On the Diffusion of Shape

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Contents

Abstract

The main objective of this thesis is to study the global geometric properties of a manifold embedded in Euclidean space, as it evolves under a stochastic flow of diffeomorphisms. The processes driving the stochastic flows are chosen to be Gaussian processes with stationary increments (in time). The most common class of Gaussian processes with stationary increments is the family of fractional Brownian motions with Hurst parameter $H \in (0, 1)$. This family encompasses a wide variety of processes with applications in the fields of oceanography, finance and telecommunications, to name a few. The fact that these processes possess stationary increments implies that the corresponding *noise* process is a stationary process, and so one can hope to obtain ergodic estimates.

In Part I of the dissertation, we study the evolution of a codimension one manifold embedded in Euclidean space, under an isotropic and volume preserving Brownian flow. In particular we obtain expressions describing the expected rate of growth of the Lipschitz-Killing curvatures, or intrinsic volumes, of the manifold evolving under the flow. These results shed new light on the some of the intriguing growth properties of flows from a global perspective, rather than the local perspective, on which there is much larger literature.

In Part II, we deviate from the setting of standard Brownian flows, whose analysis was primarily based on the Markovian character of the flow, and move to stochastic flows driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. Adopting a pathwise approach, we obtain estimates for the growth of the Hausdorff measure of an m dimensional manifold embedded in $\mathbb{R}^n$.

List of Symbols

M	smooth compact manifold
T_xM	tangent space to the manifold M at $x \in M$
λ_i	i-th Lyapunov exponent
$\mathcal{L}_k(M)$	k-th Lipschitz-Killing curvature of the manifold M
$\mathbb{R}^n$	n-dimensional Euclidean space
Φ_t	stochastic flow of diffeomorphisms
B^H	fractional Brownian motion with Hurst parameter H
TM	tangent bundle to the manifold M
$\nabla_X Y$	covariant derivative of Y in the direction of X
$\mathcal{H}_m(\cdot)$	m-dimensional Hausdorff measure
N_xM	normal cone to the manifold M at $x \in M$
$\{U_\gamma\}_{\gamma\in\mathbb{N}}$	collection of deterministic vector fields defined on $\mathbb{R}^n$
D_H	Malliavin derivative operator
D_H^k	k-th iteration of Malliavin derivative operator
δ_H	divergence operator
$L^p(a,b)$	the space of Lebesgue measurable functions $f:(a,b)\to\mathbb{R}$ such that $\|f\|_{L^p(a,b)} < \infty$
I^α_{a+}	left sided Riemann-Liouville integral of order $\alpha > 0$
I^α_{b-}	right sided Riemann-Liouville integral of order $\alpha > 0$
D^α_{a+}	left sided Riemann-Liouville derivative of order $\alpha > 0$
D^α_{b-}	right sided Riemann-Liouville derivative of order $\alpha > 0$
$C^\lambda(0,T;\mathbb{R}^d)$	the space of λ-Hölder continuous functions taking values from $[0,T]$ to $\mathbb{R}^d$

Chapter 1

Introduction

A subject of recent research activity in stochastic processes has been the study of the dynamics of randomly evolving manifolds under a stochastic flow. An appreciable amount of progress has already been made by studying stochastic flows where explicit calculations are possible. For example, isotropic Brownian flows have been studied in Baxendale and Harris [6], Le Jan [20, 21], and isotropic and volume preserving flows have been studied in Cranston and Le Jan [12].

Baxendale and Harris [6] studied, in detail, the characterization of a Brownian flow using its covariance function. They also established results related to the two point motion of the flow, and studied in detail the various dynamic properties of the tangent vectors and tangent flows.

Since the beginning of the study of stochastic flows, probabilists and geometers have been searching for appropriate parameters to characterize the flow, in particular its dynamic properties. In the last two decades, researchers have shown considerable interest in Lyapunov exponents as a tool to explain the asymptotic behavior of some of the characteristics of the flow, as, for example, in the study of statistical equilibrium and the two point motion of flows in Baxendale [5]. The first step in relating the Lyapunov exponents to stochastic flows was probably made by Carverhill [8, 9]. In [9], Carverhill proved a version of the multiplicative ergodic theorem of Oseledec [31] for stochastic (Brownian) flows of diffeomorphisms of a compact smooth manifold M establishing the existence of numbers $\lambda_1 > \cdots > \lambda_k$, called the Lyapunov exponents, and random subspaces $\{V_i^{x,\omega}\}_{i=1}^k$ of the tangent space T_xM

for any x in the manifold M, such that $V_i^{x,\omega} \subset V_{i+1}^{x,\omega}$ for $1 \leq i \leq k$ and that the i-th Lyapunov exponent is determined by any $v \in V_i^{x,\omega} \backslash V_{i-1}^{x,\omega}$. (The ω in the superscript emphasizes the randomness of the subspaces.)

About the same time Le Jan [20] also proved similar results, including an explicit expression for the Lyapunov exponents of an isotropic Brownian flow defined on the Euclidean space $\mathbb{R}^n$.

It was long believed that the Lyapunov exponents could explain the recurrence of the second fundamental form at any point of a manifold, evolving randomly under a stochastic flow. Cranston [10], however, indicated that an extension to the continuous case of the results obtained in Cranston and Le Jan [11] would show that the intuition employed to explain the recurrence of the second fundamental form using the Lyapunov exponents was not true. The above observation leads one to believe that the finite time behavior of the geometric evolution of the flow may be better studied by some other characteristics of the flow. This lead Cranston and Le Jan in [12] to study the unfolding of the symmetric polynomials of the principle curvatures, including the mean and the Gaussian curvature of an $(n-1)$-dimensional manifold M embedded in $\mathbb{R}^n$, evolving under an isotropic and volume preserving flow on $\mathbb{R}^n$. They obtained an Itô formula for the symmetric polynomials of the principle curvatures and hence deduced that while the vector of all the symmetric polynomials of the principle curvatures is a diffusion, the same was not true for any proper subset of the vector.

In Part I of this thesis, we extend this by looking at the dynamic behavior of the Lipschitz-Killing curvatures of randomly evolving manifolds under isotropic and volume preserving Brownian flows on $\mathbb{R}^n$. The Lipschitz-Killing curvatures $\{\mathcal{L}_k(M)\}_{k=0}^{\dim(M)}$, also known as curvature measures, can be regarded as extensions of intrinsic volumes and hence can also be called generalized volumes. This is a natural but significant extension of Cranston and Le Jan [12], as Lipschitz-Killing curvatures can be represented as the average of the symmetric polynomials of the principle curvatures over the manifold.

Unlike the information furnished by the random filtration obtained by studying Lyapunov exponents, which basically give local information, Lipschitz-Killing curvatures describe the global geometry of randomly evolving manifolds. This is yet another motivation to work with Lipschitz-Killing curvatures. Part I of the thesis tackles this problem by studying the stochastic evolution of the Lipschitz-Killing curvatures of a manifold evolving randomly under an isotropic and volume preserving Brownian flow. The results there, are an extension of the results in [12], and have recently been published in [40].

Our main result in Chapter 4, will be a stochastic evolution equation for the Lipschitz-Killing curvatures of a randomly evolving manifold under an isotropic and volume preserving Brownian flow, an important consequence of which is an explicit, and quite simple, expression for their expected values as a function of time. In simple words, let M_t be the image, under the flow Φ_t, of an $(n-1)$-dimensional compact smooth manifold M, embedded in $\mathbb{R}^n$. Moreover, let $\mathcal{L}_m(M_t)$ denote the m-th Lipschitz-Killing curvature of the manifold M_t, for $0 \leq m \leq (n-1)$. Then we shall prove the following result.

Theorem 1.0.1 *Let Φ_t is an isotropic and volume preserving Brownian flow of C^2 diffeomorphsims of $\mathbb{R}^n$. Then, for $0 \leq m \leq n-1$, the expected rate of growth of the Lipschitz-Killing curvatures is given by*

$$\mathbb{E}\{\mathcal{L}_m(M_t)\} = \mathcal{L}_m(M) \exp(C\, t),$$

where C is a constant independent of t.

Chapter 2 is devoted towards setting up the notation and developing the required background for the results that follow in the subsequent chapters. Throughout this, and the following Chapter 3, we borrow heavily from the work of Cranston and Le Jan [12]. Chapter 3 redevelopes many results from [12] which are needed to prove the new results in Chapter 4.

An important feature of Brownian flows which is crucial for the analysis in Part I is the Markovian character of the one point motion of the flows.

Part II, therefore. moves out of this setting to see what can be done in the somewhat harder setting of non-Markovian, non-diffusive, flows.

More precisely, in Part II we study stochastic flows driven by fractional Brownian motion with Hurst parameter $H > 1/2$. The reason behind the choice of fractional

Brownian motion is two fold. Firstly, they are of independent interest, having appeared in a number of applications (see [15, 35]). Secondly, while they are no longer Markovian, their inherent Gaussian structure provides a framework in which some calculations are still possible.

Fractional Brownian motion $\{B^H(t),\ t \geq 0\}$ with Hurst parameter $H \in (0,1)$, is the zero mean Gaussian process with stationary increments, which satisfies a scaling property called *self-similarity* with index H. More precisely,

$$(B^H(t) - B^H(s)) \stackrel{\mathcal{L}}{=} B^H(|t-s|),$$

and

$$B^H(t) \stackrel{\mathcal{L}}{=} t^H B^H(1),$$

for any $s,\, t \geq 0$.

Note that for $H = 1/2$, B^H is the standard Brownian motion, which is a Markov process and also a martingale. However for $H \neq 1/2$, B^H is neither a Markov process nor a semi-martingale.

The study of fractional Brownian motion, the various ways of defining stochastic integrals with respect to this process and the study of flows generated by fractional Brownian motion with $H > 1/2$, forms the bulk of Part II.

Chapter 5 is mainly aimed at summarizing various properties of fractional Brownian motion as a process, and providing a literature review of various attempts at defining integrals with respect to this process. Then, in Chapters 6 and 7, we provide details of two different ways of defining stochastic integrals with respect to fractional Brownian motion, the Wiener integral and the pathwise approaches. Finally, in Chapter 8, we present the main result of Part II, namely the estimates on the growth of the Hausdorff measure of randomly evolving manifolds under a stochastic flow driven by fractional Brownian motion.

The main result of Part II can be stated, in short, as follows:

Theorem 1.0.2 *Let M_t be the image under the fractional flow Φ_t of an m-dimensional smooth manifold M, embedded in $\mathbb{R}^n$ for some $m < n$, and let $\mathcal{L}_m(M_t)$ be the m-dimensional Hausdorff measure of the manifold M_t. Then there exist constants c_1*

and C_1, such that

$$\sup_{t\in[0,T]} \mathcal{L}_m(M_t) \;\leq\; c_1\, \mathcal{L}_m(M)\, 2^{C_1\, T\, \|B^H\|_{\beta,T}^{1/\beta}}.$$

where $\|B^H\|_{\beta,T}$ is the β-Hölder norm of the driving process B^H, and $\beta$is a parameter to be defined later.

The thesis concludes with a brief chapter on open problems and directions for future research.

Part I

Brownian flows

Chapter 2

Background

In this chapter, we shall introduce the notion of a stochastic flow and develop the basic geometric aspects concerning Riemannian manifolds needed to study the connection between the two.

2.1 Stochastic flows

We start with a family of random mappings $\Phi_{st},\ 0 \leq s \leq t < \infty$, of $\mathbb{R}^n$ into itself, such that

- Φ_{st}, for each $s \leq t$ is a diffeomorphism of $\mathbb{R}^n$ into itself.
- $\Phi_{ut} \circ \Phi_{su} = \Phi_{st}$, for all $s \leq u \leq t < \infty$.
- Φ_{tt} is the identity map on $\mathbb{R}^n$ for all t.
- $\Phi_{s_1 t_1}, \Phi_{s_2 t_2}, \dots, \Phi_{s_n t_n}$ are independent if $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$.
- For each $x \in \mathbb{R}^n$, $\Phi_{st}(x)$, $\Phi_{st}^{-1}(x)$, $D\Phi_{st}(x) \equiv \left(\frac{\partial \Phi_{st}^i(x)}{\partial x^j}\right)$ and $D\Phi_{st}^{-1}(x)$ are jointly continuous in $0 \leq s \leq t < \infty$.

Such a family of random mappings is called a *stochastic (Brownian) flow*.

Writing Φ_t for Φ_{0t}, we can construct a *Brownian flow* on $\mathbb{R}^n$ by solving the equation

$$x_t = \Phi_t(x) = x + \int_0^t \partial U_s(\Phi_s(x)), \tag{2.1}$$

where ∂ denotes the Stratonovich interpretation of a stochastic derivative and $U_t(x)$ is a vector field valued Brownian motion with spatial covariance structure given by

$$EU_t^i(x)U_t^j(y) = (t \wedge s)C^{ij}(x-y), \quad 1 \le i,j \le n, \tag{2.2}$$

and where the C^{ij} takes the specific form

$$C^{kl}(z) = \int_0^\infty \int_{S^{n-1}} e^{i\rho\langle z,t\rangle}(\delta_l^k - t^k t^l)\sigma_{n-1}(dt)F(d\rho), \tag{2.3}$$

for a nonnegative measure F on $\mathbb{R}^+$ and normalized Lebesgue measure σ_{n-1} on S^{n-1}.

Furthermore we denote the various spatial derivatives of U as follows

$$W_j^i = \frac{\partial U^i}{\partial x^j}, \tag{2.4}$$

$$B_{jk}^i = \frac{\partial^2 U^i}{\partial x^j \partial x^k}. \tag{2.5}$$

Writing $\langle \cdot, \cdot \rangle$ for quadratic covariation, we have

$$\langle dW_j^i(t,y), dW_l^k(t,y)\rangle = C_{jl}^{ik}dt, \tag{2.6}$$

where, C_{jl}^{ik} can be obtained by taking the partial derivatives of the covariance function C (cf [1]). In our case, because of the specific choice of the covariance function in (2.3), C_{jl}^{ik} is given by

$$C_{jl}^{ik} = \frac{\mu_2}{n(n+2)}[(n+1)\delta_k^i\delta_l^j - \delta_j^i\delta_l^k - \delta_l^i\delta_j^k]. \tag{2.7}$$

In [6] and [20] isotropic and volume preserving flows are characterised in terms of the vector field U and the corresponding covariance function C.

A vector field $\{U(t,x) : t \in \mathbb{R}^+,\ x \in \mathbb{R}^n\}$ defined on the Euclidean space $\mathbb{R}^n$, is called an *isotropic* vector field if for T_y, a translation by $y \in \mathbb{R}^n$, $T_yU(t, T_{-y}x)$ and $U(t,x)$ have the same law, and moreover for R, an n-dimensional unitary matrix, $RU(t, R^{-1}x)$ and $U(t,x)$ are identical in law. A stochastic flow Φ_t is called isotropic if its corresponding vector field is isotropic. For Gaussian vector fields the conditions stated above boil down to the following condition on the spatial covariance function C,

$$C(x) = G^*C(Gx)G,$$

for any real orthogonal matrix G.

Subsequently, a necessary and sufficient condition for isotropy, in the case of Gaussian vector fields, is that the partial derivatives of the covariance function at *zero* have the form

$$C^{ik}_{jl} = a\ \delta^i_k \delta^j_l + b\ \delta^i_j \delta^k_l + c\ \delta^i_l \delta^k_j,$$

where $a+c$, $a-c$, $a+c+nb$ are nonnegative. Moreover, a flow Φ_t is said to be *volume preserving* if and only if

$$\operatorname{div} U = \sum_i W^i_i = 0,$$

almost surely, or, equivalently, if

$$E(\sum_i W^i_i)^2 = 0.$$

Hence it follows that the covariance function determined by (2.7) is that of an isotropic and volume preserving stochastic flow.

The particular choice of the covariance functions in (2.3), made also by Cranston and LeJan [12], simplifies many of the computations to follow, as certain Itô correction terms disappear. The computations are still difficult but, under (2.3), become feasible.

Furthermore, under the above assumpions, we have

$$\langle dB^i_{jk}(t,y), dW^p_q(t,y)\rangle = 0$$

for any $1 \leq i,j,k,p,q \leq n$ and

$$\langle\langle dB(u,u),v\rangle, \langle dB(u,u),v\rangle\rangle = \frac{3\mu_4}{n(n+2)(n+4)}[(n+3)\|u\|^4\|v\|^2 - 4\langle u,v\rangle^2\|u\|^2]dt,$$

for all vectors $u, v \in \mathbb{R}^n$.

Throughout the remainder of this thesis we shall assume, without further comment, that the covariance function corresponding to U is determined by (2.3) and (2.7).

2.2 Tensors

Before we can turn to the geometry of the flow, which is of central importance for us, we need to recall some terms from tensor analysis.

A *k-covariant, l-contravariant tensor* T, is defined as a multilinear map,

$$T : \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R},$$

where V is a finite dimensional vector space and V^* is the dual of V. Writing (k,l)-tensors for k-covariant and l-contravariant tensors,we denote $\mathcal{T}_l^k(V)$ as the collection of all (k,l)-tensors defined on V.

