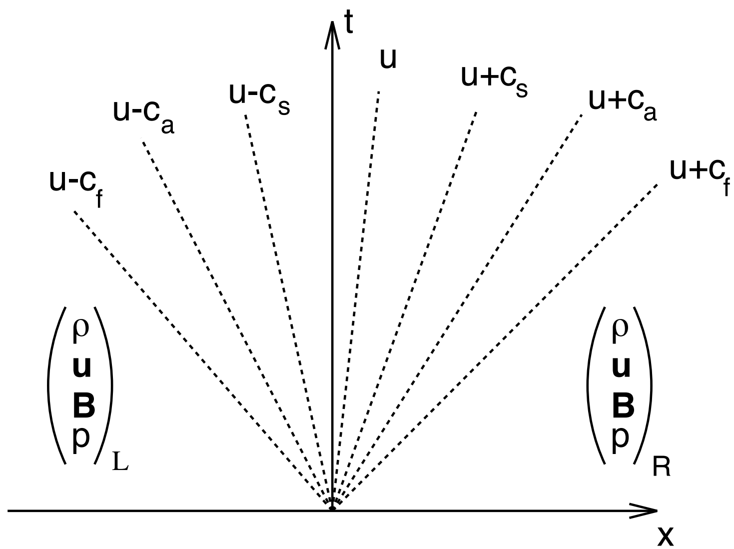
$$\boldsymbol{U}_j^{n+1} = \boldsymbol{U}_j^n - \frac{\Delta t}{\Delta x}[\boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)) - \boldsymbol{F}(\boldsymbol{U}_R(0; \boldsymbol{U}_{j-1}^n, \boldsymbol{U}_j^n))]$$

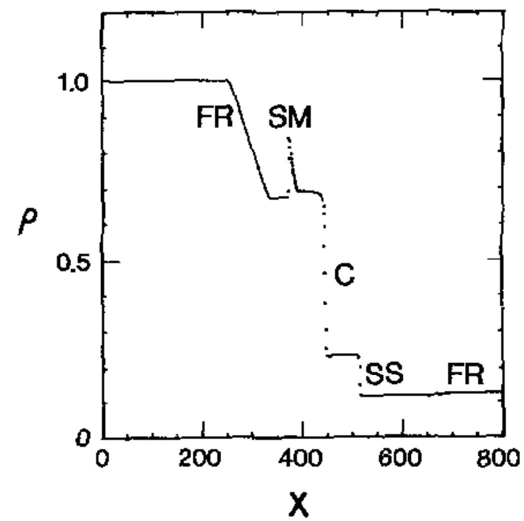
RP→Evolve→Average: Godunov finite volume scheme

MHD Riemann problem

7 waves

$u \pm c_f$	fast shock/rarefaction wave	Neither genuinely non-linear nor linearly degenerate
$u \pm c_a$	rotational discontinuity	Linearly degenerate
$u \pm c_s$	slow shock/rarefaction wave	Neither genuinely non-linear nor linearly degenerate
u	contact discontinuity	Linearly degenerate





Brio-Wu

$c_s \leq c_a \leq c_f$: Wave speeds can coincide → non-strictly hyperbolic

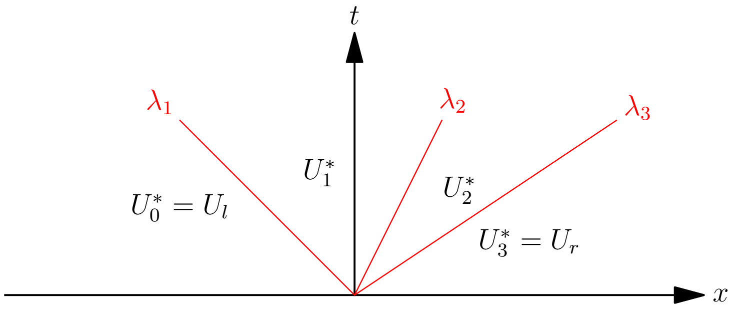
Non-regular waves: compound waves, over-compressive intermediate shocks possible

Riemann solution is not always unique

Approximate Riemann solver: Roe

Idea of P. L. Roe: replace non-linear with linear conservation law

$$\frac{\partial \boldsymbol{U}}{\partial t} + A(\boldsymbol{U}_j^n, \boldsymbol{U}_{j+1}^n)\frac{\partial \boldsymbol{U}}{\partial x} = 0$$



Simple waves

1. $A(\boldsymbol{U}, \boldsymbol{U}) = A(\boldsymbol{U})$
2. $A(\boldsymbol{U}_L, \boldsymbol{U}_R)$ has all real eigenvalues and full set of eigenvectors.
3. $A(\boldsymbol{U}_L, \boldsymbol{U}_R)(\boldsymbol{U}_R - \boldsymbol{U}_L) = \boldsymbol{F}(\boldsymbol{U}_R) - \boldsymbol{F}(\boldsymbol{U}_L)$

Exactly solve the linear problem to estimate flux

$$\boldsymbol{F}_{j+\frac{1}{2}} = \underbrace{\frac{1}{2}(\boldsymbol{F}_j + \boldsymbol{F}_{j+1})}_{\text{Central}} - \underbrace{\frac{1}{2}R_{j+\frac{1}{2}}|\Lambda_{j+\frac{1}{2}}|L_{j+\frac{1}{2}}(\boldsymbol{U}_{j+1} - \boldsymbol{U}_j)}_{\text{Upwinding, dissipative}}$$

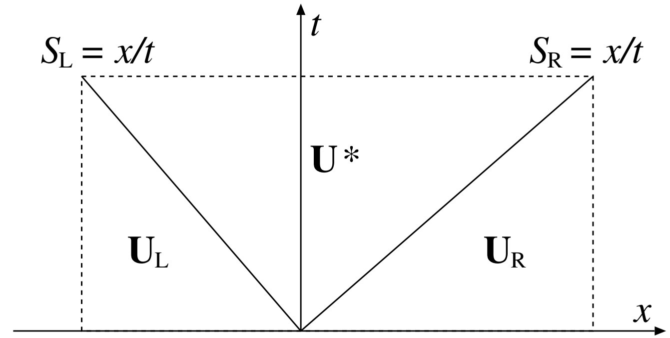
Isolated discontinuities captured exactly in 1-D

For MHD

- ❑ Roe matrix for $\gamma = 2$ by (Brio & Wu)
- ❑ general case by (Cargo & Gallice, (1997))

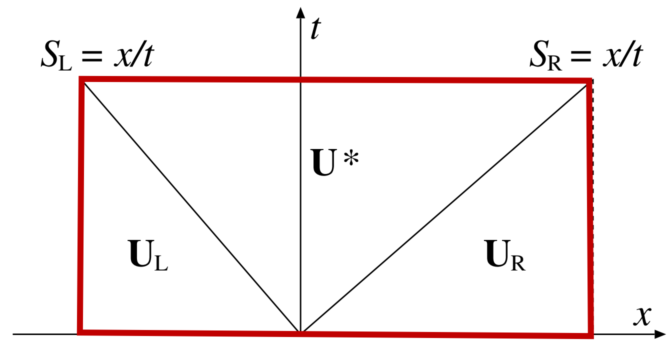
Approximate Riemann solver: HLL

Model Riemann solution with slowest and fastest wave only (Harten, Lax, van Leer)