Let $S \in \mathcal{T}_0^k(V)$ and $T \in \mathcal{T}_0^l$. Then their tensor product

$$S \otimes T : \underbrace{V \times \cdots \times V}_{(k+l) \text{ copies}} \to \mathbb{R}$$

is defined as

$$(S \otimes T)(X_1, \ldots, X_{k+l}) = S(X_1, \ldots, X_k)T(X_{k+1}, \ldots, X_{k+l}).$$

Clearly, $(S \otimes T) \in \mathcal{T}_0^{k+l}$. An illuminating example of a 3-covariant tensor is the determinant of the corresponding vectors. This example also serves as an example of an *alternating tensor*, in the sense that interchanging any of the arguments results in a change in the sign of the determinant. More precisely, an alternating tensor of order k is defined as a covariant k-tensor T on a finite dimensional vector space with the property that

$$T(X_{\sigma_1}, \ldots, X_{\sigma_k}) = (-1)^\sigma T(X_1, \ldots, X_k), \ \ \forall \sigma \in S(k),$$

where $S(k)$ is the symmetric group of permutations of k letters and $(-1)^\sigma$ denotes the sign of the permutation σ. Similarly we define a *symmetric tensor* of order k as a covariant k-tensor T on a finite dimensional vector space with the property that

$$T(X_{\sigma_1}, \ldots, X_{\sigma_k}) = T(X_1, \ldots, X_k), \ \ \forall \sigma \in S(k).$$

For $k \geq 0$, we denote by $\Lambda^k(V)$ (respectively, $\text{Sym}(\mathcal{T}_0^k(V))$) the set of all alternating (symmetric) covariant k-tensors. We can also define a natural projection $A : \mathcal{T}_0^k(V) \to \Lambda^k(V)$, by

$$(AT)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S(k)} (-1)^\sigma T(X_{\sigma(1)}, \ldots, X_{\sigma(k)}).$$

This definition helps in defining a crucial operation in tensor calculus, the *wedge product* of alternating tensors, which is given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} A(\alpha \otimes \beta).$$

Since $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$, it follows that $\alpha \wedge \beta \in \Lambda^{k+l}(V)$.

Writing $\Lambda^{m,n}(V)$ for the linear span of the image of $\Lambda^m(V) \times \Lambda^n(V)$ under the operation $\otimes$, define $\Lambda^*(V) \otimes \Lambda^*(V) = \oplus_{m,n=0}^{\infty} \Lambda^{m,n}(V)$. Then we can also define a *double wedge product* for $\gamma \in \Lambda^{m,n}(V)$ and $\theta \in \Lambda^{p,q}(V)$ by

$$\begin{aligned}
&(\gamma \cdot \theta)((u_1, \dots, u_{m+p}), (v_1, \dots, v_{n+q})) = \\
&\quad \frac{1}{m!\, n!\, p!\, q!} \sum_{\sigma \in S(m+p),\, \rho \in S(n+q)} (-1)^{\sigma+\rho} [\gamma((u_{\sigma_1}, \dots, u_{\sigma_m}), (v_{\rho_1}, \dots, v_{\rho_n})) \\
&\quad \times \theta((u_{\sigma_{m+1}}, \dots, u_{\sigma_{m+p}}), (v_{\rho_{n+1}}, \dots, v_{\rho_{n+q}}))]
\end{aligned}$$

so that $(\gamma \cdot \theta) \in \Lambda^{m+p,n+q}(V)$.

These constructions will turn out to be crucial for the *tube formulae* that we shall meet later.

2.3 Riemannian manifolds

We let M be an $(n-1)$-dimensional compact C^2 manifold embedded in $\mathbb{R}^n$. We write $T_x M$ for the tangent space at point $x \in M$ and $T(M) = \bigsqcup_{x \in M} T_x M$ for the tangent bundle. In general, a *(smooth) k-dimensional vector bundle* is a pair of smooth manifolds E and M, together with a surjective map $\pi : E \to M$, such that the following conditions are satisfied.

- E is $((n-1)+k)$ dimensional smooth manifold.
- $\pi^{-1}(x)$, for each $x \in M$, called the *fiber* of E over x, is endowed with the structure of a vector space.
- For every $x \in M$, there exists a neighborhood U of x and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\pi_1(\varphi(\pi^{-1}(U))) = U$, where π_1 is the projection onto the first factor.

- Finally, the restriction of φ to each fiber, $\varphi : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^k$, is a linear isomorphism.

Another related concept, a *smooth section* is defined as a smooth map $\sigma : M \to E$ such that, $\pi \circ \sigma = \mathrm{Id}_M$, where Id_M is the identity map on M.

Going back to the previous section, recall that we defined all tensorial objects with respect to a vector space V. We now move this to the context of manifolds so that we can make use of tensor analysis in the setup of Riemannian manifolds. Writing T_xM for V, we replicate all the definitions of the previous section. Then we define the *bundle of (k,l)-tensors* $\mathcal{T}_l^k M$ on M as

$$\mathcal{T}_l^k M = \bigsqcup_{x \in M} \mathcal{T}_l^k(T_xM)$$

where $\bigsqcup$ denotes the disjoint union. Similarly, the *bundle of k-forms* is

$$\Lambda^k M = \bigsqcup_{x \in M} \Lambda^k(T_xM).$$

We also assume that M is a Riemannian manifold equipped with a Riemannian metric. Formally, a *Riemannian metric* g on a smooth manifold M is a smooth section of $\mathrm{Sym}(\mathcal{T}_0^2(M))$, such that for each $x \in M$, g_x is *positive definite* (i.e., $g_x(X, X) > 0$ if $0 \neq X \in T_xM$).

Loosely speaking a Riemannian metric determines an inner product on each tangent space T_xM. Therefore, we shall write $g_x(X, Y) \equiv \langle X, Y \rangle$, a natural choice for inner products. Enigmatic as it may appear now, this choice of notation is actually very natural. Nevertheless, it is important to remember the dependence of the Riemannian metric on the position in the manifold.

Now we shall move on to an extremely crucial concept in differential geometry, that of differentiating vector fields and the notion of a *connection*. Writing $\mathcal{E}(M)$ for the space of smooth sections of E (from the definition of vector bundle), and $\mathcal{T}(M)$ for the space of all the vector fields, a *canonical connection* in E is defined as a map

$$\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M),$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

- $\nabla_X Y$ is linear over $C^\infty(M)$ in X, i.e.,

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \quad \text{for } f, g \in C^\infty(M).$$

- $\nabla_X Y$ is linear over $\mathbb{R}$ in Y, i.e.,

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \quad \text{for } a, b \in \mathbb{R}.$$

- ∇ satisfies product rule, i.e.,

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \quad \text{for } f \in C^\infty(M),$$

  where Xf is interpreted as the derivative of f in the direction X.

$\nabla_X Y$ is also called the *covariant derivative of Y in the direction of X*. It is noteworthy that even if we choose X, Y as vector fields taking values in the tangent bundle, the usual derivative of Y in the direction X need not lie in the tangent bundle. However, by projecting the resultant derivative onto the tangent space we get the linear connection which does lie in the tangent space.

As we did for M, we also define a connection on the ambient manifold $\mathbb{R}^n$. We write ∇ for the connection on the tangent bundle of M and $\widetilde{\nabla}$ for the connection on the ambient space $\mathbb{R}^n$.

Furthermore it is natural to request that a connection satisfy the following properties in addition to the ones already mentioned. Connections that do so are called *Levi Civita connections.*

- ∇ is torsion free, i.e., $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, where $[X, Y]f = (XY)f - (YX)f$, is the Lie bracket.
- ∇ is compatible with the metric $\langle \cdot, \cdot \rangle$, i.e., $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

All the machinery developed so far goes into defining one of the central aspects of Riemannian geometry, *curvature.* The *Riemannian curvature operator* is defined as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where X, Y are vector fields taking values in the tangent bundle of the manifold. The *Riemannian curvature tensor* is defined as

$$\begin{aligned} R(X,Y,Z,W) &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle \\ &= \langle R(X,Y)Z, W \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric.

Another important tool often used in differential geometry is the *(scalar) second fundamental form*, defined as

$$\begin{aligned} S_\nu(X,Y) &= \langle (I - \Pi)\widetilde{\nabla}_X Y, \nu \rangle \\ &= \langle \widetilde{\nabla}_X Y - \nabla_X Y, \nu \rangle, \end{aligned} \tag{2.8}$$

where $\Pi : T_x\mathbb{R}^n \to T_x M$ is orthogonal projection onto the tangent space of the manifold, ν is the unit normal vector field on the manifold and X, Y are vector fields taking values in the tangent bundle. Writing

$$S(X,Y) = (I - \Pi)\widetilde{\nabla}_X Y,$$

called the *second fundamental form*, we get

$$S_\nu(X,Y) = \langle S(X,Y), \nu \rangle.$$

It follows from the definition of the second fundamental form that $S(X,Y)$ is orthogonal to the tangent space $T_x M$ for each $x \in M$.

Remark 2.3.1 *In the definition* (2.8) *of the scalar second fundamental form, we use the fact that* $\Pi\widetilde{\nabla}_X Y = \nabla_X Y$*, without any explanation. Though it is not immediately apparent from the definition of a connection, note that connections depend on the underlying Riemannian metric. Some computations, together with the above fact, imply* $\Pi\widetilde{\nabla}_X Y = \nabla_X Y$*. We direct interested readers to [22] for complete computations and explanations.*

Remark 2.3.2 *Despite its name, the (scalar) second fundamental form should not be confused with a differential form. It derives its name from the fact that it is a* bilinear form.

We can use the (scalar) second fundamental form to induce a linear operator $S^{(k)}$ on the exterior algebra $\Lambda^k(T_xM)$ for $k \leq (n-1)$. This is done as follows. Define $S^{(1)}$ as $X \to \langle S(X,\cdot),\nu\rangle = S_\nu(X,\cdot)$. This satisfies the condition of a linear operator on $\Lambda^1(T_xM)$. Then define $S_\nu^{(k)}$ as

$$S_\nu^{(k)}(u_1 \wedge \ldots \wedge u_k,\ v_1 \wedge \ldots \wedge v_k) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{j=1}^{k} S_\nu(u_{\sigma(j)}, v_j),$$

where S_k is the collection of all k-permutations σ, and as earlier, we use $(-1)^\sigma$ for the sign of the permutation. This gives rise to a linear operator $u_1 \wedge \ldots \wedge u_k \to S^{(k)}(u_1 \wedge \ldots \wedge u_k, \cdot)$, where $u_1, \ldots, u_{n-1} \in T_xM$ is a basis of T_xM.

The last, and the most important, remaining definition is that of the *trace* of $S_\nu^{(k)}$. For this, however, we need some more notations. Define the index set I_k by

$$I_k = \{\vec{m} \in \{1, \ldots, n-1\}^k :\ m_1 < m_2 < \cdots < m_k\},$$

for $1 \leq k \leq (n-1)$.

Then, for $\vec{l} \in I_k$ define

$$\begin{aligned} |\vec{l}| &= l_1 + \cdots + l_k, \\ \alpha_{\vec{l}} &= u_{l_1} \wedge \cdots \wedge u_{l_k}, \\ \alpha^{\vec{l}} &= (-1)^{|\vec{l}|+k} u_1 \wedge \cdots \wedge \hat{u}_{l_1} \wedge \cdots \wedge \hat{u}_{l_k} \wedge \cdots \wedge u_{n-1}, \end{aligned} \tag{2.9}$$

and

$$\alpha = u_1 \wedge \cdots \wedge u_{n-1},$$

where $u_1, \ldots, u_{n-1}$ is, as defined earlier, a basis of T_xM and a vector labeled by a $\hat{\ }$ is understood to be omitted from the wedge product. Now, for $\vec{l}, \vec{m} \in I_k$, define

$$\langle \alpha_{\vec{l}}, \alpha_{\vec{m}} \rangle = \det(\langle u_{l_i}, u_{m_j} \rangle), \tag{2.10}$$

and, similarly, define

$$\|\alpha\| = \det(\langle u_i, u_j \rangle).$$

Now we are well armed, with all the tools required, to define the all important $TrS^{(k)}$ as

$$TrS^{(k)} = S^{(k)}(\alpha_{\vec{l}}, \alpha_{\vec{m}}) \langle \alpha^{\vec{l}}, \alpha^{\vec{m}} \rangle \|\alpha\|^{-2},$$

where the Einstein summation convention is used over the indices $\vec{l}, \vec{m} \in I_k$. An interesting and quite useful property of the trace, which we shall use later, is that $TrS^{(k)}$ is independent of the choice of basis.

We shall now explain why $TrS^{(k)}$ is important in the study of manifolds. As earlier, we write ν for the unit normal vector field in M. Then for $x \in M$, and ν, the unit normal vector field on the manifold M, the *shape operator* $S^*_{x,\nu}(\cdot)$ of M is defined as a map,

$$S^*_{x,\nu} : T_xM \to T_xM$$

such that,

$$S^*_{x,\nu}(Y) = -\widetilde{\nabla}_Y \nu,$$

where $Y \in T_xM$. A simple calculation shows that the shape operator is a linear operator on the tangent space of M at x. An extremely important property of the shape operator is that

$$\begin{aligned}
\langle S^*_\nu(X), Y\rangle &= -\langle \widetilde{\nabla}_X \nu, Y\rangle \\
&= \langle \nu, \widetilde{\nabla}_X Y\rangle \\
&= \langle \nu, (I - \Pi)\widetilde{\nabla}_X Y\rangle + \underbrace{\langle \nu, \nabla_X Y\rangle}_{=0} \\
&= \langle S(X, Y), \nu\rangle \\
&= S_\nu(X, Y),
\end{aligned}$$

where X, Y are the vector fields taking values in the tangent bundle and the underbraced part is zero as $\nabla_X Y$ lies in the tangent bundle, by definition, hence its inner product with ν vanishes. This observation is a consequence of what is known as the *Weingarten equation*. So we observe that the second fundamental form can be retrieved from the shape operator.

Being a linear operator on the tangent bundle of the manifold, the shape operator has eigenvalues and eigenvectors. The eigenvalues of the shape operator $\{\lambda_k\}_{k=1}^{n-1}$ are called the *principal curvatures* and the corresponding eigenvectors are called the *principal curvature directions*.

Writing $u_1, \ldots, u_{n-1}$ for the principal curvature directions at a point $x \in M$ and $\lambda_1, \ldots, \lambda_{n-1}$ for the corresponding principal curvatures, we find $S^*_\nu(u_j) = \lambda_j u_j$ and, by using the property of the shape operator mentioned above we conclude that

$$\langle S(u_j, v), \nu\rangle = \lambda_j \langle u_j, v\rangle.$$

Now making use of the fact that the principal curvature direction vectors are orthogonal we find that,

$$S^{(k)}(\alpha_{\vec{l}}, \alpha_{\vec{l}}) = \lambda_{l_1} \dots \lambda_{l_k}.$$

Since $TrS^{(k)}$ is independent of the choice of the basis, we can, and so shall, evaluate it for the principal curvature direction vectors. Therefore, we observe that

$$TrS^{(k)} = \sum_{\vec{l} \in I_k} \lambda_{l_1} \dots \lambda_{l_k} \triangleq P_k(\lambda_1, \dots, \lambda_{n-1}),$$

which is the k-th symmetric polynomial of the principal curvatures for $1 \leq k \leq (n-1)$. Observe that the first symmetric polynomial is the *mean curvature* and the last symmetric polynomial is the *Gaussian curvature.*

In the following section we shall see an example of why symmetric k-polynomials of the principal curvatures are very important in differential geometry.

2.4 Tube formula and Lipschitz-Killing curvatures

Perhaps surprisingly, the problem of finding the volume of the tube around a manifold, when the manifold is inflated by some amount, has its roots in statistics. In stochastic processes, the *tube formula* is used to evaluate the maximal distributions of Gaussian processes (cf [1]). The tube formula is credited to Hotelling ([19]) and Weyl ([41]). This celebrated result gives the volume of the tube around a manifold in terms of geometric invariants of the manifold itself. More precisely, if M is an $(n-1)$-dimensional smooth manifold embedded in $\mathbb{R}^n$ and endowed with the canonical Riemannian structure on $\mathbb{R}^n$, then we shall define the tube of radius ρ around M as

$$Tube(M, \rho) = \{x \in \mathbb{R}^n : \ d(x, M) \leq \rho\},$$

where

$$d(x, M) = \inf_{y \in M} \|x - y\|.$$

Writing $\mu(Tube(M,\rho))$ for the volume of the tube, Weyl's tube formula states that there exists a $\rho_c \geq 0$, known as the *critical radius*, such that for $\rho \leq \rho_c$, the volume of the tube is given by

$$Vol(Tube(M,\rho)) = \sum_{i=0}^{n-1} \rho^{n-i}\omega_{n-i}\mathcal{L}_i(M), \tag{2.11}$$

where ω_i is the volume of the i-dimensional unit ball and $\mathcal{L}_i(M)$ denotes the i^{th}-Lipschitz-Killing curvature.

It is shown in [1] that the volume of the tube can be explicitly calculated by using the tools we have seen in the previous sub-sections and a little more. Consequently, given this calculation and then comparing coefficients of ρ^i on both the sides of (2.11), one can obtain precise expression for Lipschitz-Killing curvatures $\mathcal{L}_i(M)$.

In particular, it follows that the Lipschitz-Killing curvatures of a smooth $(n-1)$-dimensional hypersurface M embedded in $\mathbb{R}^n$ can be written as

$$\begin{aligned}\mathcal{L}_{n-k-1}(M) &= K_{n,k}\int_M \int_{S(\mathbb{R})} Tr(S_\nu^{(k)}) \\ &\quad \times 1_{N_xM}(-\nu)\mathcal{H}_0(d\nu)\mathcal{H}_{n-1}(dx),\end{aligned}$$

where, $K_{n,k} = \frac{1}{(2\pi)^{(k+1)/2}}\Gamma\left(\frac{(k+1)}{2}\right)$, $\mathcal{H}_k(dx)$ is the k-dimensional Hausdorff measure, and N_xM is the *normal cone* to the manifold M at point x. Loosely speaking, a *normal cone* N_xM for $x \in M$, a *smooth* manifold, is the vector space generated by the set of vectors normal to the tangent space. If the codimension of the manifold M is 1, then it is easy to see that the normal cone N_xM comprises of two unit vectors pointing in opposite directions, which are orthogonal to the tangent space T_xM.

Chapter 3

Geometry of the flow

In the previous chapter we developed the background required for the geometric analysis of stochastic flows. In this chapter we shall review and rederive some results due to Cranston and Le Jan in [12] concerning the evolution of the scalar second fundamental form S_ν, the induced k-form $S_\nu^{(k)}$ and finally the trace of the induced k-form. Essentially all the results we give here, together with their proofs, can be found in [12]. Nevertheless we collect them here, sometimes with additional details, since they are needed for what follows in Chapter 4, where we present new results.

Before we start proving theorems, we need some notations which are basically extensions of what we have already seen in previous chapter. Recall that we assumed M to be an $(n-1)$-dimensional manifold embedded in $\mathbb{R}^n$ and we have defined a stochastic flow with its covariance function in Chapter 2. The special choice of the covariance function made the stochastic flow volume preserving and isotropic.

We define M_t as $\{\Phi_t(x) : x \in M\}$ or equivalently $\Phi_t(M)$ and $T_{x_t}M_t$ as the tangent space of M_t at x_t and write $u_1, \dots, u_{n-1}$ for a basis of T_xM, and $u_1(t), \dots, u_{n-1}(t)$ for a basis of $T_{x_t}M_t$. At this point we require no connection between the u_j and $u_j(t)$.

Here is a quick overview of some of the geometric objects redefined for M_t:

- The orthogonal projection onto the tangent space is written as

$$\Pi_t \ : T_{x_t}\mathbb{R}^n \to T_{x_t}M_t. \tag{3.1}$$

- The second fundamental form for M_t is denoted by

$$S_t(u(t), v(t)) = (I - \Pi_t)\widetilde{\nabla}_{u(t)}(v(t)), \tag{3.2}$$

and the scalar second fundamental form is given by

$$S_{\nu_t}(u(t), v(t)) = \langle S_t(u(t), v(t)), \nu_t \rangle, \tag{3.3}$$

where $u(t), v(t) \in T_{x_t}M_t$ and ν_t is a unit normal vector field of M_t.

- Finally, the k-form $S^{(k)}_{\nu_t}$ induced by the second fundamental form is given by,

$$S^{(k)}_{\nu_t}(v_1(t) \wedge \cdots \wedge v_k(t), w_1(t) \wedge \cdots \wedge w_k(t)) = \sum_{\sigma \in S_k} (-1)^{\sigma} \prod_{j=1}^{k} S_{\nu_t}(v_{\sigma(j)}(t), w_j(t)), \tag{3.4}$$

where $(-1)^{\sigma}$ is as defined earlier, $v_i(t), w_i(t) \in T_{x_t}M_t$, and S_k is the collection of all k-permutations.

We know the exact expression for the Lipschitz-Killing curvatures for M, now we shall define them for the moved manifold M_t. There is no change, whatsoever, in the definition of the Lipschitz-Killing curvatures, but the constituent terms change with the flow.

$$\begin{aligned}\mathcal{L}_{n-k-1}(M_t) &= K_{n,k} \int_{M_t} \int_{S(\mathbb{R})} Tr^{T_{x_t}M_t}(S^{(k)}_{\nu_t}) \\ &\quad \times 1_{N_{x_t}M_t}(-\nu_t)\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx_t)\end{aligned}$$

Although it is natural that the exterior integral here is over M_t, it will be convenient for us to have the integral over the original manifold M, which we can obtain via a standard change of variables formula to be

$$\begin{aligned}
\mathcal{L}_{n-k-1}(M_t) &= K_{n,k}\int_M\int_{S(\mathbb{R})} Tr^{T_{\Phi_t(x)}\Phi_t(M)}(S^{(k)}_{\nu_t})\sqrt{\det(\langle u_i(t), u_j(t)\rangle)} \\
&\quad \times 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\,\mathcal{H}_{n-1}(dx) \\
&= K_{n,k}\int_M\int_{S(\mathbb{R})} Tr^{T_{\Phi_t(x)}\Phi_t(M)}(S^{(k)}_{\nu_t})\|\alpha_t\| \\
&\quad \times 1_{N_{x_t}M_t}(-\nu_t)\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx) \\
&= K_{n,k}\int_M\int_{S(\mathbb{R})} S^{(k)}_{\nu_t}(\alpha_{\vec{l},t}, \alpha_{\vec{m},t})\langle \alpha_t^{\vec{l}}\alpha_t^{\vec{m}}\rangle\|\alpha_t\|^{-2}\|\alpha_t\| \\
&\quad \times 1_{N_{x_t}M_t}(-\nu_t)\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx) \\
&= K_{n,k}\int_M\int_{S(\mathbb{R})} S^{(k)}(\alpha_{\vec{l},t}, \alpha_{\vec{m},t})\langle \alpha_t^{\vec{l}}\alpha_t^{\vec{m}}\rangle\|\alpha_t\|^{-1} \\
&\quad \times 1_{N_{x_t}M_t}(-\nu_t)\mathcal{H}_0(d\nu_t)\mathcal{H}_{n-1}(dx). \qquad (3.5)
\end{aligned}$$

Now we are in a position to develop some results for isotropic and volume preserving stochastic flows which we shall, in the following chapter, apply towards realising our main goal, that of finding an Itô formula for Lipschitz-Killing curvatures.

3.1 Itô formula for the second fundamental form

Clearly, it follows from (2.1) that,

$$dD\Phi_t(x) = \partial W(x_t)D\Phi_t(x), \qquad (3.6)$$

where we remind the reader that ∂ denotes the Stratonovich interpretation of a stochastic derivative, D is the space derivative as mentioned earlier while defining stochastic flows in Section 2.1 and, finally, d denotes the Itô interpretation of a stochastic derivative. It is important to note that $D\Phi_t(x)$ is a full rank matrix and, hence, is invertible due to the diffeomorphic nature of the flow $\Phi_t(x)$.

Assuming $u \in T_xM$, we define

$$u(t) = D\Phi_t(x)u.$$

Then by a simple calculation of the *push-forward* (see [23] for complete computations), it follows that $u(t) \in T_{x_t}M_t$ and moreover,

$$du(t) = \partial W u(t) = dW u(t). \qquad (3.7)$$

The correction term in the above expression is $\frac{1}{2}dWdWu_t$, which vanishes due to the choice of the covariance function in (2.3) and (2.7). Furthermore, if we choose $\{u_1, \ldots, u_{n-1}\}$ as the basis for T_xM, then it follows that $\{u_1(t), \ldots, u_{n-1}(t)\}$ forms a basis for $T_{x_t}M_t$ by a simple application of the *push-forward*.

Recalling that we are using $\widetilde{\nabla}$ for the *canonical connection* on Euclidean space, we have the following result.

Lemma 3.1.1 *Let u, v be vectors belonging to T_xM and set $u(t) = D\Phi_t(x)u$, $v(t) = D\Phi_t(x)v$. Then, $u(t), v(t) \in T_{x_t}M_t$ and*

$$d\widetilde{\nabla}_{u(t)}(v(t)) = \partial W \widetilde{\nabla}_{u(t)}(v(t)) + \partial B(u(t), v(t)).$$

Proof: Let $u, v \in T_xM$. Extend v to a smooth vector field V in a neighbourhood of $x \in M$ from which it follows that $V_t(x_t) = v(t)$. Set

$$V_t(y) = (D\Phi_t)(\Phi_t^{-1}(y))V(\Phi_t^{-1}(y)),$$

where $y \in M_t$, which implies $\Phi_t^{-1}(y) \in M$ and hence $V(\Phi_t^{-1}(y))$ is well defined. Now denote $Z_t \equiv \widetilde{\nabla}_{u(t)}V_t(x_t)$, then take γ to be a curve taking values on the manifold M, with $\gamma(0) = x$, $\gamma^{'}(0) = u$, thereafter, define $\gamma_t(\cdot) = \Phi_t(\gamma(\cdot))$, so that $\gamma_t^{'}(0) = D\Phi_t(\gamma(0))\gamma^{'}(0) = D\Phi_t(x)u = u(t)$. Then,

$$\begin{aligned}
\widetilde{\nabla}_{u(t)}V_t &= \lim_{s\to 0} s^{-1}(V(\gamma_t(s)) - V_t(\gamma_t(0))) \\
&= \lim_{s\to 0} s^{-1}(D\Phi_t(\gamma(s))V(\gamma(s)) - D\Phi_t(x)v) \\
&= \lim_{s\to 0} s^{-1}[(D\Phi_t(\gamma(s)) - D\Phi_t(x))V(\gamma(s)) \\
&\quad + D\Phi_t(x)(V(\gamma(s)) - v)] \\
&= D^2\Phi_t(x)(u,v) + D\Phi_t(x)\widetilde{\nabla}_u v.
\end{aligned}$$

However, by (3.6),

$$dD\Phi_t(x) = \partial W D\Phi_t(x)$$

and

$$dD^2\Phi_t(x)(u,v) = \partial B(D\Phi_t(s)u, D\Phi_t(x)v) + \partial W D^2\Phi_t(x)(u,v).$$

Therefore,

$$\begin{aligned}
dZ_t &= dD^2\Phi_t(x)(u,v) + dD\Phi_t(x)\widetilde{\nabla}_u v \\
&= \partial B(D\Phi_t(s)u, D\Phi_t(x)v) + \partial W D^2\Phi_t(x)(u,v) + \partial W D\Phi_t(x)\widetilde{\nabla}_u v \\
&= \partial B(u(t), v(t)) + \partial W D^2\Phi_t(x)(u,v) + \partial W(Z_t - D^2\Phi_t(x)(u,v)) \\
&= \partial B(u(t), v(t)) + \partial W Z_t,
\end{aligned}$$

which proves the lemma. □

From here on we shall always assume $u(t)$ to be given by $u(t) = D\Phi_t(x)u$, where $u \in T_xM$, unless mentioned otherwise.

Now recall, $\Pi_t : T_{x_t}R^n \to T_{x_t}M_t$ was defined as the orthogonal projection in (3.1). Hence we can write

$$\Pi_t = Y_t(Y_t{}'Y_t)^{-1}Y_t{}',$$

where $Y_t = (u_1(t), \ldots, u_{n-1}(t))$, is an $n \times (n-1)$ matrix generated by the basis $\{u_i(t)\}_{i=1}^{n-1}$ of $T_{x_t}M_t$ and Y_t' is the transpose of Y_t. Considering Π_t as a matrix, we have the following Itô formula for the orthogonal projection, describing the evolution of Π_t as a function of time t.

Lemma 3.1.2

$$d\Pi_t = (I - \Pi_t)\partial W \Pi_t + \Pi_t \partial W'(I - \Pi_t)$$

Proof: We have $dY_t = \partial W Y_t$ or equivalently $dY_t{}' = Y_t{}'\partial W'$, where W' denotes the transpose of W. Using the simple product rule we get,

$$\partial(Y_t{}'Y_t) = Y_t{}'\partial W' Y_t + Y_t{}'\partial W Y_t = Y_t{}'(\partial W + \partial W')Y_t. \tag{3.8}$$

Since

$$(Y_t{}'Y_t)(Y_t{}'Y_t)^{-1} = I,$$

where I is the identity matrix, we have

$$\partial[(Y_t{}'Y_t)(Y_t{}'Y_t)^{-1}] = 0, \tag{3.9}$$

and so, again by the product rule, we obtain

$$\partial[(Y_t'Y_t)(Y_t'Y_t)^{-1}] = [\partial(Y_t'Y_t)](Y_t'Y_t)^{-1} + (Y_t'Y_t)\partial[(Y_t'Y_t)^{-1}]. \tag{3.10}$$

Using (3.9) and (3.10) we therefore have

$$\partial(Y_t'Y_t)^{-1} = -(Y_t'Y_t)^{-1}\partial(Y_t'Y_t)(Y_t'Y_t)^{-1}.$$

Hence,

$$\begin{aligned}
d\Pi_t &= \partial\Pi_t \\
&= \partial W Y_t(Y_t'Y_t)^{-1}Y_t' + Y(Y_t'Y_t)^{-1}Y_t'\partial W' + Y_t\partial(Y_t'Y_t)^{-1}Y_t' \\
&= \partial W\Pi_t + \Pi_t\partial W' - Y(Y_t'Y_t)^{-1}\partial(Y_t'Y_t)(Y_t'Y_t)^{-1}Y_t' \\
&= \partial W\Pi_t + \Pi_t\partial W' - \Pi_t(\partial W' + \partial W)\Pi_t \\
&= (I-\Pi_t)\partial W\Pi_t + \Pi_t\partial W'(I-\Pi_t),
\end{aligned}$$

which is what we wanted to prove. □

Now we define $dP_t, dQ_t, d\lambda_t, d\mu_t$ as follows:

$$(I-\Pi_t)\partial W = (I-\Pi_t)dW + d\lambda_t = dP_t + d\lambda_t,$$

$$\Pi_t\partial W' = \Pi_t dW' + d\mu_t = dQ_t + d\mu_t.$$

Then,

$$d\Pi_t = dP_t\Pi_t + d\lambda_t\Pi_t + dQ_t(I-\Pi_t) + d\mu_t(I-\Pi_t), \tag{3.11}$$

where the correction terms indicated by λ_t, μ_t are expressed as

$$2d\lambda_t = d(I-\Pi_t)dW,$$

and

$$2d\mu_t = d\Pi_t dW'.$$

Following the expression of the second fundamental form in (3.2), we can rewrite it as

$$S_t(u(t), v(t)) = \widetilde{\nabla}_{u(t)}(v(t)) - \Pi_t\widetilde{\nabla}_{u(t)}(v(t)) = \widetilde{\nabla}_{u(t)}(v(t)) - R_t,$$

and it follows that $R_t \in T_{x_t}M_t$.

Theorem 3.1.1 *The Itô formula for the second fundamental form is given by*

$$\begin{aligned} dS_t(u(t),v(t)) &= (I-\Pi_t)dB(u(t),v(t)) + (dP_t - dQ_t)S_t(u(t),v(t)) \\ &\quad + d(\lambda_t - \mu_t)S_t(u(t),v(t)) + \frac{1}{2}(dP_t - dQ_t)^2 S_t(u(t),v(t)), \end{aligned}$$

where $u(t), v(t) \in T_{x_t}M_t$ *and furthermore* $v(t)$ *is extended to a vector field in a similar way as in Lemma 3.1.1.*

Proof:

$$\begin{aligned} dS(u(t),v(t)) &= (I-\Pi_t)\partial\widetilde{\nabla}_{u(t)}v(t) - (\partial\Pi_t)\widetilde{\nabla}_{u(t)}v(t) \\ &= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)\partial W\widetilde{\nabla}_{u(t)}v(t) - (\partial\Pi_t)\widetilde{\nabla}_{u(t)}v(t) \\ &= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)\partial W S_t(u(t),v(t)) \\ &\quad + \underbrace{(I-\Pi_t)\partial W R_t - (\partial\Pi_t)R_t}_{=0} - (\partial\Pi_t)S_t(u(t),v(t)) \\ &= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)\partial W S_t(u(t),v(t)) \\ &\quad - (\partial\Pi_t)S_t(u(t),v(t)) \\ &= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)\partial W S_t(u(t),v(t)) \\ &\quad - \Pi_t\partial W' S_t(u(t),v(t)), \end{aligned}$$

where the underbraced part equals zero as a consequence of Lemma 3.1.2.

Replacing the Stratonovich derivative by the Itô derivative, which is done by introducing a correction term, we obtain

$$
\begin{aligned}
dS_t(u(t),v(t)) &= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)\partial W S_t(u(t),v(t)) - \Pi_t\partial W' S_t(u(t),v(t)) \\
&= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)dW S_t(u(t),v(t)) \\
&\quad + \frac{1}{2}[d(I-\Pi_t)dW S_t(u(t),v(t)) + (I-\Pi_t)dW dS_t(u(t),v(t))] \\
&\quad - \Pi_t dW' S_t(u(t),v(t)) - \frac{1}{2}[d\Pi_t dW' S_t(u(t),v(t)) + \Pi_t dW' dS_t(u(t),v(t))] \\
&= (I-\Pi_t)dB(u(t),v(t)) + (I-\Pi_t)dW S_t(u(t),v(t)) \\
&\quad + d\lambda_t S_t(u(t),v(t)) + \frac{1}{2}(I-\Pi_t)dW dS_t(u(t),v(t)) \\
&\quad - \Pi_t dW' S_t(u(t),v(t)) - d\mu_t S_t(u(t),v(t)) - \frac{1}{2}\Pi_t dW' dS_t(u(t),v(t)) \\
&= (I-\Pi_t)dB(u(t),v(t)) + (dP_t - dQ_t)S_t(u(t),v(t)) + (d\lambda_t - d\mu_t)S_t(u(t),v(t)) \\
&\quad + \frac{1}{2}[dP_t dS_t(u(t),v(t)) - dQ_t dS_t(u(t),v(t))] \\
&= (I-\Pi_t)dB(u(t),v(t)) + (dP_t - dQ_t)S_t(u(t),v(t)) \\
&\quad + (d\lambda_t - d\mu_t)S_t(u(t),v(t)) + \frac{1}{2}(dP_t - dQ_t)^2 S_t(u(t),v(t)),
\end{aligned}
$$

which is what had to be proved. □

The above expression involves correction terms which can be simplified by a specific choice of an orthonormal basis of $T_{x_t}\mathbb{R}^n$.