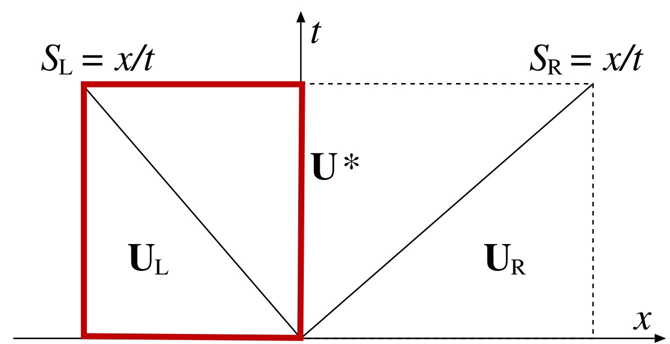
Intermediate state

$$\boldsymbol{U}_* = \frac{S_R \boldsymbol{U}_R - S_L \boldsymbol{U}_L - (\boldsymbol{F}_R - \boldsymbol{F}_L)}{S_R - S_L}$$



Numerical flux

$$\boldsymbol{F}_* = \frac{S_R \boldsymbol{F}_L - S_L \boldsymbol{F}_R + S_L S_R (\boldsymbol{U}_R - \boldsymbol{U}_L)}{S_R - S_L}$$

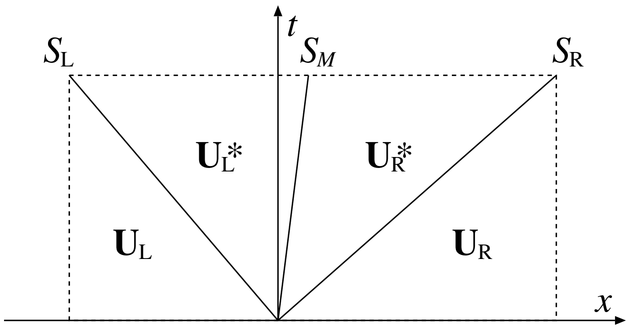


Very simple for any conservation law
Only need estimates of wave speeds: S_L, S_R

Contact wave not included.
Scheme diffuses contacts

Multi-state Riemann solver

HLLC: include contact wave
(Toro, Spruce, Spears)

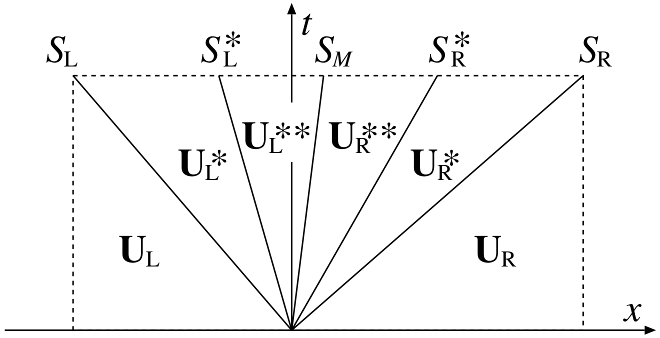


Intermediate states computed by satisfying jump conditions

For MHD: no unique way to determine intermediate states (Gurski (2004), Li (2005))

5-wave version (HLLD) developed by
Miyoshi & Kusano (2005)

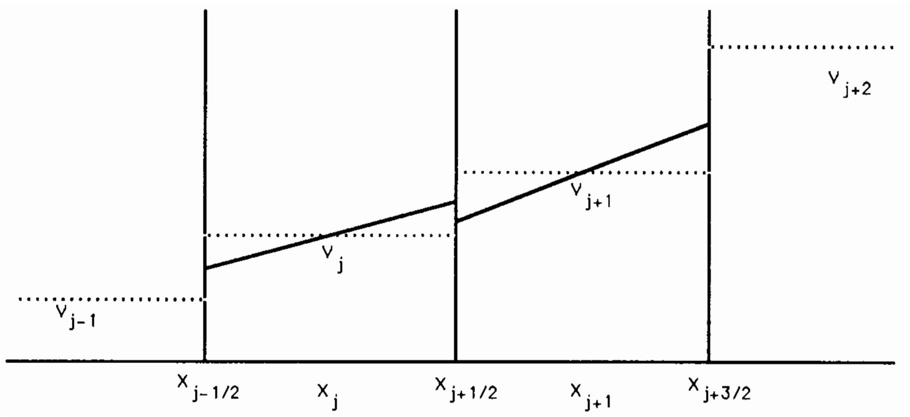
Includes Alfven waves which are also linear
waves.



Higher order finite volume schemes

First order: $|U - U_h| = O(\Delta x)$; Higher order: $|U - U_h| = O(\Delta x^p), p > 1$

From cell averages, reconstruct piecewise linear approximation



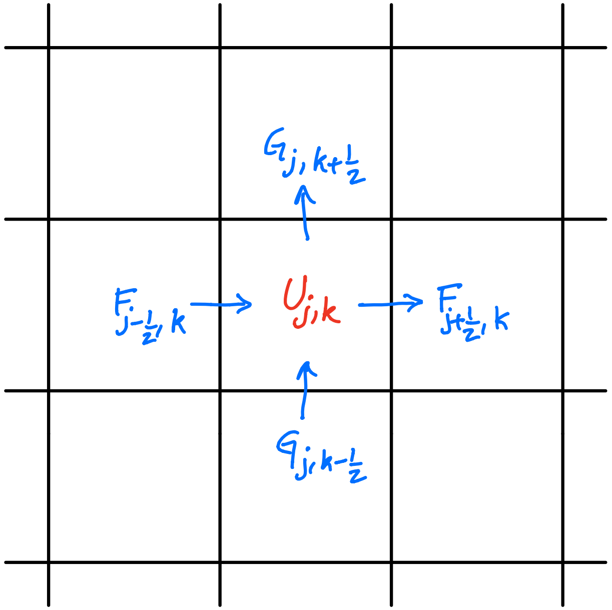
$$\boldsymbol{U}(x) = \boldsymbol{U}_j + \frac{1}{\Delta x}(x - x_j)\boldsymbol{\delta}_j, \qquad \boldsymbol{\delta}_j = minmod(\boldsymbol{U}_j - \boldsymbol{U}_{j-1}, \boldsymbol{U}_{j+1} - \boldsymbol{U}_j)$$

$$minmod(a,b) = \begin{cases} s\min(|a|,|b|) & s = sign(a) = sign(b) \\ 0 & \text{otherwise} \end{cases}$$

Need some non-linear limiter function to control numerical oscillations.

Higher order time accuracy using Runge-Kutta scheme.