Lemma 3.1.3 *If $\zeta_t \in N_{x_t}M_t$ is a semi-martingale, then*

$$
\begin{aligned}
d\lambda_t\zeta_t &= \frac{(n-1)\mu_2}{2n(n+2)}\zeta_t dt, &(3.12)\\
d\mu_t\zeta_t &= \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\zeta_t dt, &(3.13)\\
\frac{1}{2}(dP_t - dQ_t)^2\zeta_t &= -\frac{n(n-1)\mu_2}{2n(n+2)}\zeta_t dt. &(3.14)
\end{aligned}
$$

Proof: Using the earlier definitions and Theorem 3.1.2,

$$2d\lambda_t = d(I-\Pi_t)dW = -(I-\Pi_t)dW\Pi_t dW - \Pi_t dW'(I-\Pi_t)dW$$

$$2d\mu_t = d\Pi_t dW' = (I-\Pi_t)dW\Pi_t dW' + \Pi_t dW'(I-\Pi_t)dW'.$$

Let's choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for $T_{x_t}R^n$ such that $e_n \in N_{x_t}M_t$. Then clearly,

$$\Pi_t = \sum_{i=1}^{n-1} e_i {e_i}',$$
$$(I - \Pi_t) = e_n {e_n}'.$$

Therefore,

$$\begin{aligned}
2d\lambda_t\zeta_t &= -(I-\Pi_t)dW\Pi_t dW\zeta_t - \Pi_t dW'(I-\Pi_t)dW\zeta_t \\
&= -\sum_{i=1}^{n} e_n {e_n}' dW e_i {e_i}' dW\zeta_t \\
&\quad -\sum_{i=1}^{n-1} e_i {e_i}' dW' e_n {e_n}' dW\zeta_t \\
&= -\sum_{i=1}^{n-1} \langle dW_n^i, dW_i^n \rangle \zeta_t \\
&= -\sum_{i=1}^{n-1} C_{ni}^{in}\zeta_t dt \\
&= \frac{(n-1)\mu_2}{n(n+2)}\zeta_t dt.
\end{aligned}$$

Here we have used the fact that $\langle dW_i^n, dW_n^n \rangle = 0,\ 1 \leq i \leq (n-1)$ (from (2.6) and (2.7)). Similarly we obtain the other two expressions. □

A trivial consequence of the above lemma is the following.

Corollary 3.1.1

$$\begin{aligned}
dS_t(u(t), v(t)) &= (I-\Pi_t)dB(u,v) + (dP_t - dQ_t)S_t(u(t), v(t)) \\
&\quad - \frac{(n-1)\mu_2}{(n+2)} S_t(u(t), v(t))dt. \qquad (3.15)
\end{aligned}$$

Proof: Apply the results of Lemma 3.1.3 to Theorem 3.1.1. □

We now choose a particularly convenient basis to work with. In particular, let $(u_1, \ldots, u_n)$ be an orthonormal basis of $T_x\mathbb{R}^n$ such that the first $(n-1)$ vectors form a basis of T_xM. Then, as we saw earlier, $\{u_1(t), \ldots, u_{n-1}(t)\}$ forms a basis for $T_{x_t}M_t$. Clearly this new set of vectors need not remain orthonormal, hence $u_n(t)$ need not belong to $N_{x_t}M_t$. Therefore we define a unit normal vector

$$\nu(t) = \frac{(I - \Pi_t)u_n(t)}{\|(I - \Pi_t)u_n(t)\|}. \tag{3.16}$$

Theorem 3.1.2 *With the above choice of the* $\{u_j\}$, *let* ν_t *be the unit normal vector field for* M_t *determined by* (3.16). *Then the Itô formula for* ν_t *is given by:*

$$d\nu_t = -dQ_t\nu_t - \frac{(n^2-1)\mu_2}{2n(n+2)}\nu_t dt.$$

Proof: Let $v_n(t) = (I - \Pi_t)u_n(t)$, so that $\nu_t = v_n(t)\|v_n(t)\|^{-1}$. We now start developing expressions that we shall need to compute the Itô formula for ν_t. The first is

$$\begin{aligned}
dv_n(t) &= (d(I-\Pi_t))u_n(t) + (I-\Pi_t)du_n(t) \\
&= -(d\Pi_t)u_n(t) + (I-\Pi_t)\partial W u_n(t) \\
&= -(I-\Pi_t)\partial W \Pi_t u_n(t) - \Pi_t\partial W'(I-\Pi_t)u_n(t) + (I-\Pi_t)\partial W u_n(t) \\
&= (I-\Pi_t)\partial W(v_n(t) - u_n(t)) - \Pi_t\partial W' v_n(t) + (I-\Pi_t)\partial W u_n(t) \\
&= (I-\Pi_t)\partial W v_n(t) - (I-\Pi_t)\partial W u_n(t) - \Pi_t\partial W' v_n(t) + (I-\Pi_t)\partial W u_n(t) \\
&= (I-\Pi_t)\partial W v_n(t) - \Pi_t\partial W' v_n(t) \\
&= (I-\Pi_t)dW v_n(t) + \tfrac{1}{2}d(I-\Pi_t)dW v_n(t) + \tfrac{1}{2}(I-\Pi_t)\, dW\, dv_n(t) \\
&\quad - \Pi_t dW' v_n(t) - \tfrac{1}{2}d\Pi_t dW' - \tfrac{1}{2}\Pi_t dW' dv_n(t) \\
&= (dP_t - dQ_t)v_n(t) + (d\lambda_t - d\mu_t)v_n(t) + \tfrac{1}{2}(dP_t - dQ_t)^2 v_n(t) \\
&= (dP_t - dQ_t)v_n(t) - \frac{(n-1)\mu_2}{(n+2)}v_n(t)dt.
\end{aligned}$$

Itô's formula gives us the second as

$$\begin{aligned}
d\|v_n(t)\|^2 &= 2\langle v_n(t), dv_n(t)\rangle + \langle (dP_t - dQ_t)v_n(t)\rangle \\
&= 2\langle v_n(t), (dP_t - dQ_t)v_n(t)\rangle - \frac{2(n-1)\mu_2}{(n+2)}\|v_n(t)\|^2 dt \\
&\quad + \langle dW_n^n v_n(t) - \sum_{i=1}^{n-1} e_i dW_n^i \langle v_n(t), e_n\rangle\rangle \\
&= 2\|v_n(t)\|^2 dW_n^n - \frac{2(n-1)\mu_2}{(n+2)}\|v_n(t)\|^2 dt \\
&\quad + dW_n^n dW_n^n \|v_n(t)\|^2 + \sum_{i=1}^{n-1} dW_n^i dW_n^i \langle v_n(t), e_n\rangle^2 \\
&= 2\|v_n(t)\|^2 dW_n^n - \frac{2(n-1)\mu_2}{(n+2)}\|v_n(t)\|^2 dt \\
&\quad + \frac{(n-1)\mu_2}{n(n+2)}\|v_n(t)\|^2 dt + \sum_{i=1}^{n-1} \frac{(n+1)\mu_2}{n(n+2)}\|v_n(t)\|^2 dt \\
&= 2\|v_n(t)\|^2 dW_n^n - \frac{(n-1)(n-2)\mu_2}{n(n+2)}\|v_n(t)\|^2 dt.
\end{aligned}$$

Finally, the third is given by

$$\begin{aligned}
d\|v_n(t)\|^{-1} &= d((\|v_n(t)\|^2)^{-1/2}) \\
&= \frac{(-1)d\|v_n(t)\|^2}{2(\|v_n(t)\|^2)^{3/2}} + (\frac{3}{8})\frac{d\langle \|v_n(t)\|^2\rangle}{(\|v_n(t)\|^2)^{5/2}} \\
&= -\|v_n(t)\|^{-1} dW_n^n + \frac{(n-1)(n-2)\mu_2}{2n(n+2)}\|v_n(t)\|^{-1} dt \\
&\quad + (\frac{3}{8})\frac{4(n-1)\mu_2}{n(n+2)}\|v_n(t)\|^{-1} dt \\
&= -\|v_n(t)\|^{-1} dW_n^n + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\|v_n(t)\|^{-1} dt.
\end{aligned}$$

Hence, amalgamating the last three calculations, we find

$$\begin{aligned}
d\nu_t &= d(\frac{v_n(t)}{\|v_n(t)\|}) \\
&= \frac{dv_n(t)}{\|v_n(t)\|} + v_n(t)d\|v_n(t)\|^{-1} + \langle dv_n(t), d\|v_n(t)\|^{-1}\rangle \\
&= ((dP_t - dQ_t)v_n(t) - \frac{(n-1)\mu_2}{(n+2)}v_n(t)dt)\|v_n(t)\|^{-1} + v_n(t)(-\|v_n(t)\|^{-1}dW_n^n \\
&\quad + \frac{(n-1)(n+1)\mu_2}{2n(n+2)}\|v_n(t)\|^{-1}dt) - \langle (dP_t - dQ_t)v_n(t), \|v_n(t)\|^{-1}dW_n^n\rangle \\
&= -dQ_t\nu_t - \frac{(n-1)^2\mu_2}{2n(n+2)}\nu_t\, dt - \frac{(n-1)\mu_2}{n(n+2)}\nu_t\, dt \\
&= -dQ_t\nu_t - \frac{(n^2-1)\mu_2}{2n(n+2)}\nu_t\, dt,
\end{aligned}$$

which proves the theorem. □

Now we are well equipped to compute the Itô formula for the *scalar* second fundamental form.

Theorem 3.1.3 *Let $u(t) = D\Phi_t u$ and $v(t) = D\Phi_t v$, for $u, v \in T_xM$, and ν_t be the unit normal vector field* (3.16)*. Then,*

$$\begin{aligned}
dS_{\nu_t}(u(t), v(t)) &= \langle dB(u(t), v(t)), \nu_t\rangle + S_{\nu_t}(u(t), v(t))dW_n^n \\
&\quad - \frac{(n-1)^2\mu_2}{2n(n+2)}S_{\nu_t}(u(t), v(t))dt.
\end{aligned}$$

Proof: From (3.15) we know that

$$\begin{aligned}
dS_t(u(t), v(t)) &= (I - \Pi_t)dB(u(t), v(t)) + (dP_t - dQ_t)S_t(u(t), v(t)) \\
&\quad - \frac{(n-1)\mu_2}{(n+2)}S_t(u(t), v(t))dt.
\end{aligned}$$

Using (2.6) and (2.7), we note that

$$\begin{aligned}
-\langle (dP_t - dQ_t)S_t(u(t),v(t)), dQ_t\nu_t\rangle &= \langle dQ_t S_t(u(t),v(t)), dQ_t\nu_t\rangle \\
&= \langle S_t(u(t),v(t)),\nu_t\rangle \sum_{i=1}^{n-1} dW_n^i dW_n^i \\
&= S_{\nu_t}(u(t),v(t)) \sum_{i=1}^{n-1} C_{nn}^{ii} dt \\
&= \frac{(n^2-1)\mu_2}{n(n+2)} S_{\nu_t}(u(t),v(t))dt.
\end{aligned}$$

Now, using the relation (3.3) and standard bivariate Itô formula, we obtain

$$\begin{aligned}
dS_{\nu_t}(u(t),v(t)) &= \langle dS_t(u(t),v(t)),\nu_t\rangle + \langle S_t(u(t),v(t)), d\nu_t\rangle + \langle dS_t(u(t),v(t)), d\nu_t\rangle \\
&= \langle dB(u(t),v(t)),\nu_t\rangle + \langle (dP_t - dQ_t)S_t(u(t),v(t)),\nu_t\rangle \\
&\quad - \frac{(n-1)\mu_2}{(n+2)}\langle S_t(u(t),v(t)),\nu_t\rangle dt - \underbrace{\langle S_t(u(t),v(t)), dQ_t\nu_t\rangle}_{=0} \\
&\quad - \langle S_t(u(t),v(t)),\nu_t\rangle \frac{(n^2-1)\mu_2}{2n(n+2)} dt - \langle (dP_t - dQ_t)S_t(u(t),v(t)), dQ_t\nu_t\rangle \\
&= \langle dB(u(t),v(t)),\nu_t\rangle + \langle (dW_n^n S_t(u(t),v(t)) \\
&\quad - \sum_{i=1}^{n-1} e_i dW_n^i \langle S_t(u(t),v(t)), e_n\rangle),\nu_t\rangle \\
&\quad - \frac{(n-1)\mu_2}{(n+2)} S_{\nu_t}(u(t),v(t))dt - \frac{(n^2-1)\mu_2}{2n(n+2)} S_{\nu_t}(u(t),v(t))dt \\
&\quad + \frac{(n^2-1)\mu_2}{n(n+2)} S_{\nu_t}(u(t),v(t))dt \\
&= \langle dB(u(t),v(t)),\nu_t\rangle + S_{\nu_t}(u(t),v(t))dW_n^n - \frac{(n-1)^2\mu_2}{2n(n+2)} S_{\nu_t}(u(t),v(t))dt,
\end{aligned}$$

which completes the proof. □

Note that essentially the same proof gives the quadratic covariation terms.

Corollary 3.1.2 *With the notation of the above theorem,*

$$\begin{aligned}
\langle dS_{\nu_t}(u_i(t),u_j(t)), dS_{\nu_t}(u_k(t),u_l(t))\rangle &= \langle\langle dB(u_i(t),u_j(t)),\nu_t\rangle, \langle dB(u_k(t),u_l(t)),\nu_t\rangle\rangle \\
&\quad + \frac{(n-1)\mu_2}{n(n+2)}\langle S_{\nu_t}(u_i(t),u_j(t)) S_{\nu_t}(u_k(t),u_l(t))dt,
\end{aligned}$$

where $u_i(t) = D\Phi_t u_i \in T_{x_t}M_t$.

With this we end the preliminaries required for the next chapter, where we shall prove the main result of Part I of the thesis.

Chapter 4

Main result: Itô formula for the Lipschitz-Killing curvatures.

In Chapters 2 and 3 we developed the foundation on which we build our main result. However, before getting to this we shall present one more result each from [12] and [20]. Although these results are related to those of Chapter 3, they are presented in this chapter due to their importance for our main result, which appears in Theorem 4.1.4. Some direct consequences of our main result are listed as a remark and a corollary, immediately following Theorem 4.1.4.

4.1 Itô formula

We shall go back to [12] and [20], to prove the last two results necessary for our main result. For this, we retain the notation of Chapters 2 and 3, and start with $\{u_1(t), \ldots, u_{n-1}(t)\}$, a basis of $T_{x_t}M_t$, and in the spirit of Section 2.3, define

$$\alpha_{\vec{l_p}}(t) = (-1)^{p+1} u_{l_1}(t) \wedge \cdots \wedge \hat{u}_{l_p}(t) \wedge \cdots \wedge u_{l_k}(t), \tag{4.1}$$

for $1 \leq k \leq (n-1)$, where $\vec{l} \in I_k$, $\vec{l_p} \in I_{k-1}$ and $1 \leq p \leq k$.

Rewriting the above expression as

$$\alpha_{\vec{l_p}}(t) = (-1)^{p+1} u^{(p)}_{l_1}(t) \wedge \cdots \wedge u^{(p)}_{l_{k-1}}(t), \tag{4.2}$$

defines $u^{(p)}_l$.

We can now formulate the following result.

Theorem 4.1.1 *The Itô formula for the k-form $S^{(k)}_{\nu_t}$, induced by the second fundamental form as in* (3.4)*, is given by*

$$dS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t)) = \sum_{i,p=1}^{k} S^{(k-1)}_{\nu_t}(\alpha_{\vec{l_p}}(t),\alpha_{\vec{m_i}}(t))\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle + kS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))dW^n_n - \frac{k(n-k)(n-1)\mu_2}{2n(n+2)}S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))dt$$

where $\alpha_{\vec{l}}(t) = u_{l_1}(t)\wedge\cdots\wedge u_{l_k}$ and $\alpha_{\vec{m}}(t)$ is defined similarly.