Two dimensional case



MHD model in conservation form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} = 0$$

Finite volume method

$$\Delta x \Delta y \frac{\mathrm{d}\boldsymbol{U}_{j,k}}{\mathrm{d}t} + [\boldsymbol{F}_{j+\frac{1}{2},k} - \boldsymbol{F}_{j-\frac{1}{2},k}]\Delta y + [\boldsymbol{G}_{j,k+\frac{1}{2}} - \boldsymbol{G}_{j,k-\frac{1}{2}}]\Delta x = 0$$

Divergence constraint

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

Use 1-D Riemann solver to estimate fluxes

But $(\boldsymbol{B} \cdot \boldsymbol{n})_L \neq (\boldsymbol{B} \cdot \boldsymbol{n})_R$

Two classes of methods → Approximately div-free
→ Exactly div-free

Projection methods

Use a standard scheme, say Godunov FV with Roe solver, to update solution

$$\boldsymbol{B}^n \to \boldsymbol{B}^* \qquad \text{but} \qquad \nabla \cdot \boldsymbol{B}^* \neq 0$$

Correct $\boldsymbol{B}^*$ by removing the divergence (Brackbill & Barnes (1980))

$$\boldsymbol{B}^* = \nabla \times \boldsymbol{A} + \nabla \phi$$

Divergence contained here

Solve for the potential $\quad \Delta \phi = \nabla \cdot \boldsymbol{B}^*$

$$\min_{\boldsymbol{B}} \int (\boldsymbol{B} - \boldsymbol{B}^*)^2 dx$$

such that $\nabla \cdot \boldsymbol{B} = 0$

Solution at next time $\quad \boldsymbol{B}^{n+1} = \boldsymbol{B}^* - \nabla \phi$

$$\mathcal{E}^{n+1} = \mathcal{E}^* - \frac{1}{2}|\boldsymbol{B}^*|^2 + \frac{1}{2}|\boldsymbol{B}^{n+1}|^2$$

Need to solve a matrix problem

Internal energy/temperature changed; add some correction

Conservation of magnetic flux is lost; does not affect results

Hyperbolic divergence cleaning

Mixed GLM

Dedner et al.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + PI - \boldsymbol{B} \otimes \boldsymbol{B}) = 0$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [(\mathcal{E} + P)\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}] = 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{v}) + \nabla \psi = 0$$

$$\frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \boldsymbol{B} = -\frac{c_h^2}{c_p^2}\psi$$

Generalized Lagrange multiplier

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{c_h^2}{c_p^2}\frac{\partial \psi}{\partial t} = c_h^2 \Delta \psi$$

Damps divergence

Transports divergence

Hyperbolic system: 9 real eigenvalues

$$-c_h, \quad u - c_f, \quad u - c_a, \quad u - c_s, \quad u, \quad u + c_s, \quad u + c_a, \quad u + c_f, \quad +c_h$$

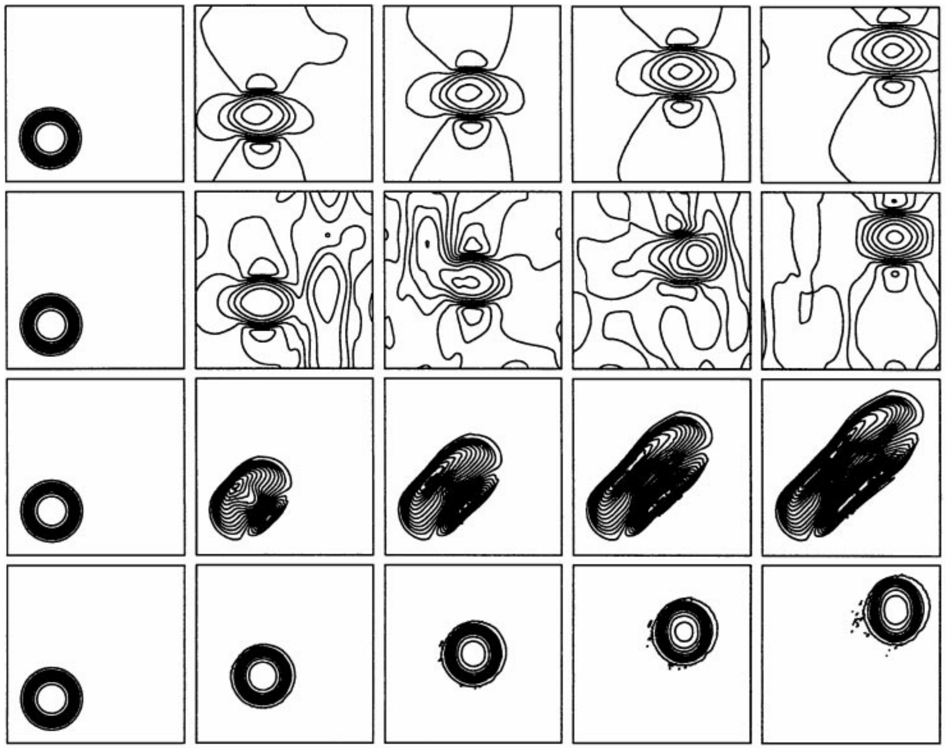
Carry jumps in $\boldsymbol{B} \cdot \boldsymbol{n}$ and ψ

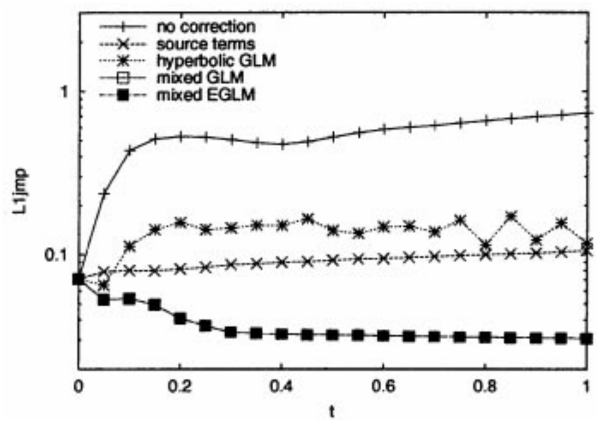
c_h = maximum wave speed in whole domain

$c_p = O(\sqrt{c_h})$: equal hyperbolic and parabolic time scales

Hyperbolic divergence cleaning

Propagation of divergence error





Toth (2000)

Peak in B_x ($\gamma = 5/3$)
Computational domain: $[-0.5, -0.5] \times [1.5, 1.5]$
Boundaries: periodic

ρ	u_x	u_y	u_z	B_x	B_y	B_z	p
1.0	1.0	1.0	0.0	$r(x^2+y^2)/\sqrt{4\pi}$	0.0	$1/\sqrt{4\pi}$	6.0

$r(s) := 4096\, s^4 - 128\, s^2 + 1$

8-wave Riemann solver

In one dimension $(B_x)_L = (B_x)_R$ but not true in multi-D

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \mathcal{E}\\ B_x\\ B_y\\ B_z\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ P+\rho u^2 - B_x^2\\ \rho uv - B_x B_y\\ \rho uw - B_x B_z\\ (\mathcal{E}+P)u - (\boldsymbol{v}\cdot\boldsymbol{B})B_x\\ 0\\ uB_y - vB_x\\ uB_z - wB_x\end{bmatrix} = 0$$

There is a zero eigenvalue !!!
Corresponding mode is undamped !!!

Powell modified MHD equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} = -\begin{bmatrix}0\\ \boldsymbol{B}\\ \boldsymbol{v}\cdot\boldsymbol{B}\\ \boldsymbol{B}\end{bmatrix}\nabla\cdot\boldsymbol{B}$$

Galilean invariant

Eigenvalues are

$$u - c_f \le u - c_a \le u - c_s \le u = u \le u + c_s \le u + c_a \le u + c_f$$

New eigenvalue u corresponds to a divergence wave

$$\frac{\partial D}{\partial t} + \boldsymbol{v}\cdot\nabla D = 0, \qquad D := \frac{1}{\rho}\nabla\cdot\boldsymbol{B}$$

Construct Riemann solver, e.g. Roe type

Divergence errors are advected away.

Entropy stable schemes

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}_\alpha}{\partial x_\alpha} = 0$$

Entropy pair: $\eta(U), f_\alpha(U)$

η is strictly convex function

$f_\alpha'(U) = \eta'(U) F_\alpha{}'(U)$

Smooth solutions

$$\frac{\partial \eta}{\partial t} + \frac{\partial f_\alpha}{\partial x_\alpha} = 0$$

Discontinuous solutions

$$\frac{\partial \eta}{\partial t} + \frac{\partial f_\alpha}{\partial x_\alpha} \le 0$$

Symmetrizability $\Leftrightarrow$ Existence of convex entropy (Mock, Godunov)

$$\boldsymbol{U} \to \boldsymbol{V} = \eta'(\boldsymbol{U}) \quad \Longrightarrow \quad \underbrace{A_0}_{SPD} \frac{\partial \boldsymbol{V}}{\partial t} + \underbrace{A_\alpha}_{Sym} \frac{\partial \boldsymbol{V}}{\partial x_\alpha} = 0$$

Entropy variables

For MHD, we have the thermodynamic entropy $s = \ln(p\rho^{-\gamma})$

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \boldsymbol{v}) + (\gamma - 1)\frac{\rho(\boldsymbol{v} \cdot \boldsymbol{B})}{p} \nabla \cdot \boldsymbol{B} = 0$$

Entropy equation follows if $\nabla \cdot \boldsymbol{B} = 0$, may not hold at numerical level !!!

Entropy stable schemes

Also, the change of variables $\boldsymbol{U} \to \boldsymbol{V} = \eta'(\boldsymbol{U})$ fails to symmetrize the MHD model !!!

Godunov showed how to symmetrize conservation laws with an involution constraint

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}_\alpha}{\partial x_\alpha} + \phi'(\boldsymbol{V})^\top \nabla \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{V} \cdot \phi'(\boldsymbol{V}) = \phi(\boldsymbol{V})$$

Entropy equation follows without requiring divergence constraint

$$\frac{\partial \eta}{\partial t} + \frac{\partial f_\alpha}{\partial x_\alpha} = 0, \qquad \eta = -\frac{\rho s}{\gamma - 1}, \qquad f_\alpha = -\frac{\rho s v_\alpha}{\gamma - 1}$$

This modification is identical to Powell's work !!!

Finite volume scheme

$$\frac{\mathrm{d}\boldsymbol{U}_{j,k}}{\mathrm{d}t} + \frac{\boldsymbol{F}_{j+\frac{1}{2},k} - \boldsymbol{F}_{j-\frac{1}{2},k}}{\Delta x} + \frac{\boldsymbol{G}_{j,k+\frac{1}{2}} - \boldsymbol{G}_{j,k-\frac{1}{2}}}{\Delta y} + \phi'(\boldsymbol{V}_{j,k})^\top \nabla_h \cdot \boldsymbol{B}_{j,k} = 0$$

Entropy stable schemes

Entropy conserving scheme (following Tadmor's ideas)

$$(\boldsymbol{V}_{j+1,k} - \boldsymbol{V}_{j,k}) \cdot \boldsymbol{F}_{j+\frac{1}{2},k} = f^*_{j+1,k} - f^*_{j,k} - (\phi_{j+1,k} - \phi_{j,k})(\bar{B}_x)_{j+\frac{1}{2},k}$$

$$f^* = \boldsymbol{V} \cdot \boldsymbol{F} + \phi B_x - f$$

Such fluxes for MHD can be constructed (C & Klingenberg (2016), Winters & Gassner (2016))

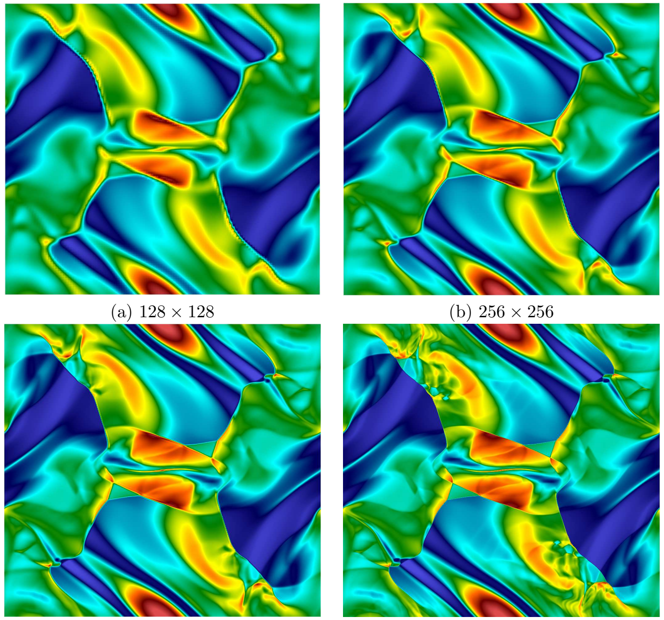
For discontinuous solutions, need to generate entropy at jumps; numerical flux

$$\boldsymbol{F}_{j+\frac{1}{2},k} - \frac{1}{2} R|\Lambda|R^\top (\boldsymbol{V}_{j+1,k} - \boldsymbol{V}_{j,k})$$

Jumps in entropy variables generate entropy; compare with Roe scheme.

Stable scheme without exactly making the divergence zero.

Summation-by-parts DG schemes (Gassner et al.)