Proof: Using Theorem 3.1.3, Corollary 3.1.2, and the multivariate Itô formula we see that

$$\begin{aligned} dS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t)) &= \sum_{\sigma\in S_k}(-1)^\sigma\sum_{i=1}^{k}[\prod_{j=1,j\neq i}^{k} S_{\nu_t}(u_{l_{\sigma(j)}}(t),u_{m_j}(t))]\langle dB(u_{l_{\sigma(i)}}(t),u_{m_i}(t)),\nu_t\rangle \\ &\quad + kS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))dW^n_n - \frac{k(n-1)^2\mu_2}{2n(n+2)}S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))dt \\ &\quad + \frac{1}{2}\sum_{\sigma\in S_k}(-1)^\sigma\sum_{1\leq i\neq p\leq k}^{k}[\prod_{j=1,j\neq\{i,p\}}^{k} S_{\nu_t}(u_{l_{\sigma(j)}}(t),u_{m_j}(t))] \\ &\quad \langle dB^n(u_{l_{\sigma(i)}}(t),u_{m_i}(t)),dB^n(u_{l_{\sigma(p)}}(t),u_{m_p}(t))\rangle \\ &\quad + \frac{k(k-1)(n-1)\mu_2}{2n(n+2)}S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\,dt. \end{aligned}$$

The fourth term of the sum vanishes as, for each $\sigma\in S_k$, there exists exactly one $\eta\in S_k$ such that, $\{\sigma(i),\sigma(p)\}=\{\eta(i),\eta(p)\}$ and $\sigma(j)=\eta(j)$ for $j\notin\{i,p\}$ and $(-1)^\sigma = -(-1)^\eta$.

We simplify the first term as follows.

$$\begin{aligned} &\sum_{i,p=1}^{k}\sum_{\sigma\in S_k,\sigma(i)=p}(-1)^\sigma[\prod_{j=1,j\neq i}^{k} S_{\nu_t}(u_{l_{\sigma(j)}}(t),u_{m_j}(t))]\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle \\ &= \sum_{i,p=1}^{k}[\sum_{\sigma\in S_{k-1}}(-1)^{i+p}(-1)^\sigma\{\prod_{j=1,j\neq i}^{k} S_{\nu_t}(u^{(p)}_{l_{\sigma(j)}}(t),u^{(i)}_{m_j}(t))\}]\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle \\ &= \sum_{i,p=1}^{k} S^{(k-1)}_{\nu_t}(\alpha_{\vec{l_p}}(t),\alpha_{\vec{m_i}}(t))\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle. \end{aligned}$$

The third and the fifth terms combine to give the final expression. □

Before we move further in realising our main goal of obtaining an Itô formula for the Lipschitz-Killing curvatures, we recall some of the results and notations from Le Jan [20], which are essential for a thorough comprehension of our main result.

Let $\beta(t)$ be an m-form written as $\beta(t) = \beta_1(t) \wedge \ldots \wedge \beta_m(t)$, and define

$$\tau_l^j \beta(t) = e^j \wedge \imath(e^l)\beta(t),$$

where $\imath(e^l)\beta(t) = \sum_{k=1}^{m}(-1)^{k+1}\langle\beta_k(t), e^l\rangle\beta_1(t)\wedge\cdots\wedge\hat{\beta}_k(t)\wedge\cdots\wedge\beta_m(t)$ and $\{\beta_j(t)\}_{j=1}^m$ are vectors in $T_{x_t}M_t$ with $\{e^k\}_{k=1}^n$ being the standard basis of $\mathbb{R}^n$

Theorem 4.1.2 *Let $\xi(t) = u_{i_1}(t)\wedge\ldots\wedge u_{i_k}(t)$ and $\psi(t) = u_{j_1}(t)\wedge\ldots\wedge u_{j_k}(t)$, where $u_{i_j}(t) \in T_{x_t}M_t$. Then,*

$$\begin{aligned} d\langle\xi(t), \psi(t)\rangle &= \sum_{l,\,j}(\langle\tau_i^j\xi(t), \psi(t)\rangle + \langle\xi(t), \tau_i^j\psi(t)\rangle)dW_j^i(t) \\ &\quad + \frac{k(n-k)\mu_2}{n}\langle\xi(t), \psi(t)\rangle dt, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is as defined in (2.10).

Proof: We know from (3.7) that $du_i(t) = dWu_i(t)$. Then using the product rule of differentiation,

$$\begin{aligned} d\psi(t) &= \sum_{j=1}^{k} u_{i_1}(t)\wedge\ldots\wedge du_{i_j}(t)\wedge\ldots\wedge u_{i_k}(t) + \text{correction} \\ &= \sum_{j} u_{i_1}(t)\wedge\ldots\wedge\{\sum_{p=1}^{n-1} e_p du_{i_j}^p(t)\}\wedge\ldots\wedge u_{i_k}(t) + \text{correction} \\ &= \sum_{j,\,p} u_{i_1}(t)\wedge\ldots\wedge \underbrace{e_p}_{j^{th}} \wedge\ldots\wedge u_{i_k}(t) du_{i_j}^p(t) + \text{correction} \\ &= \sum_{j,\,p} u_{i_1}(t)\wedge\ldots\wedge \underbrace{e_p}_{j^{th}} \wedge\ldots\wedge u_{i_k}(t)\{\sum_{l=1}^{n} dW_l^p u_{i_j}^l(t)\} + \text{correction} \\ &= \sum_{j}\sum_{p,\,l}(-1)^{j+1}u_{i_j}^l(t)e_p\wedge u_{i_1}(t)\wedge\ldots\wedge\hat{u}_{i_j}(t)\wedge\ldots\wedge u_{i_k}(t)dW_l^p + \text{correction} \\ &= \sum_{p,\,l} e_p\wedge\imath(e_l)\psi(t)dW_l^p + \text{correction} \\ &= \sum_{p,\,l}\tau_l^p\psi(t)dW_l^p + \text{correction.} \end{aligned}$$

Using (2.6) and (2.7) the corresponding correction term is given by

$$\begin{aligned} 2(\text{correction}) &= \sum_{p,\, l,\, q,\, m} \tau_l^p \tau_m^q \psi(t) C_{lm}^{pq} dt \\ &= \sum_{p,\, q} ((n+1)\tau_q^p \tau_q^p \psi(t) - \tau_p^p \tau_q^q \psi(t) - \tau_q^p \tau_p^q \psi(t)) \frac{\mu_2 dt}{n(n+2)}. \end{aligned}$$

In general,

$$\tau_l^p \tau_m^q = (\delta_l^q \tau_m^p - e_p \wedge e_q \wedge \imath(e_l)\imath(e_m))\psi(t).$$

Exploiting the linearity of the wedge product we restrict our attention to $\psi(t)$ of the form $e_{i_1} \wedge \ldots \wedge e_{i_k}$ and find that

$$\begin{aligned} 2(\text{correction}) &= \frac{((n+1)k - k^2 - (n-k-1)k)\psi(t)\mu_2 dt}{n(n+2)} \\ &= 0. \end{aligned}$$

Utilising the above results, we obtain the Itô formula for $\langle \xi(t), \psi(t) \rangle$.

$$\begin{aligned} d\langle \xi(t), \psi(t) \rangle &= \langle d\xi(t), \psi(t) \rangle + \langle \xi(t), d\psi(t) \rangle + \langle d\xi(t), d\psi(t) \rangle \\ &= \sum_{l,\, j} (\langle \tau_j^l \xi(t), \psi(t) \rangle + \langle \xi(t), \tau_j^l \psi(t) \rangle) dW_j^l \\ &\quad + \sum_{l,\, j,\, p,\, q} \langle \tau_j^l \psi(t), \tau_q^p \xi(t) \rangle C_{jq}^{lp} dt \\ &= \sum_{l,\, j} (\langle \tau_j^l \xi(t), \psi(t) \rangle + \langle \xi(t), \tau_j^l \psi(t) \rangle) dW_j^l(t) \\ &\quad + \text{correction}. \end{aligned}$$

where the correction term can again be simplified by using the linearity of the wedge product and hence by restricting ourselves to the wedge products of standard basis vectors $\{e_k\}$. Following this argument we observe

$$\begin{aligned} \text{correction} &= \sum_{l,\, j} ((n+1)\langle \tau_j^l \psi(t), \tau_j^l \xi(t) \rangle - \langle \tau_l^l \psi(t), \tau_j^j \xi(t) \rangle - \langle \tau_j^l \psi(t), \tau_l^j \xi(t) \rangle) \frac{\mu_2 dt}{n(n+2)} \\ &= \frac{\mu_2 dt}{n(n+2)} ((n+1)k(n-k+1) - k^2 - k) \\ &= \frac{\mu_2 dt}{n(n+2)} k((n+1)(n-k+1) - k - 1) \\ &= \frac{k(n-k)\mu_2 dt}{n}. \end{aligned}$$

Hence the required result. $\square$

Applying the argument of (3.7) in (3.6), we write,

$$dD\Phi_t(x) = \partial W D\Phi_t(x) = dW D\Phi_t(x).$$

Now a simple application of Theorem 4.1.2 will give us the following result:

$$d\|D\Phi_t(x)\| = \sum_{i,j=1}^{n} \langle \tau_i^j \beta, e^1 \wedge \cdots \wedge e^n \rangle dW_j^i, \tag{4.3}$$

where $\beta = \bigwedge_{i=1}^{n} (D\Phi_t)^{(i)}$ and $(D\Phi_t)^{(i)}$ denotes the ith column of the Jacobian matrix $D\Phi_t(x)$.

Going back to the definition of $\tau_i^j(\cdot)$, we see that due to linearity of the inner product and the wedge product, it suffices to simplify (4.3) using an orthonormal basis $\{e^k\}_{k=1}^n$. Hence,

$$\langle \tau_j^i \beta, e^1 \wedge \cdots \wedge e^n \rangle = \delta_{ij} \|D\Phi_t(x)\|.$$

Therefore, we can rewrite the Itô formula in (4.3) in a more comprehensible way as

$$d\|D\Phi_t(x)\| = \|D\Phi_t(x)\| \sum_{i=1}^{n} dW_i^i.$$

If we now assume the flow to be *divergence free*, then

$$\sum_{i=1}^{n} dW_i^i = 0, \quad \text{a.s.}$$

Therefore, for a divergence free flow, the Jacobian $D\Phi_t(x)$ is almost surely a constant, which in our case is 1, as $\|D\Phi_0(x)\| = 1$ almost surely.

Note that it follows from this observation that divergence free property is equivalent to the volume preserving characteristic of a flow. To see this, let M^* be an n-dimensional manifold with the ambient manifold being $\mathbb{R}^n$ and $M_t^* = \Phi_t(M^*)$. For example, take M^* to be an open ball in $\mathbb{R}^3$. Then,

$$Vol(M_t^*) = \int_{M_t^*} dx_t = \int_{M^*} dx = Vol(M),$$

which establishes equivalence between the divergence free and the volume preserving properties.

In [20], Theorem 4.1.2 is a crucial step towards computing the Lyapounov exponents of the flow. Another application of Theorem 4.1.2, needed for our main result, is the following.

Corollary 4.1.1 *Let* $\alpha = u_1(t) \wedge \cdots \wedge u_{n-1}(t)$*, where* $(u_1(t), \ldots, u_{n-1}(t))$ *is any basis of* $T_{x_t}M_t$*. Write* $\|\alpha(t)\| = \det(\langle u_i(t), u_j(t)\rangle)$*. Then*

$$d\|\alpha(t)\|^{-1} = \|\alpha(t)\|^{-1}\Big(- \sum_{i=1}^{n-1} dW_i^i - \frac{(n-1)^2\mu_2}{2n(n+2)}dt\Big).$$

Proof: Applying Theorem 4.1.2 to $\|\alpha(t)\|^2$ we find

$$d\|\alpha(t)\|^2 = \|\alpha(t)\|^2\Big(2\sum_{i=1}^{n-1} dW_i^i + \frac{(n-1)\mu_2}{n}dt\Big).$$

Now using the standard Itô formula and the relations (2.6) and (2.7) we obtain

$$\begin{aligned} d\|\alpha(t)\|^{-1} &= d(\|\alpha(t)\|^2)^{-\frac{1}{2}} \\ &= -\frac{1}{2\|\alpha(t)\|^3}\|\alpha(t)\|^2\Big(2\sum_{i=1}^{n-1} dW_i^i + \frac{(n-1)\mu_2}{n}dt\Big) \\ &\quad + \frac{3\|\alpha(t)\|^4}{8\|\alpha(t)\|^5}\Big(2\sum_{i=1}^{n-1} dW_i^i\Big)^2 \\ &= -\frac{1}{2}\|\alpha(t)\|^{-1}\Big(\sum_{i=1}^{n-1} dW_i^i + \frac{(n-1)\mu_2}{n}dt\Big) \\ &\quad + \frac{3(n-1)\mu_2}{2n(n+2)}\|\alpha(t)\|^{-1}dt \\ &= \|\alpha(t)\|^{-1}\Big(- \sum_{i=1}^{n-1} dW_i^i - \frac{(n-1)^2\mu_2}{2n(n+2)}dt\Big). \end{aligned}$$

This proves the result. □

Now define,

$$Y^{(k)}(t) = S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle \|\alpha(t)\|^{-1}, \tag{4.4}$$

where Einstein's summation convention is used over the indices $\vec{l}$ and $\vec{m}$.

Finally, we compute the last ingredient necessary to obtain our main result. Although the following result does not appear anywhere in the references but it can simply be obtained as a consequence of Theorem 4.1.1, Theorem 4.1.2 and Corollary 4.1.1.

Theorem 4.1.3

$$\begin{aligned} dY^{(k)}(t) &= [\sum_{i,p=1}^{k} S_{\nu_t}^{(k-1)}(\alpha_{\vec{l_p}}(t), \alpha_{\vec{m}_i}(t))\langle dB(u_{l_p}(t), u_{m_i}(t)), \nu_t\rangle]\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle \|\alpha(t)\|^{-1} \\ &\quad + Y^{(k)}(t)[kdW_n^n - \sum_{i=1}^{n-1} dW_i^i] \\ &\quad + S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))(\langle \tau_i^j \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle + \langle \alpha^{\vec{l}}(t), \tau_i^j \alpha^{\vec{m}}(t)\rangle)dW_j^i \|\alpha(t)\|^{-1} \\ &\quad + \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)} Y^{(k)}(t)dt, \end{aligned} \tag{4.5}$$

for $1 \le k \le n-1$, where $\alpha_{\vec{l}}(t)$, $\alpha_{\vec{l_p}}(t)$ and all other terms are as defined earlier.

Proof: From Theorem 4.1.2 and Corollary 4.1.1 we have

$$\begin{aligned} d\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle &= (\langle \tau_i^j \alpha^{\vec{l}}(t), \alpha^{\vec{l}}(t)\rangle + \langle \alpha^{\vec{l}}(t), \tau_i^j \alpha^{\vec{m}}(t)\rangle)dW_j^i \\ &\quad + \frac{(k+1)(n-k-1)\mu_2}{n}\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle dt, \\ \frac{d\|\alpha(t)\|^{-1}}{\|\alpha(t)\|^{-1}} &= -\sum_{i=1}^{n-1} dW_i^i - \frac{(n-1)^2\mu_2}{2n(n+2)}dt. \end{aligned}$$

Recall that,

$$Y^{(k)}(t) = S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle \|\alpha(t)\|^{-1}.$$

Hence, by the multivariate Itô formula,

$$\begin{aligned}
dY^{(k)}(t) &= (dS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t)))\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle \|\alpha(t)\|^{-1} \\
&\quad + S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))(d\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle)\|\alpha(t)\|^{-1} \\
&\quad + TrS^{(k)}_{\nu_t}\frac{d\|\alpha(t)\|^{-1}}{\|\alpha(t)\|^{-1}} \\
&\quad + \langle dS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t)), d\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle\rangle \|\alpha(t)\|^{-1} \\
&\quad + \langle dS^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t)), d\|\alpha(t)\|^{-1}\rangle \langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}\rangle(t) \\
&\quad + S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))\langle d\langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle, d\|\alpha(t)\|^{-1}\rangle \\
&= I + II + III + IV + V + VI.
\end{aligned}$$

We shall simplify the above expression term by term.