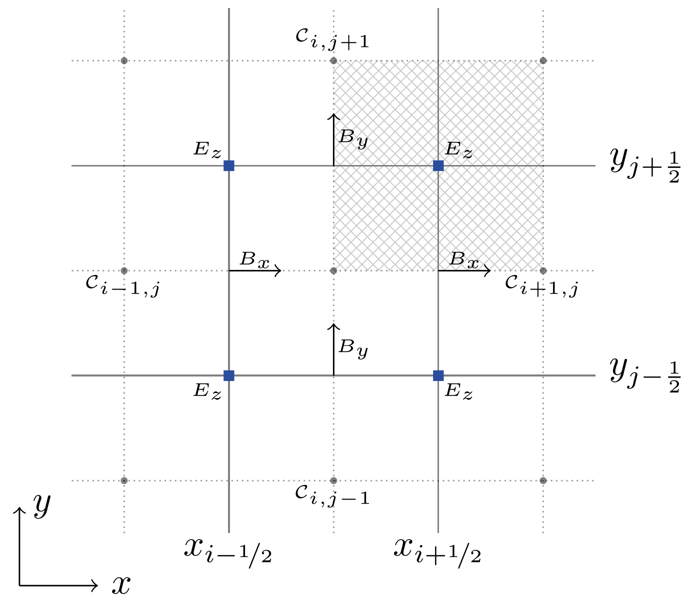
(a) 128×128 (b) 256×256 (c) 512×512 (d) 1024×1024

FIG. 3. *Density at $t = 0.5$ for Orszag-Tang test case on different meshes. The density range is 0.09 to 0.48*

Constrained transport schemes

Staggered storage of variables
(Maxwell: Yee (1967))

Remaining variables
stored at cell center



Store normal component $\boldsymbol{B} \cdot \boldsymbol{n}$ on each face

Induction equation has 1-D form

$$\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} = 0, \qquad \frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0$$

Define finite difference scheme on the faces

$$\frac{(B_x)^{n+1}_{i+\frac{1}{2},j} - (B_x)^{n}_{i+\frac{1}{2},j}}{\Delta t} + \frac{(E_z)^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)^{n}_{i+\frac{1}{2},j-\frac{1}{2}}}{\Delta y} = 0$$

$$\frac{(B_y)^{n+1}_{i,j+\frac{1}{2}} - (B_y)^{n}_{i,j+\frac{1}{2}}}{\Delta t} - \frac{(E_z)^{n}_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)^{n}_{i-\frac{1}{2},j+\frac{1}{2}}}{\Delta x} = 0$$

Constrained transport schemes

Measure divergence at cell center

$$(\nabla_h \cdot \boldsymbol{B})_{i,j} = \frac{(B_x)_{i+\frac{1}{2},j} - (B_x)_{i-\frac{1}{2},j}}{\Delta x} + \frac{(B_y)_{i,j+\frac{1}{2}} - (B_y)_{i,j-\frac{1}{2}}}{\Delta y}$$

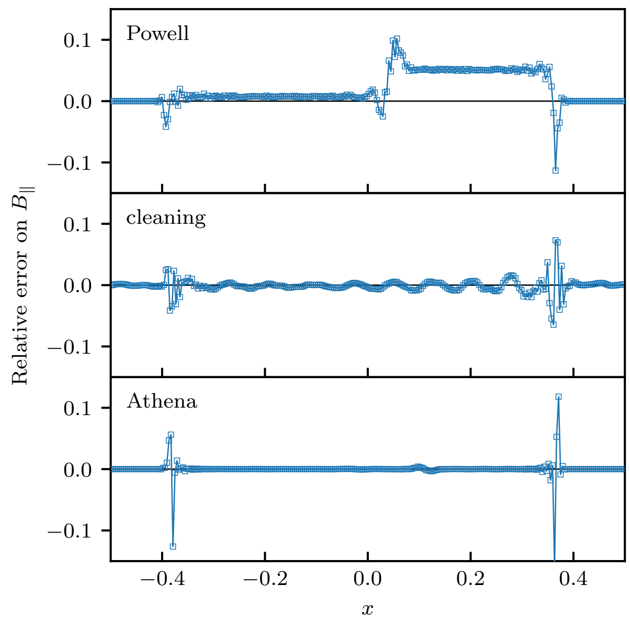
$$(\nabla_h \cdot \boldsymbol{B})^{n+1}_{i,j} = (\nabla_h \cdot \boldsymbol{B})^{n}_{i,j} - \frac{\Delta t}{\Delta x \Delta y}\Big[(E_z)_{i+\frac{1}{2},j+\frac{1}{2}} - (E_z)_{i+\frac{1}{2},j-\frac{1}{2}} - (E_z)_{i-\frac{1}{2},j+\frac{1}{2}} + (E_z)_{i-\frac{1}{2},j-\frac{1}{2}}$$
$$ - (E_z)^n_{i+\frac{1}{2},j+\frac{1}{2}} + (E_z)^n_{i-\frac{1}{2},j+\frac{1}{2}} + (E_z)^n_{i+\frac{1}{2},j-\frac{1}{2}} - (E_z)^n_{i-\frac{1}{2},j-\frac{1}{2}}\Big]$$

All the E_z cancel !!! $\Longrightarrow$ $(\nabla_h \cdot \boldsymbol{B})^{n+1}_{i,j} = (\nabla_h \cdot \boldsymbol{B})^{n}_{i,j}$

Discrete approximation of $\nabla \cdot \boldsymbol{B}$ is kept zero.

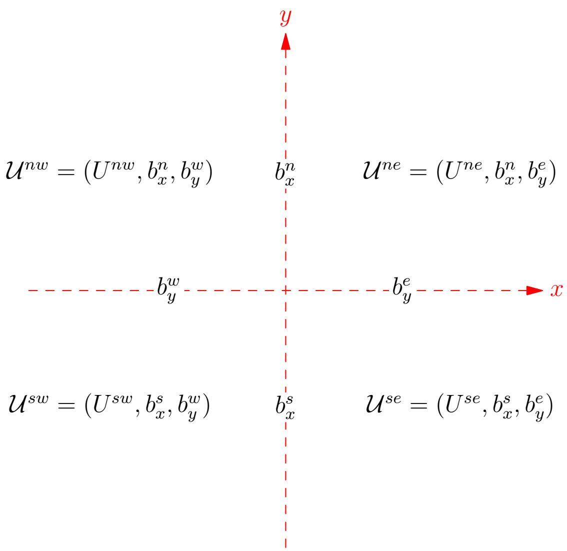
How to estimate the emf at the cell corner ?