The first term can be rewritten as a consequence of Theorem 4.1.1.

$$\begin{aligned}
I &= \sum_{i,p=1}^{k} S^{(k-1)}_{\nu_t}(\alpha_{\vec{l_p}}(t), \alpha_{\vec{m_i}}(t))\langle dB(u_{l_p}(t), u_{m_i}(t)), \nu_t\rangle \langle \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle \|\alpha(t)\|^{-1} \\
&\quad + k\, Y^{(k)}(t) dW^n_n - \frac{k(n-k)(n-1)\mu_2}{2n(n+2)} Y^{(k)}(t)\, dt. \qquad (4.6)
\end{aligned}$$

Using Theorem 4.1.2 we obtain

$$\begin{aligned}
II &= \sum_{i,j=1}^{n} S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t), \alpha_{\vec{m}}(t))(\langle \tau^j_i \alpha^{\vec{l}}(t), \alpha^{\vec{m}}(t)\rangle + \langle \alpha^{\vec{l}}(t), \tau^j_i \alpha^{\vec{m}}(t)\rangle)\, dW^i_j \|\alpha(t)\|^{-1} \\
&\quad + \frac{(k+1)(n-k-1)\mu_2}{n} Y^{(k)}(t)\, dt. \qquad (4.7)
\end{aligned}$$

An application of Corollary 4.1.1 gives us that

$$III = -Y^{(k)}(t)\sum_{i=1}^{n-1} dW^i_i - \frac{(n-1)^2\mu_2}{2n(n+2)} Y^{(k)}(t)\, dt. \qquad (4.8)$$

Other terms are simplified along the similar lines using Theorem 4.1.1, Theorem 4.1.2 and Corollary 4.1.1.

$$
\begin{aligned}
IV &= k\,S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i,j=1}^{n}(\langle\tau_i^j\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^j\alpha^{\vec{m}}(t))\,\langle dW_j^i,dW_n^n\rangle\|\alpha(t)\|^{-1}\\
&= k\,S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i,j=1}^{n}(\langle\tau_i^j\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^J\alpha^{\vec{m}}(t)\rangle)\|\alpha(t)\|^{-1}C^{in}_{jn}\,dt\\
&= \frac{k\mu_2}{n(n+2)}S^{(k)}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i,j=1}^{n}(\langle\tau_i^j\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^j\alpha^{\vec{m}}(t)\rangle)\\
&\quad [(n+1)\delta_n^i\delta_n^j-\delta_j^i\delta_n^n-\delta_n^i\delta_n^j]\|\alpha(t)\|^{-1}\,dt\\
&= \frac{k\mu_2}{n(n+2)}S^{(k)}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i=1}^{n}(\langle\tau_i^i\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^i\alpha^{\vec{m}}(t)\rangle)\\
&\quad [(n+1)\delta_n^i\delta_n^i-\delta_i^i\delta_n^n-\delta_n^i\delta_n^i]\|\alpha(t)\|^{-1}\,dt\\
&= -\frac{2k(n-k-1)\mu_2}{n(n+2)}Y^{(k)}(t)\,dt. \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
V &= -k\,S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i=1}^{n-1}\langle dW_n^n,dW_i^i\rangle\langle\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1}dt\\
&= -\frac{k\mu_2}{n(n+2)}S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i=1}^{n-1}\langle\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1}[(n+1)\delta_n^i\delta_n^i-\delta_i^i\delta_n^n-\delta_n^i\delta_n^i]\,dt\\
&= \frac{k(n-1)\mu_2}{n(n+2)}Y^{(k)}(t)dt. \qquad (4.10)
\end{aligned}
$$

$$
\begin{aligned}
VI &= -S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i,j=1,p=1}^{n,n-1}(\langle\tau_i^j\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^j\alpha^{\vec{m}}(t)\rangle)\langle dW_j^i,dW_p^p\rangle\|\alpha(t)\|^{-1}\\
&= -\frac{\mu_2}{n(n+2)}S^{(k)}_{\nu_t}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\sum_{i=1,p=1}^{n,n-1}(\langle\tau_i^i\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle+\langle\alpha^{\vec{l}}(t),\tau_i^i\alpha^{\vec{m}}(t)\rangle)\\
&\quad [(n+1)\delta_p^i\delta_p^i-\delta_i^i\delta_p^p-\delta_p^i\delta_p^i]\|\alpha(t)\|^{-1}dt\\
&= -\frac{2(n-k-1)\mu_2}{n(n+2)}Y^{(k)}(t)dt. \qquad (4.11)
\end{aligned}
$$

Adding the terms from (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11), we get the required result. □

We now have everything we need to present the main new result of Part I.

Theorem 4.1.4 *Let M be a smooth $(n-1)$-dimensional manifold embedded in $\mathbb{R}^n$ and M_t its image at time t under the stochastic, isotropic, and volume preserving flow Φ_t described in Section 2.1. Let $\mathcal{L}_k$ be the Lipschitz-Killing curvatures defined in* (3.5)*, $S_{\nu_t}^{(k)}$ be the induced k-form defined in* (3.4)*, where ν_t is the unit normal vector field defined in* (3.16)*, $\alpha^{\vec{l}}$, $\alpha_{\vec{l_p}}$ be as defined in* (2.9) *and* (4.2) *respectively and $Y^{(k)}$ be as defined in* (4.4)*. Furthermore, let W_q^p and B_{jk}^i, defined in* (2.4) *and* (2.5) *respectively, be the spatial derivatives of the vector field driving the flow. Then the Itô formula for the Lipschitz-Killing curvatures is given by*

$$\begin{aligned} d\mathcal{L}_{n-k-1}(M_t) &= \Big[K_{n,k}\int_{M_t}\int_{S(\mathbb{R})}\Big([\sum_{i,\,p=1}^{k} S_{\nu_t}^{(k-1)}(\alpha_{\vec{l_p}}(t),\alpha_{\vec{m}_i}(t))\langle dB(u_{l_p}(t),u_{m_i}(t)),\nu_t\rangle] \\ &\quad \langle\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1} \\ &\quad + Y^{(k)}(t)[kdW_n^n - \sum_{i=1}^{n-1} dW_i^i]\|\alpha(t)\| \\ &\quad + \sum_{i,\,j} S^{(k)}(\alpha_{\vec{l}},\alpha_{\vec{m}})(\langle\tau_i^j\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle + \langle\alpha^{\vec{l}}(t),\tau_i^j\alpha^{\vec{m}}(t)\rangle)dW_j^i\|\alpha(t)\|^{-1}\Big) \\ &\quad \times 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\,\mathcal{H}_{n-1}(dx_t)\Big] \\ &\quad + \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)}\mathcal{L}_{n-k-1}(M_t)dt, \end{aligned} \tag{4.12}$$

where $\mathcal{H}_k$ is the k-dimensional Hausdorff measure and $N_{x_t}M_t$ is the normal cone to M_t at $x_t \in M_t$.

Proof: Using (3.5) and (4.4) we rewrite the Lipschitz-Killing curvatures as,

$$\begin{aligned} \mathcal{L}_{n-k-1}(M_t) &= K_{n,k}\int_{M}\int_{S(\mathbb{R})} S_{\nu_t}^{(k)}(\alpha_{\vec{l}}(t),\alpha_{\vec{m}}(t))\langle\alpha^{\vec{l}}(t),\alpha^{\vec{m}}(t)\rangle\|\alpha(t)\|^{-1} \\ &\quad \times 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\,\mathcal{H}_{n-1}(dx) \\ &= K_{n,k}\int_{M}\int_{S(\mathbb{R})} Y^{(k)}(t)\times 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\,\mathcal{H}_{n-1}(dx). \end{aligned}$$

Hence,

$$d\mathcal{L}_{n-k-1}(M_t) = K_{n,k}\int_{M}\int_{S(\mathbb{R})}(dY^{(k)}(t))\times 1_{N_{x_t}M_t}(-\nu_t)\,\mathcal{H}_0(d\nu_t)\,\mathcal{H}_{n-1}(dx).$$

Now using Theorem 4.1.3 we obtain the desired result. □

Remark 4.1.1 *It is clear from the results obtained by Cranston and Le Jan [12] and the expression obtained in the above Theorem 4.1.4, that the vector of Lipschitz-Killing curvatures is not a diffusion due to the presence of the first term in* (4.12)*.*

An immediate consequence of the above theorem is an exact expression for the mean of Lipschitz-Killing curvatures.

Corollary 4.1.2 *Under the conditions of Theorem 4.1.4,*

$$E(\mathcal{L}_{n-k-1}(M_t)) = \mathcal{L}_{n-k-1}(M) \exp\Big(\frac{(n-k-1)(n+1)(k+1)\mu_2 t}{2n(n+2)}\Big). \tag{4.13}$$

In particular, for $k = (n-1)$*, we have* $E(\mathcal{L}_0(M_t)) = \mathcal{L}_0(M)$*, for all* t*, which is what we expect, as* $\mathcal{L}_0(M)$ *is the Euler characteristic of* M *and so is invariant under diffeomorphisms.*

Proof: In (4.12), we note that except for the last term, the other terms are zero mean martingales. Therefore, taking expectations of (4.12), after taking the integral over time t, will yield

$$E(\mathcal{L}_{n-k-1}(M_t)) = \frac{(n-k-1)(n+1)(k+1)\mu_2}{2n(n+2)} \int_0^t E(\mathcal{L}_{n-k-1}(M_s))ds.$$

Solving this linear differential equation we obtain the required result. □

Part II

Fractional Brownian motion and stochastic flows

Chapter 5

Introduction and background

Continuing from where we left in the previous chapter, here we shall present some similar results, but in a different setting, attempting to extend the results available for the diffusive Brownian flow to non-diffusive flows. By non-diffusive flows, we mean flows *driven* by a non-diffusive process, in particular, a fractional Brownian motion.

The fractional Brownian motion $\{B^H(t),\ t \geq 0\}$ with Hurst parameter $H \in (0,1)$, is the zero mean Gaussian process with covariance function

$$E[B^H(s)B^H(t)] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \tag{5.1}$$

Note that $E|B^H(t) - B^H(s)|^2 = |t-s|^{2H}$, and hence the process B^H has stationary increments. Moreover, when $H = \frac{1}{2}$, the process also has independent increments and this case corresponds to the standard Brownian motion. A simple application of a Garsia-Rodemich-Rumsey type of inequality, together with (5.1), implies that the process B^H has α-Holder continuous paths for all $\alpha \in (0,H)$. (See [17] for the original inequality, or [30] for the application of the inequality to fractional Brownian motion.) It is also important to note that the process B^H, for $H \neq \frac{1}{2}$, is neither a semimartingale nor a Markov process.

In order to construct a non-diffusive flow, we start with a collection of independent fractional Brownian motions, $\{B_\gamma^H\}_{\gamma\in\mathbb{N}}$ and define, for some set $I \subset \mathbb{N}$,

$$U_I(x,t) = \sum_{\gamma\in I} U_\gamma(x) B_\gamma^H(t), \tag{5.2}$$

where $\{U_\gamma\}_{\gamma\in\mathbb{N}}$ is a collection of deterministic vector fields defined on $\mathbb{R}^n$ such that, for $I = \mathbb{N}$, $U_{\mathbb{N}}(\cdot,\cdot)$ is an isotropic Gaussian vector field. For the moment, we do not impose any conditions on the vector fields, although we shall add various conditions as and when required.

Now we introduce a candidate for the evolution equation describing a stochastic flow driven by a fractional Brownian motion by setting

$$\Phi_t(x) = x + \sum_{\gamma\in I} \text{``} \int_0^t U_\gamma(\Phi_s(x))\, dB_\gamma^H(s)\text{''}. \tag{5.3}$$

For $H = \frac{1}{2}$, the integral appearing in the above expression can be interpreted as either an Itô or a Stratonovich integral. When $H \neq \frac{1}{2}$, although the standard semimartingale arguments cease to work, there is a plethora of literature available on various ways to define an integral $\int_a^b f(s)\, dB^H(s)$, where $f(s)$ denotes some random process and B^H the fractional Brownian motion. See, for instance, [2, 7, 13, 24, 16].

In the remaining part of this chapter, we shall present an overview of the various available results on defining integrals with respect to fractional Brownian motion, without rigorously defining the various terms involved. The details will be presented in Chapters 6 and 7. In Chapter 8, we shall return to the main results of this part of the thesis, and develop analogues of the results of Chapter 4, for non-diffusive flows.

One of the earliest efforts at dealing with integral/differential equations driven by fractional Brownian motion for $H > \frac{1}{2}$ can be attributed to Lyons [25] in 1994. The idea was primarily based on Young's [42] striking analysis of integrals driven by non-smooth functions. The existence and uniqueness of the solution of an integral equation (5.3) was proven by using the p-variation of iterated integrals of the process B^H to derive Lipschitz type behaviour for various iterations in a Picard's iteration scheme. Later, in 1998, Lyons generalized the argument in [26] and extended the theory to various other cases, including $H = \frac{1}{2}$. This was further generalized to the case $H > \frac{1}{4}$ by Unterberger [37].

Around the same time, Zähle [43, 44], defined pathwise integrals of the form $\int_0^t u_s dB^H(s)$, for $H > \frac{1}{2}$, where u and B^H were considered to be elements of fractional Sobolev space. Apart from proving an Itô type formula, connections were established between the pathwise definition of the stochastic integral with various other interpretations of integrals. Elsewhere, Ruzmaikina [33] independently, obtained similar results by approximating the stochastic integral by a Riemann-Stieltjes sum.

Following the above method of using the pathwise definition of a stochastic integral driven by fractional Brownian motion, Nualart and Răşcanu ([30]) proved existence and uniqueness of the solutions of multidimensional stochastic differential equations of the form

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, dB^H(s) + \int_0^t b(s, X_s)\, ds,$$

where B^H is a fractional Brownian motion with $H > \frac{1}{2}$, and the integral with respect to B^H is a pathwise Riemann-Stieltjes integral, as defined by Zähle and others. Later, continuing in the same vein, Decreusefond and Nualart in [14] proved the existence of a homeomorphic stochastic flow driven by fractional Brownian motion. We shall have more to say about this in detail in Chapter 7.

While the pathwise approach towards defining integrals driven by fractional Brownian motion was gaining momentum, Decreusefond and Üstünel ([16]), at essentially the same time, approached the problem in an altogether different way by utilizing the Gaussian character of fractional Brownian motion, and employing the stochastic calculus of variations called the *Malliavin calculus*[1]. They extended the Malliavin calculus, which was primarily designed for the Wiener process, to fractional Brownian motion and developed the stochastic calculus for fractional Brownian motion. This was further extended in [2], where stochastic calculus with respect to *Volterra processes* of the form, $Y_t = \int_0^t K(t,s)\, dW(s)$, was developed, where W is a standard Wiener process and $K(t,s)$, a square integrable kernel, is called the Volterra kernel. This covered the case of fractional Brownian motion for a specific choice of the kernel $K(\cdot,\cdot)$. The analysis of the stochastic calculus for the fractional Brownian motion was further refined in [3] by Nualart et.al., where some L^p estimates of the divergence

[1]For a quick introduction to Wiener space, we refer the reader to the appendix on "An Introduction to Malliavin Calculus" in [39], whereas for an excellent detailed account on the same topic, we refer the refer the reader to [38], and finally, to [27] for a more analytical treatment and its various applications.

integral were presented. Chapter 6 is devoted to the study of this approach towards defining stochastic integrals, with the aim of convincing the reader that an attempt to define a stochastic flows with the integral interpreted as a divergence operator is unlikely to work. Subsequently, in Chapter 7, we shall present the background required for defining pathwise integrals and we shall give a brief argument for the existence and uniqueness of stochastic flows in a pathwise sense.

Finally, once we settle with a definition to interpret (5.3), we shall start analyzing the evolution of the geometric characteristics of a randomly evolving manifold under the flow, which will be the central theme of Chapter 8. The main goal is to achieve reasonable upper bounds for the basic characteristics of the randomly evolving manifolds, *viz.* the appropriate Hausdorff measure of the manifold. We shall show that the $(n-1)$-dimensional Hausdorff measure of an $(n-1)$-dimensional manifold evolving under an n-dimensional stochastic flow, exhibits a growth which is almost surely bounded by an exponential function with the exponent depending on the appropriate Hölder norm of the fractional Brownian motion.

Chapter 6

Stochastic calculus of variations

As described in the previous chapter, there is a significant literature available on the topic of defining integrals driven by fractional Brownian motion using the Malliavin calculus or stochastic calculus of variations. (See [28] for a survey of the various results available on this topic.) For a variety of technical reasons, we shall restrict our attention to the case $H > \frac{1}{2}$. These are processes that are smoother than standard Brownian motion.

This section is devoted to reviewing the results related to the *divergence integral*, and trying to implement them to interpret the integral appearing in (5.3).

6.1 Preliminaries on fractional Brownian motion and the Wiener integral

We shall start with $\mathcal{S}$ as the set of step functions on $[0, T]$, and denote $\mathcal{H}_H$ as the Hilbert space defined as the closure of $\mathcal{S}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}_H} = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) = R_H(s,t).$$

Now consider a Gaussian field $\{B^H(\phi) : \phi \in \mathcal{H}_H\}$, with its covariance function given by

$$E[B^H(\phi)B^H(\psi)] = \alpha_H \int_0^T \int_0^T \phi(r)\psi(s)|r-s|^{2H-2}\, drds \triangleq \langle \phi, \psi \rangle_{\mathcal{H}_H}. \qquad (6.1)$$

where $\alpha_H = H(2H-1)$.

Note that the map $\phi \to B^H(\phi)$ defines an isometry between $\mathcal{H}_H$ and the Gaussian space $H(B)$, which is the space of random L^2 random variables of the form $B^H(\phi)$. It is customary to interpret $B^H(\phi)$ as the *divergence integral* of $\phi \in \mathcal{H}_H$ with respect to B^H.