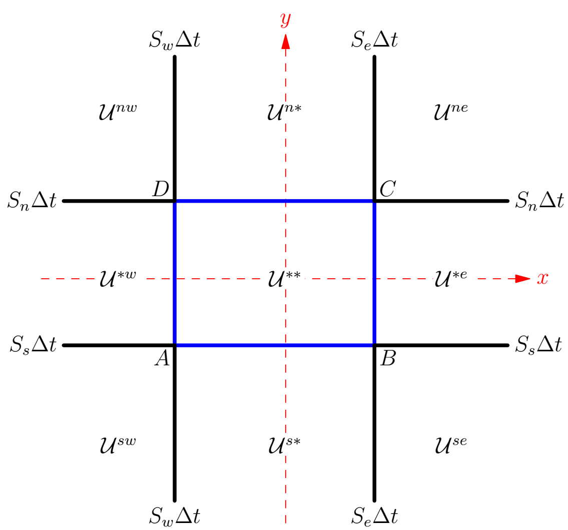
- Interpolation of primary data
- Interpolate solution of 1-D Riemann problems
- Solve 2-D Riemann problem

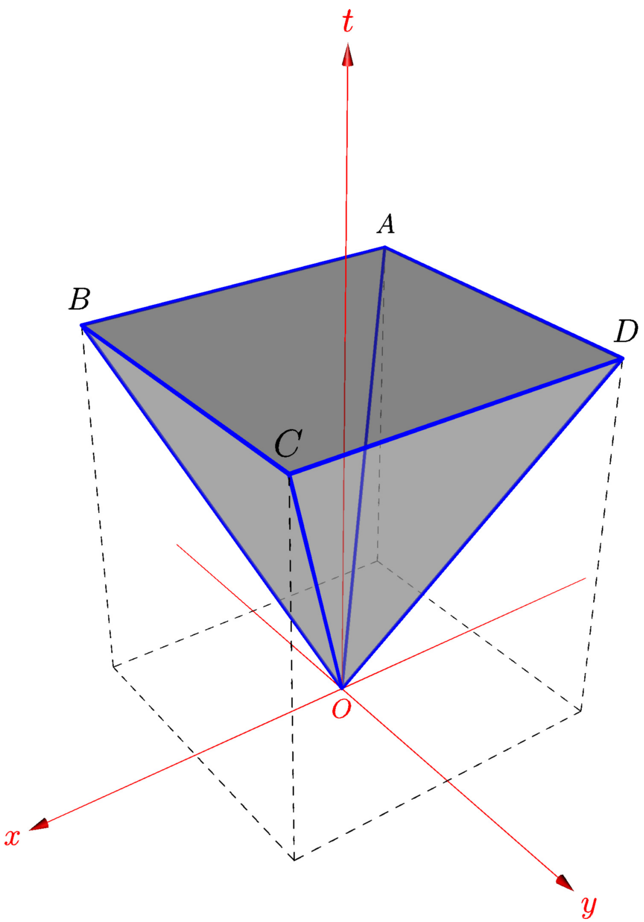


Guillet et al.

2-D Riemann problem

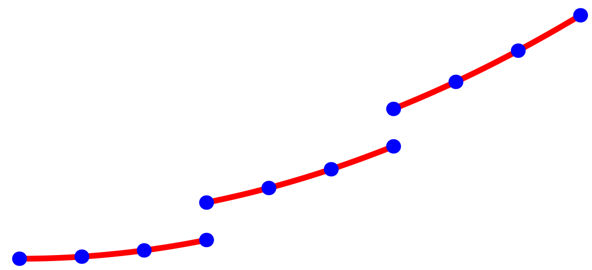






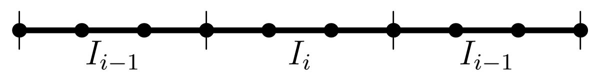
Discontinuous Galerkin Method

Piecewise discontinuous
polynomial approximation



Jumps help to stabilize the method

Compact stencil, good for parallel computing



$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

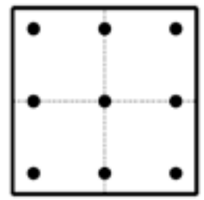
(Semi-discrete DG scheme) Find $u_h(\cdot, t) \in V_h^k$ such that for all $v_h \in V_h^k$

$$\int_{I_i} \frac{\partial u_h}{\partial t} v_h dx - \int_{I_i} f(u_h) \frac{\partial v_h}{\partial x} dx + \hat{f}_{i+\frac{1}{2}}(t) v_h(x^-_{i+\frac{1}{2}}) - \hat{f}_{i-\frac{1}{2}}(t) v_h(x^+_{i-\frac{1}{2}}) = 0 \quad (3.1)$$

Entire polynomial is evolved forward in time
Fluxes across elements obtained from Riemann solvers
Very high order accuracy can be achieved

Discontinuous Galerkin Method

Gauss-Legendre nodes for
Lagrange basis



$(\boldsymbol{B} \cdot \boldsymbol{n})_L \neq (\boldsymbol{B} \cdot \boldsymbol{n})_R$
use 8-wave Riemann solver

Locally divergence-free basis for $\boldsymbol{B}$
(Li et al., Pablo et al.)

- $\nabla \cdot \boldsymbol{B} = 0$ inside each cell.
- But $(\boldsymbol{B} \cdot \boldsymbol{n})_L \neq (\boldsymbol{B} \cdot \boldsymbol{n})_R$
- Use 8-wave Riemann solver or entropy stable fluxes.

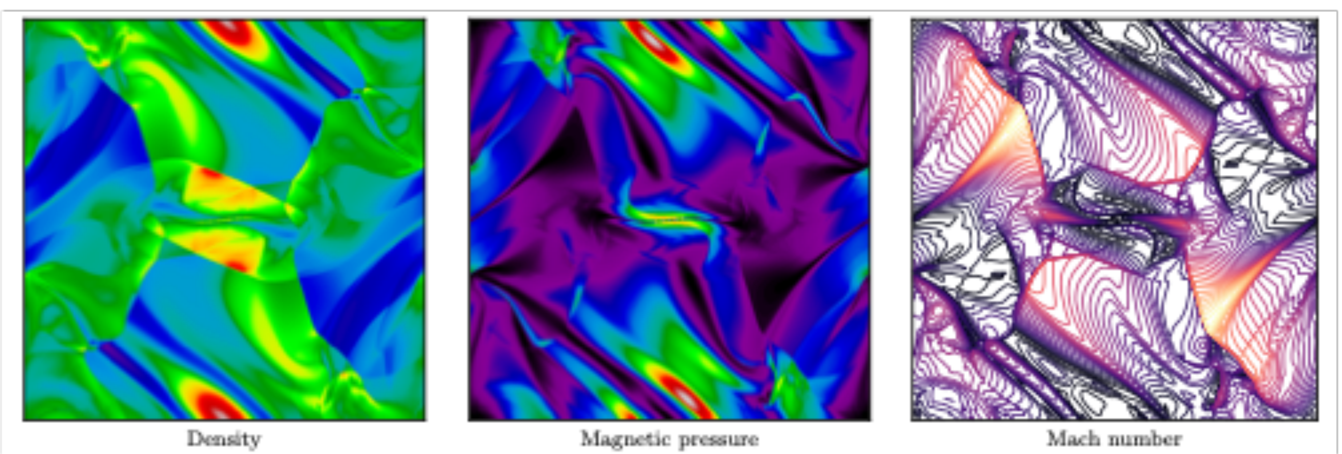
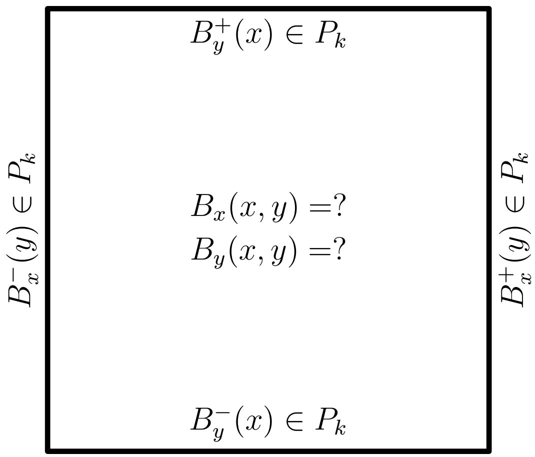


Figure 10. *Orszag–Tang vortex test problem at* $t = 0.5$. The density, pressure and Mach number are shown on a 512^2 grid, computed using the third-order DG scheme with the Powell method.

Guillet et al.

Discontinuous Galerkin Methods

Divergence-free reconstruction (Balsara)



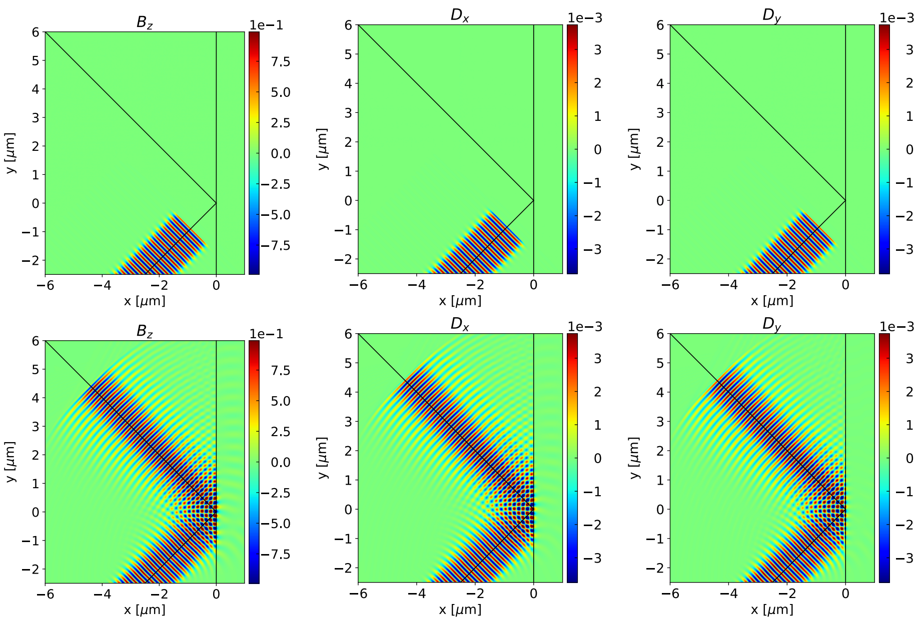
Find B_x, B_y such that

$$\nabla \cdot \boldsymbol{B} = 0$$

$\boldsymbol{B} \cdot \boldsymbol{n}$ agrees with value on face

$\boldsymbol{B}$ is Brezzi-Douglas-Marini polynomial.
Beyond third order (k>2), need information of $\nabla \times \boldsymbol{B}$ to perform reconstruction (Hazra et al.)

Hybrid DG scheme on faces/cells



Total internal reflection of electromagnetic wave by Maxwell's equation

4'th order div-free scheme (Hazra et al. (2019))

Only using H(div) elements

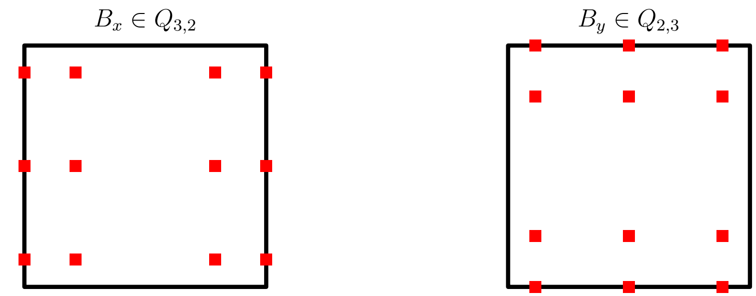
Discontinuous Galerkin Methods

Constraint preserving DG schemes (C, J. Sci. Comp., 2018 for induction equation)

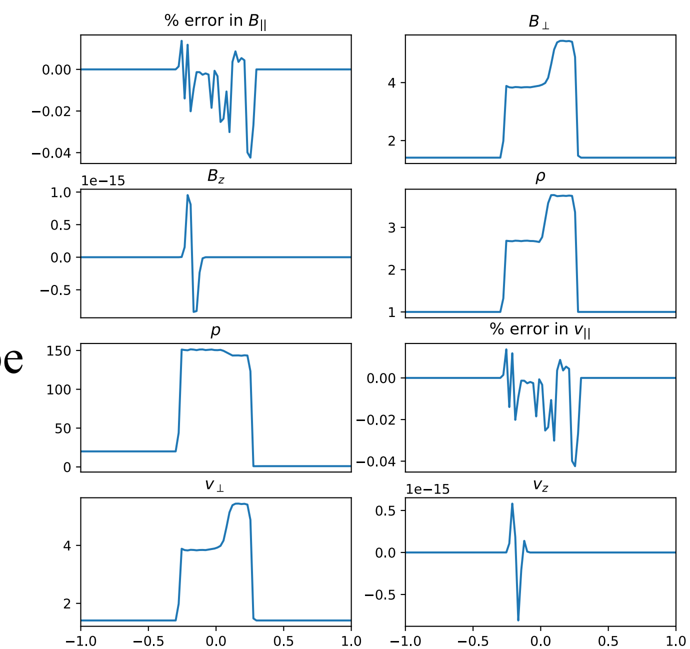
B is Raviart-Thomas polynomial

Hybrid DG scheme on faces and cells; Petrov-Galerkin type

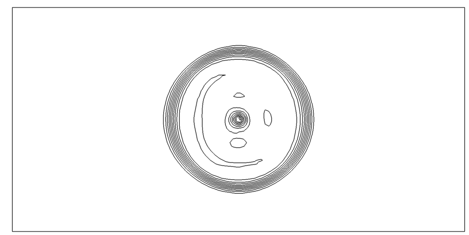
$\nabla \cdot \boldsymbol{B} = 0$ everywhere, $\boldsymbol{B} \cdot \boldsymbol{n}$ cts.