Remark 6.1.1 *For the case $H = \frac{1}{2}$, the divergence integral is the Wiener integral, and it is for this case that the standard Malliavin calculus was designed. Also note that for this case* (6.1) *takes the familiar form*

$$E[B^{1/2}(\phi)B^{1/2}(\psi)] = \int_0^T \phi(s)\psi(s)\, ds.$$

Remark 6.1.2 *The Hilbert space $\mathcal{H}_H$ is also called the reproducing kernel Hilbert space for fractional Brownian motion with Hurst parameter H. Although in some books the same space is also referred to as the Cameron-Martin space, we shall make a distinction between these two spaces and we shall denote the later by $\mathcal{H}_H^*$. A closer look at these two spaces (see, for instance, [16] and [3]), shows that one is a mere transformation of the other. We shall explain this with reasonable generality in the following section. A typical element of $\mathcal{H}_H$ is generally not a function, but rather a distribution of negative order (see [2] or [32]). In fact the space $\mathcal{H}_H$ coincides with the space of distributions f such that $s^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}}(u^{H-\frac{1}{2}} f(u))(s)$ is a square integrable function, where $I_{0+}^{H-\frac{1}{2}}$ is the left-sided fractional integral of order $H - \frac{1}{2}$. This and more about deterministic fractional calculus will constitute our next Chapter 7. An appropriate definition of $\mathcal{H}_H^*$ will be formulated in the next section, which is devoted to the study of the stochastic calculus of variations.*

Remark 6.1.3 *In the special case $H = \frac{1}{2}$, $\mathcal{H}_H \equiv L^2([0,T])$, and $\mathcal{H}_H^*$ is the space of absolutely continuous functions on $[0,T]$ with square integrable derivatives.*

The key to stochastic calculus for fractional Brownian motion, or Volterra processes in general, is the representation (see [2], [28])

$$B^H(t) = \int_0^t K_H(t,s)\, dB(s),$$

where B is a standard Brownian motion and $K_H(t,s)$ is the $L^2[0,T]$ kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du,$$

for $s < t$ and 0 otherwise, and c_H is a normalizing constant such that

$$R_H(s,t) = \int_0^{s\wedge t} K_H(s,u) K_H(t,u)\, du.$$

(This, by the way, proves that $R_H(t,s)$ is nonnegative definite.)

The kernel $K_H(\cdot,\cdot)$ can itself be regarded as an operator in $L^2([0,T])$ defined via the correspondence

$$(K_H\phi)(t) = \int_0^t K_H(t,u)\phi(u)\, du,$$

for $\phi \in L^2([0,T])$.

Furthermore, it can also be used to define an operator $K_H^* : \mathcal{S} \to L^2([0,T])$, given by

$$(K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s), \tag{6.2}$$

which is extended as a linear isometry to all of $\mathcal{H}_H$. For our choice of $H > \frac{1}{2}$, the action of the operator K_H^* can be expressed as,

$$(K_H^*\phi)(s) = \int_s^T \phi(r) \frac{\partial K_H}{\partial r}(r,s)\, dr.$$

The relationship between the operators K_H and K_H^*, one as the adjoint of the other, can be seen in the following lemma.

Lemma 6.1.1 *For any $\phi \in \mathcal{S}$ and $h \in L^2([0,T])$, we have*

$$\int_0^T (K_H^*\phi)(t)h(t)\, dt = \int_0^T \phi(t)(K_H h)(dt),$$

where the integral on the right hand side is interpreted as a standard Riemann-Stieltjes integral with respect to the function $K_H h$.

For the proof of this lemma and a general treatment of integrals with respect to Gaussian processes, we refer the reader to [2].

An immediate consequence of the isometry K_H^*, between $\mathcal{H}_H$ and $L^2([0,T])$ is the transfer rule between the standard Wiener integral and the Gaussian process $B(\phi)$ defined by (6.1), given by

$$B^H(\phi) = W(K_H^*\phi), \tag{6.3}$$

for any $\phi \in \mathcal{H}_H$, and where $W(\cdot)$ is the Wiener integral defined on $L^2([0,T])$.

Consequently, it is clear from (6.2) and (6.3) that the divergence integral with respect to the fractional Brownian motion is an *anticipative* integral. Thus interpreting the integral appearing in (5.3) as a divergence integral would result in an anticipative stochastic differential equation, which is difficult to solve except in some simple cases. A brief argument explaining the difficulty in solving such stochastic differential equations will be the focus of our next subsection.

6.2 Malliavin calculus

As we mentioned above in Remark 6.1.2, corresponding to the Hilbert space $\mathcal{H}_H$ we can associate another space $\mathcal{H}_H^*$, called the Cameron-Martin space, such that for each element h^* in $\mathcal{H}_H^*$, there exists an $h \in \mathcal{H}_H$, such that

$$h^*(t) = \int_0^t K_H(t,s)(K_H^* h)(s)\, ds. \tag{6.4}$$

Therefore the space $\mathcal{H}_H^*$ can also be identified as a Hilbert space with the inner product induced by $\mathcal{H}_H$, i.e., for $h^*, g^* \in \mathcal{H}_H^*$

$$\langle h^*, g^* \rangle_{\mathcal{H}_H^*} \triangleq \langle h, g \rangle_{\mathcal{H}_H}.$$

For a comprehensive study of the space $\mathcal{H}_H^*$, we refer the reader to [16].

Now let us define $\mathcal{E}$ to be the set of smooth cylindrical random variables of the form

$$F(B^H) = f(B^H(\phi_1^*), \ldots, B^H(\phi_n^*)), \tag{6.5}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,, $\phi_i^* \in \mathcal{H}_H^*$, and $B^H(\phi_i^*) \triangleq B^H(\phi_i)$, where ϕ_i is the element in $\mathcal{H}_H$ corresponding to $\phi_i^* \in \mathcal{H}_H^*$. Recall that $B^H(\phi)$ for $\phi \in \mathcal{H}_H$ was defined as a Gaussian process with the covariance function given by (6.1).

The above definition of cylindrical random variables can also be considered as a map $B^H \mapsto f(B^H(\phi_1^*), \ldots, B^H(\phi_n^*)) = F(B^H) \equiv F$. The idea being, to study the calculus of variation with respect to the underlying "randomness" induced by B^H.

Let W denote the Banach space of continuous functions from $[0, 1]$ to $\mathbb{R}$, and equip it with the measure induced by fractional Brownian motion B^H. Then it is natural to expect that differentiation on this space is of a Fréchet kind. However, the existence of such a derivative requires the mapping $w \mapsto F(w)$, for $w \in W$, to be continuous in the norm topology of W. However, for most, at very least, F this is not true. For instance, a general diffusion with reasonably smooth coefficients is not continuous with respect to the underlying Brownian motion, due to the presence of the *correction term* marking the difference between the Stratonovich and Itô representations of the stochastic integral.

Moreover, the fact that almost all functionals in probability theory are defined up to equivalence classes induced by the underlying measure suggests that an appropriate definition of the derivative of the Wiener functionals must be well defined up to the equivalence classes, given by the Cameron-Martin theorem, of the Wiener functionals. (See [16] or [39], for an exact formulation of the Cameron-Martin theorem for abstract Wiener space.) Hence, a Sobolev type of differentiation rule is better suited, which is well-defined for these equivalence classes.

A standard way of defining derivatives in abstract spaces is by choosing a direction in which the perturbation is introduced, and then defining the resultant limit as the gradient in the chosen direction. However, since not all the directions are *feasible*, as noted above, only the directions belonging to the Cameron-Martin space qualify for defining the derivative. For a F defined as above, its gradient in the direction $h^* \in \mathcal{H}_H^*$ is given by

$$\nabla_{h^*}^H F(w) = \frac{d}{d\lambda} F(w + \lambda h^*)|_{\lambda=0} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\phi_1^*), \ldots, B^H(\phi_n^*)) \langle h^*, \phi_i^* \rangle_{\mathcal{H}} \quad (6.6)$$

Considering this relationship as a linear, continuous functional on $\mathcal{H}_H^*$, there exists a map $w \mapsto (D_H \phi(w))(s)$ with values in $\mathcal{H}_H$, as a consequence of Riesz representation theorem, such that

$$\nabla_{h^*}^H F(w) = \int_0^T \int_0^T (D_H F(w))(s) h(r) |r-s|^{2H-2} \, dr ds, \tag{6.7}$$

where h is the element in $\mathcal{H}_H$ corresponding to $h^* \in \mathcal{H}_H^*$.

We shall use the symbols $(D_H F)(h)$ and $\nabla_h^H F$, interchangeably. Comparing the expressions in (6.6) and (6.7), we shall write

$$D_H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\phi_1^*), \dots, B^H(\phi_n^*))\phi_i. \tag{6.8}$$

This derivative operator is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}_H)$ for any $p \geq 1$, which means that if for a sequence of random variables $\{F_n\}_{n\geq 1} \subset \mathcal{E}$ converging to zero in $L^p(\Omega)$, $\{D_H F_n\}_{n\geq 1}$ is a Cauchy sequence in $L^p(\Omega; \mathcal{H}_H)$, then $\{D_H F_n\}_{n\geq 1}$ also converges to zero in $L^p(\Omega; \mathcal{H}_H)$. We refer the reader to Proposition B.3.1. in [39], for a proof of the above fact in the case of $H = \frac{1}{2}$, which in principle works for $H > \frac{1}{2}$ too.

Writing D_H^k for the iteration of the derivative operator, we define the Sobolev space $\mathbb{D}_H^{k,p}$ as the closure of $\mathcal{E}$ with respect to the norm given by

$$\|F\|_{k,p}^p = E|F|^p + \sum_{i=1}^k E\|D_H^i F\|_{\mathcal{H}_H^{\otimes i}}^p. \tag{6.9}$$

Similarly, given a Hilbert space V we shall denote by $\mathbb{D}_H^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables.

We have now reached the main point of this section, that of defining the divergence operator δ_H, which in simple words is defined as the adjoint of the derivative operator given by the duality relationship

$$E(F\delta_H(u)) = E\langle D_H F, u\rangle_{\mathcal{H}_H},$$

where u is an element in the domain of the operator δ_H, which is defined as the class of $u \in L^2(\Omega; \mathcal{H}_H)$ such that

$$|E\langle D_H F, u\rangle_{\mathcal{H}_H}| \leq c_u \|F\|_{L^2(\Omega)},$$

for any $F \in \mathcal{E}$.

Remark 6.2.1 *The divergence operator defined above using the duality relationship has its roots in the Gauss divergence theorem, which establishes a relationship between the gradient and the divergence operators, one as the negative of the dual of the other.*

Remark 6.2.2 *It is customary to write* $\delta_H(u) = \int_0^T u(s)\delta B^H(s)$*, and to make it clearer, we note that* $B^H(\phi) = \delta_H(\phi)$*, for deterministic* $\phi \in \mathcal{H}_H$*, and for non deterministic (random)* ϕ*, the divergence (integral), for* $H = \frac{1}{2}$*, coincides with the generalized Itô stochastic integral introduced by Skorohod for non adapted integrands (see [36]).*

Some of the basic properties of the divergence operator δ are listed below:

- The space $\mathbb{D}_H^{1,2}(\mathcal{H}_H)$ is contained in the domain of the divergence operator δ_H.
- For any $u \in \mathbb{D}_H^{1,2}(\mathcal{H}_H)$ we have

$$E(\delta_H(u))^2 = E\|u\|^2_{\mathcal{H}_H} + E\langle D_H u, (D_H u)^*\rangle_{\mathcal{H}_H \otimes \mathcal{H}_H}, \tag{6.10}$$

  where $(D_H u)^*$ is the adjoint of $(D_H u)$ in the Hilbert space $\mathcal{H}_H \otimes \mathcal{H}_H$.

- For any $F \in \mathbb{D}_H^{1,2}$ and any u in the domain of δ_H such that Fu and $F\delta_H(u) + \langle D_H F, u\rangle_{\mathcal{H}_H}$ are square integrable, the Fu is in the domain of δ_H and

$$\delta_H(Fu) = F\delta_H(u) - \langle D_H F, u\rangle_{\mathcal{H}_H}.$$

Embedded in the Hilbert space $\mathcal{H}_H$ is the Banach space $|\mathcal{H}_H|$ whose norm is given by

$$\|\phi\|_{|\mathcal{H}_H|} = \int_{[0,t]^2} |\phi(s)|\ |\phi(r)|\ |r-s|^{2H-2}\, dr\, ds.$$

Along similar lines we can define $|\mathcal{H}_H| \otimes |\mathcal{H}_H|$, and as before this will define a Banach space with respect to the norm $\|\cdot\|_{|\mathcal{H}_H|\otimes|\mathcal{H}_H|}$. Clearly, this space is isometric to a subspace of $\mathcal{H}_H \otimes \mathcal{H}_H$ and it is identified with this subspace. We are interested in this space as it forms a natural basis for the analysis that we are soon going to start.

Our main objective is to be able to interpret the integral appearing in (5.3) as a divergence integral. The two most common and robust methods of obtaining existence proofs, namely the Banach fixed point theorem and Picard's iteration, both hinge on the *Lipschitz* behavior of the functional involved, which in our case is the divergence operator. (Note that a contraction also exhibits Lipschitz behavior.) This is precisely what is missing in this set up. We shall see this in the following illustration of the Picard's iteration scheme.

Let us assume that $U(x,t) = U(x)B^H(t)$, and define $x^n(t)$ iteratively as follows

$$x^n(t) = x + \int_0^t U(x^{n-1}(t))\, \delta B^H(s),$$

with $x^0 \equiv x$. Then,

$$|x^{n+1}(t) - x^n(t)| = \Big| \int_0^t (U(x^n(s)) - U(x^{n-1}(s)))\, \delta B^H(s) \Big|. \tag{6.11}$$

Now we require an L^p bound on the right hand side of above expression involving terms like $|x^n(t) - x^{n-1}(t)|$ only.

By an immediate consequence of Meyer's inequalities (see, for instance [27]), for $p \geq 1$, a process $u \in \mathbb{D}_H^{1,p}(|\mathcal{H}_H|)$ belongs to the domain $Dom(\delta_H)$ of the divergence in $L^p(\Omega)$, and we have

$$E|\delta(u)| \leq C_{H,p}\big(\|Eu\|^p_{|\mathcal{H}_H|} + E\|Du\|^p_{|\mathcal{H}_H|\otimes|\mathcal{H}_H|}\big).$$

This together with (6.10), implies that it is not likely that our requirement of the bound on the right side of (6.11) is fulfilled, unless there are bounds available for the gradient $D_H F$ in terms of F for any $F \in \mathcal{E}$. This, of course, is not generally the case. Hence interpreting the integral in (5.3) as a divergence integral is not possible, due to our inability to find answers to questions concerning the existence of a solution.

Remark 6.2.3 *It is worth noting here that, at least for some simple U, it is possible to construct the integral equation using the divergence integral. For example, for $U(x) = x$, a simple chaos expansion of the solution exists, whereas for the case of general U, the solution can be shown to exist for small t, using the Taylor series expansion of U around 0, and hence reducing it to the* linear *case, with the added conditions that the higher order terms in the Taylor series expansion do not contribute much. (See [4].) However, neither of these cases are of any help in our scenario.*

Chapter 7

A pathwise approach

Another natural way to define the integrals with respect to fractional Brownian motion is the *pathwise* approach using the deterministic fractional calculus. Primarily, as noted in the introduction, this method is based on Young's way of defining integrals with respect to Hölder functions. Although there is a plethora of literature available on this topic, we shall closely follow Zähle's approach as it appeared in [43], and we shall borrow heavily from Nualart's various papers on the subject.

We start by listing some of the basic formulae required from the deterministic fractional calculus, and the fractional spaces associated with them.

For $a, b \in \mathbb{R}$, $a < b$, let $L^p(a,b)$, $p \geq 1$, be the space of Lebesgue measurable functions $f : [a,b] \to \mathbb{R}$ with $\|f\|_{L^p(a,b)} < \infty$, where

$$\|f\|_{L^p(a,b)} = \begin{cases} (\int_a^b |f(x)|^p \, dx)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty \\ ess \sup |f(x)| : x \in [a,b], & \text{if} \quad p = \infty. \end{cases}$$

The left sided fractional Riemann-Liouville integral of $f \in L^1(a,b)$ of order $\alpha > 0$ is given by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) \, dy,$$

for almost all $x \in (a,b)$, where $\Gamma(\alpha)$ is the standard Euler function. Similarly, the right sided fractional integral is defined, for almost all $x \in (a,b)$, as

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) \, dy,$$

where $(-1)^{-\alpha} = e^{-i\pi\alpha}$. If we consider the fractional integral I_{a+}^{α} (or I_{b-}^{α}) as an operator with domain $L^p(a,b)$, then the range space is denoted by $I_{a+}^{\alpha}(L^p(a,b))$ (or

$I^{\alpha}_{b-}(L^p(a,b)))$. Clearly, for $\alpha = 1$, I^{α}_{a+} is the standard left integral operator, and a simple calculation yields that $\lim_{\alpha\to 0}(I^{\alpha}_{a+}f)(x) = f(x-) = \lim_{\epsilon\downarrow 0} f(x-\epsilon)$, for each $x \in (a,b)$. An immediate consequence of the definition of the fractional integral is the following property:

$$I^{\alpha}_{a+}(I^{\beta}_{a+}f) = I^{\alpha+\beta}_{a+}f, \tag{7.1}$$

for all $\alpha, \beta > 0$. With some obvious variations wherever needed, all the above properties are true for the right sided fractional integrals too and they are listed as follows:

$$\begin{aligned}
(I^{1}_{b-}f)(x) &= (-1)\int_x^b f(y)\,dy,\\
\lim_{\alpha\to 0}(I^{\alpha}_{b-}f)(x) &= f(x+)\\
&= \lim_{\epsilon\downarrow 0} f(x+\epsilon), \text{ and}\\
I^{\alpha}_{b-}(I^{\beta}_{b-}f) &= I^{\alpha+\beta}_{b-}f,\ \forall\alpha,\beta > 0
\end{aligned}$$

(See [43] for these and more on fractional calculus.)