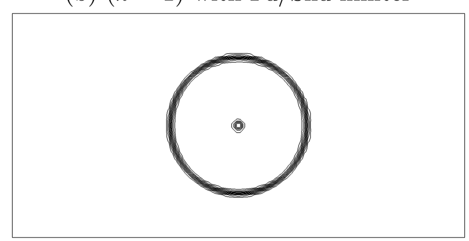
Location of dofs of Raviart-Thomas polynomial for $k = 2$



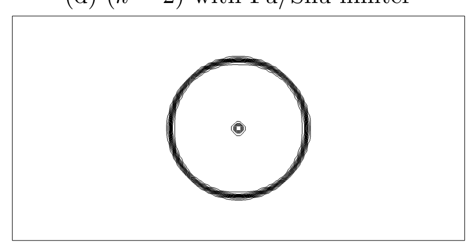
Rotated Shock tube



(b) $(k = 1)$ with Fu/Shu limiter



(d) $(k = 2)$ with Fu/Shu limiter



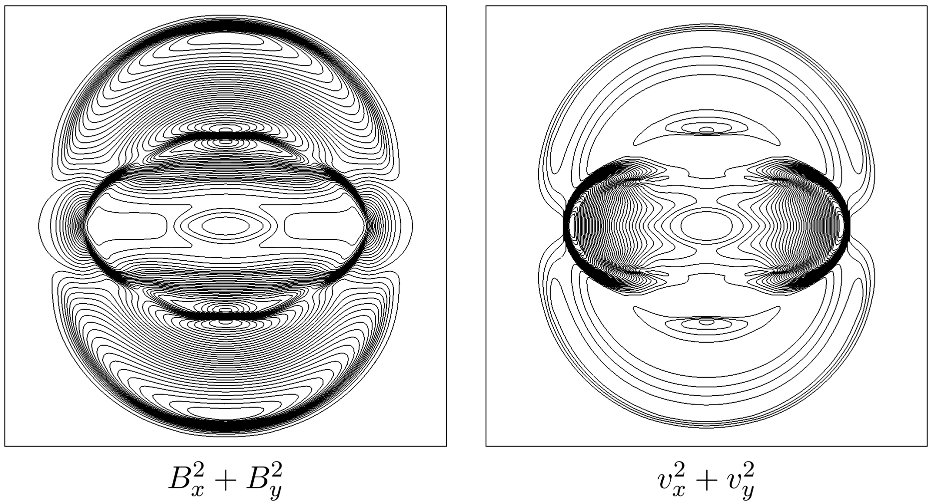
(f) $(k = 3)$ with Fu/Shu limiter

Magnetic loop advection

Positivity property

Blast wave: 200×200 cells

$$\rho = 1, \quad \boldsymbol{v} = (\ 0,\ 0,\ 0), \quad \mathfrak{B} = \frac{1}{\sqrt{4\pi}}(100,\ 0,\ 0), \quad p = \begin{cases} 1000 & r < 0.1 \\ 0.1 & r > 0.1 \end{cases}$$



RTDG div-free scheme

Positivity needs some div-free property to hold (Kailiang Wu)

Locally div-free basis
Godunov's MHD model
➔Positive scheme
(Wu & Shu, 2019)

- ❑ Cannot prove positivity of globally div-free DG scheme.
- ❑ Multi-dimensional stencil due to corner fluxes complicates analysis.

Summary

MHD model poses special challenges in its numerical solution.

Being hyperbolic conservation law, Godunov's approach using approximate Riemann solvers provides a robust strategy.

Ideas based on entropy stability theory and constrained transport lead to useful schemes, even at high order of accuracy, using discontinuous Galerkin method.

Constrained transport approaches are very elegant; DG for high order accuracy.

There are issues in Riemann solver design, especially for 2-D Riemann solvers, which are not well understood.

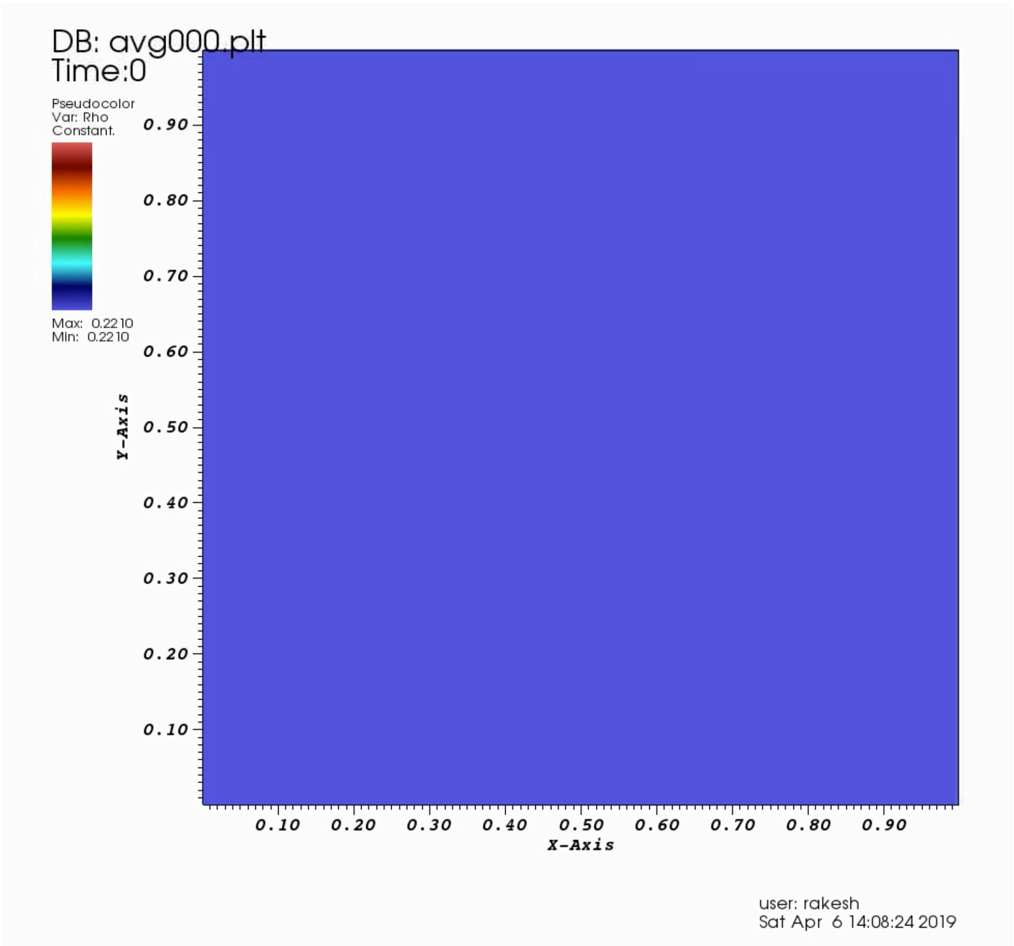
Designing provably positive schemes is difficult due to the need to satisfy a multi-dimensional constraint involving the divergence.

Limiting solution without degrading accuracy still poses challenges. How to decide where/when to limit the solution ? Techniques of machine learning are proving to be usefull (Ray & Hesthaven).

Thank You

Orszag-Tang Vortex

Second order div-free RTDG scheme



Collaborators: Dinshaw Balsara, Thomas Guillet, Arijit Hazra, Christian Klingenberg, Rakesh Kumar, Juan Pablo, Volker Springel.