In order to better understand the linear spaces $I^{\alpha}_{a+}(L^p(a,b))$, we can write $f \in I^{\alpha}_{a+}(L^p(a,b))$ if and only if $f \in L^p(a,b)$ and $\lim_{\epsilon\to 0}\int_a^{x-\epsilon}\frac{f(x)-f(y)}{(x-y)^{1+\alpha}}\,dy$ exists in $L^p(a,b)$ as a function in $x \in (a,b)$. Furthermore, if $p\alpha < 1$ then $I^{\alpha}_{a+}(L^p(a,b)) = I^{\alpha}_{b-}(L^p(a,b)) \subset L^q(a,b)$, with $q = p/(1-p\alpha)$, and if $p\alpha > 1$ then $f \in I^{\alpha}_{a+}(L^p(a,b))$ implies that f is $(\alpha - 1/p)$-Hölder continuous function on the interval (a,b). These and many more such results can be found in [34].

Having defined a fractional integral, we now define a fractional derivative as the *inverse* of the fractional integral operator, whenever it is well defined. In other words, each element f in $I^{\alpha}_{a+}(L^p(a,b))$ has a corresponding $\phi \in L^p(a,b)$, such that $I^{\alpha}_{a+}\phi = f$, which is unique in $L^p(a,b)$ and agrees almost everywhere with the appropriate fractional derivative of f. More precisely, the left sided Riemann-Liouville derivative, also called the Weyl derivative, of α^{th}-order of $f \in I^{\alpha}_{a+}(L^p)$ is defined as:

$$\begin{aligned}
D^{\alpha}_{a+}f(x) &= \left(\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(y)}{(x-y)^{\alpha}}\,dy\right)1_{(a,b)}(x)\\
&= \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{1+\alpha}}\,dy\right)1_{(a,b)}(x).
\end{aligned} \tag{7.2}$$

Equivalently, we can write $D^{\alpha}_{a+} = D(I^{1-\alpha}_{a+}f)$, where D is the standard derivative operator. Similarly, we can define the right sided Weyl derivative as $D^{\alpha}_{b-} = D(I^{1-\alpha}_{b-}f)$, and

$$\begin{aligned} D^{\alpha}_{b-}f(x) &= \Big(\frac{(-1)^{\alpha-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(y)}{(y-x)^{\alpha}}\,dy\Big)\,1_{(a,b)}(x) \\ &= \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\Big(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{1+\alpha}}\,dy\Big)\,1_{(a,b)}(x). \end{aligned} \tag{7.3}$$

As in the case of the integral operators, there is an analogue of the composition formula, given, for all $\alpha, \beta > 0$, by

$$D^{\alpha}_{a+}(D^{\beta}_{a+}f) = D^{\alpha+\beta}_{a+}f. \tag{7.4}$$

A similar formula also holds for the right sided derivatives, and is given by,

$$D^{\alpha}_{b-}(D^{\beta}_{b-}f) = D^{\alpha+\beta}_{b-}f, \tag{7.5}$$

as long as all the fractional derivatives are well defined.

We note that the linear spaces $I^{\alpha}_{a+}(L^p(a,b))$, for various choices of α and p, are Banach spaces equipped with the norms

$$\|f\|_{I^{\alpha}_{a+}(L^p(a,b))} = \|f\|_{L^p(a,b)} + \|D^{\alpha}_{a+}f\|_{L^p(a,b)},$$

and a similar norm is defined on the space $I^{\alpha}_{b-}(L^p(a,b))$.

Using the methods of fractional calculus, one can extend the standard integration by parts formula to the more general case of L^p functions. Hence the generalized *integration by parts formula* can be written as

$$\int_a^b f(x)I^{\alpha}_{a+}g(x)dx = (-1)^{\alpha}\int_a^b g(x)I^{\alpha}_{b-}f(x)dx, \tag{7.6}$$

where $f \in L^p(a,b),\ g \in L^q,\ p \geq 1,\ q \geq 1,\ 1/p + 1/q \leq 1+\alpha,\ p,q > 1$, and $1/p+1/q = 1+\alpha$. A similar formula, called the *second integration by parts formula*, holds true for derivative operators, and is given as

$$(-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f(x)g(x)dx = \int_a^b f(x)D^{\alpha}_{b-}g(x)dx, \tag{7.7}$$

where $f \in I^{\alpha}_{a+}(L^p(a,b)),\ g \in I^{\alpha}_{b-}(L^q(a,b)),\ p \geq 1,\ q \geq 1,\ 1/p + 1/q \leq 1+\alpha.$

Let $f(a+) = \lim_{\epsilon\downarrow 0} f(a+\epsilon)$, and $g(b-) = \lim_{\epsilon\downarrow 0} f(b-\epsilon)$, whenever the limit exists and is finite, and define

$$\begin{aligned} f_{a+}(x) &= (f(x) - f(a+))1_{(a,b)}(x), \\ g_{b-}(x) &= (g(x) - g(b-))1_{(a,b)}(x). \end{aligned}$$

Using these definitions and the machinery developed above, an extension of the Stieltjes integral, called the *generalized Stieltjes integral*, of f with respect to g is defined as

$$\begin{aligned}\int_a^b f(x)dg(x) &= (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x)dx \\ &\quad + f(a+)(g(b-) - g(a+)),\end{aligned} \tag{7.8}$$

where $f_{a+} \in I_{a+}^\alpha(L^p(a,b))$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q(a,b))$ for some $p, q \geq 1,\ 1/p + 1/q \leq 1,\ 0 \leq \alpha \leq 1$. Furthermore, if we impose the condition $\alpha p < 1$, then $f_{a+} \in I_{a+}^\alpha(L^p(a,b))$ implies $f \in I_{a+}^\alpha(L^p(a,b))$, in which case (7.8) can be rewritten as

$$\int_a^b f(x)dg(x) = (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x)dx. \tag{7.9}$$

This representation is sometimes also referred to as the *forward integral representation* due to the choice of left and right sided derivatives for f and g respectively. By interchanging this choice of the left and the right-sided fractional derivatives in (7.9), for the integrand and the integrator respectively, we get what is called the *backward integral representation*, given by

$$\begin{aligned}\int_a^b f(x)\, dg(x) &= (-1)^\alpha \int_a^b D_{b-}^\alpha f_{b-}(x) D_{a+}^{1-\alpha} g_{a+}(x)\, dx \\ &\quad + f(b-)(g(b-) - g(a+))\end{aligned} \tag{7.10}$$

if $f_{b-} \in I_{b-}^\alpha(L^p)$, $g_{a+} \in I_{a+}^{1-\alpha}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$.

Remark 7.0.4 *It is important to note here that in view of* (7.4)*,* (7.5)*,* (7.6) *and* (7.7)*, the definition* (7.8)*, or equivalently* (7.9)*, is independent of the choice of* α*.*

Next we define $C^\lambda(0, T; \mathbb{R}^d)$, the space of λ-Hölder continuous functions, with $\lambda \in (0, 1]$, as the space of $\mathbb{R}^d$ valued functions for some fixed $d \in \mathbb{N}$, the space of natural numbers, equipped with the norm given by

$$\|f\|_\lambda := \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\lambda} < \infty,$$

where $\|f\|_\infty = \sup_{t \in [0,T]} |f(t)|$.

In [43], Zähle proved that the conditions of the definitions (7.8) and (7.9) are met if $f \in C^\lambda(0, T; \mathbb{R})$ and $g \in C^\mu(0, T; \mathbb{R})$ for $\lambda + \mu > 1$, in which case the integral defined in (7.8) or (7.9) coincides with the Riemann-Stieltjes integral. Now we state the following well known result concerning the Hölder coefficient and exponent of fractional Brownian motion with Hurst parameter H.

Lemma 7.0.1 *For $\{B^H(t) : t \in [0,T]\}$, a fractional Brownian motion with Hurst parameter $H \in (0,1)$, there exists a positive random variable $\eta_{\epsilon,T}$, for each $0 < \epsilon < H$ and $T > 0$, such that $E(|\eta_{\epsilon,T}|^p) < \infty$ for all $p \in [1,\infty)$ and for all $s,t \in [0,T]$*

$$|B^H(t) - B^H(s)| \;\leq\; \eta_{\epsilon,T}|t-s|^{H-\epsilon} \;\; a.s.,$$

where $\eta_{\epsilon,T} = C_{H,\epsilon}T^{H-\epsilon}\xi_T$, with the $L^q(\Omega)$ norm of ξ_T bounded by $c_{\epsilon,q}T^{\epsilon}$ for $q \geq \frac{2}{\epsilon}$.

(For a proof of this, we refer the reader to [30].)

Hence, the integrals with respect to the fractional Brownian motion can be proven by the results obtained in [43]. In the same article, the corresponding stochastic calculus is also developed with an appropriate formula for change of variables. These existence results can naturally be extended to the case of vector valued integrands.

For the following definitions, we shall assume $\alpha < \frac{1}{2}$.

Define $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, as the space of measurable functions $f : [0,T] \to \mathbb{R}^d$ equipped with the norm given by

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \big(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{|t-s|^{\alpha+1}}ds\big) < \infty. \tag{7.11}$$

A trivial observation following from this definition is that

$$C^{\alpha+\epsilon}(0,T;\mathbb{R}^d) \,\subset\, W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \,\subset\, C^{\alpha-\epsilon}(0,T;\mathbb{R}^d), \tag{7.12}$$

for all $0 < \epsilon < \alpha$.

Together with $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, another space which will form the backbone of the analysis that will follow, is defined as $W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$, the space of measurable functions $g : [0,T] \to \mathbb{R}$, such that its corresponding norm is given as

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \big(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{|t-s|^{2-\alpha}}dy\big) < \infty. \tag{7.13}$$

The corresponding embedding result is

$$C^{1-\alpha+\epsilon}(0,T;\mathbb{R}) \subset W_T^{1-\alpha,\infty}(0,T;\mathbb{R}) \subset C^{1-\alpha}(0,T;\mathbb{R}), \tag{7.14}$$

for all $\epsilon > 0$. Moreover, if $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$, then $g|_{(0,t)} \in I_{t-}^{1-\alpha}(L^\infty(0,t))$ for all $t \in (0,T)$.

Now consider the integral $\int_0^t f dg$ for $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$ and f such that

$$\|f\|_{\alpha,1} = \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s)-f(y)|}{(s-y)^{1+\alpha}} dy ds < \infty. \tag{7.15}$$

Then the conditions for the existence of the integral for all $t \in [0,T]$ are trivially satisfied, and using (7.2) and (7.3) the integral can be bounded as follows:

$$\begin{aligned}
\Big|\int_0^T f dg\Big| &= \Big|\int_0^T D_{0+}^\alpha f(s) D_{T-}^{1-\alpha} g_{T-}(s) ds\Big| \\
&\leq \int_0^T |D_{0+}^\alpha f(s)| \cdot |D_{T-}^{1-\alpha} g_{T-}(s)| ds \\
&= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^T \Big|\frac{f(s)}{s^\alpha} + \alpha \int_0^s \frac{f(s)-f(y)}{(s-y)^{1+\alpha}}\, dy\Big| \\
&\quad \times \Big|\frac{g(T)-g(s)}{(T-s)^{1-\alpha}} + \alpha \int_s^T \frac{g(y)-g(s)}{(y-s)^{2-\alpha}}\, dy\Big|\, ds \\
&\leq \|f\|_{\alpha,1}\, \Lambda_\alpha(g),
\end{aligned}$$

where the last inequality is the result of (7.15) and

$$\begin{aligned}
\Lambda_\alpha(g) &= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |D_{t-}^{1-\alpha} g_{t-}(s)| \\
&\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T}.
\end{aligned} \tag{7.16}$$

Now recall the vector fields defined in (5.2), with following additional conditions:

$(A1)$ $|U_\gamma^i(x)| \leq M_\gamma$, $\forall x \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where U_γ^i denotes the i-th component of the γ-th vector field.

$(A2)$ Writing $\|\cdot\|_2$ for the Euclidean norm in $\mathbb{R}^d$, for the appropriate d, we require, $|U_\gamma^i(x) - U_\gamma^i(y)| \leq M_\gamma^{(1)} \|x-y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$.

$(A3)$ Denoting $W_\gamma(x)$ as the spatial derivative of $U_\gamma(x)$, we also require that, $|W_{\gamma,j}^i(x) - W_{\gamma,j}^i(y)| \leq M_\gamma^{(2)} \|x-y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where $W_{\gamma,j}^i(\cdot)$ is the (i,j)-th element of the matrix $W_\gamma(\cdot)$.

(A4) Finally, $M^{(1)} = \sum_{\alpha\in\mathbb{N}} M_\alpha^{(1)} < \infty$, $M^{(2)} = \sum_{\alpha\in\mathbb{N}} M_\alpha^{(2)} < \infty$, and $M^{(3)} = \sum_{\alpha\in\mathbb{N}} M_\alpha^{(3)} < \infty$.

Consider the following deterministic integral equation in $\mathbb{R}^n$

$$\psi_t(x) = x + \sum_{\gamma\in I} \int_0^t U_\gamma(\psi_s(x)) dg_\gamma(s), \tag{7.17}$$

where $g_\gamma \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$.

Then, under $(A1)-(A4)$ on the vector fields, existence and uniqueness of a solution of (7.17), in the space $C^{1-\alpha}(0,T;\mathbb{R}^n)$ is proven in [30] for $|I| < \infty$. We shall present the idea of the proof in short without delving too deeply into the technical aspects of the proof.

First, an operator G_x^I is defined on $W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$, for fixed $0 < \alpha < 1/2$, by setting

$$(G_x^I f)(t) = x + \sum_{\gamma\in I} \int_0^t U_\gamma(f(s)) dg_\gamma(s),$$

for $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ and $g_\gamma \in W_T^{1-\alpha,0}(0,T;\mathbb{R})$. Then, it is proven that $G_x^I : W_0^{\alpha,\infty}(0,T;\mathbb{R}^n) \to W_0^{\alpha,\infty}(0,T;\mathbb{R}^n)$ is a contraction. The rest of the proof hinges on a simple application of Banach fixed point theorem. The resultant solution is then shown to have $(1-\alpha)$-Hölder continuous paths.

In fact, the existence and uniqueness of the solution can be proven under far weaker conditions, but that prevents the solution from being a diffeomorphism in $\mathbb{R}^n$, while we are interested in the flow properties of smooth flows (see [14] for details).

Finally, the existence and uniqueness of the flow of diffeomorphisms defined in (5.3), using the above interpretation of the integral, is established by using the fact that the sample path of B^H belongs to the space $W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$ for $\alpha \in (1-H, \frac{1}{2})$. Appropriate estimates of the solution of the flow equation are also obtained in [30], which are improved on in [29].

So far we surveyed two ways of constructing stochastic integrals with respect to fractional Brownian motion, and have, hopefully presented enough evidence to convince the reader that the pathwise method seems more amenable for the construction of stochastic flows.

Chapter 8

The flow and its geometric properties

In this chapter, we shall use the approach developed in [29] for getting estimates on some of the basic geometric characteristics of the flow defined in (5.3), with the integral interpreted as in the previous chapter.

We start with recalling the class of vector fields $\{U_\gamma\}_{\gamma\in\mathbb{N}}$ defined in (5.2), and restate the corresponding assumptions as follows.

($A1$) $|U^i_\gamma(x)| \le M_\gamma$, $\forall x \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where U^i_γ denotes the i-th component of the γ-th vector field.

($A2$) Writing $\|\cdot\|_2$ for the Euclidean norm in $\mathbb{R}^d$, for the appropriate d, we require, $|U^i_\gamma(x) - U^i_\gamma(y)| \le M^{(1)}_\gamma \|x-y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$.

($A3$) Denoting $W_\gamma(x)$ as the spatial derivative of $U_\gamma(x)$, we also require that, $|W^i_{\gamma,j}(x) - W^i_{\gamma,j}(y)| \le M^{(2)}_\gamma \|x-y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where $W^i_{\gamma,j}(\cdot)$ is the (i,j)-th element of the matrix $W_\gamma(\cdot)$.

($A4$) Finally, $M = \sum_{\gamma\in\mathbb{N}} M_\gamma < \infty$, $M^{(1)} = \sum_{\gamma\in\mathbb{N}} M^{(1)}_\gamma < \infty$, and $M^{(2)} = \sum_{\gamma\in\mathbb{N}} M^{(2)}_\gamma < \infty$.

We consider the flow

$$\Phi_t(x) = x + \sum_{\gamma\in I} \int_0^t U_\gamma(\Phi_s(x))\, dB^H_\gamma(s), \tag{8.1}$$

where I is a fixed but generic subset of $\mathbb{N}$, such that $|I| < \infty$, where $|I|$ denotes the cardinality of the set I, and the integral appearing in the expression is interpreted in the pathwise sense defined in the previous chapter.

Beginning in a similar way as in Part I, with M, an m-dimensional smooth manifold embedded in the n-dimensional Euclidean space, we denote T_xM as the tangent space of M at $x \in M$. Let $v \in T_xM$. Then its push-forward under the flow Φ_t is given as

$$v_t = D\Phi_t(x)v,$$

and $v_t \in T_{x_t}M_t$.

From here on we shall write x_t for $\Phi_t(x)$. Another notational nuance that we shall often use in the subsequent proofs is the following definition of the norm for a $\mathbb{R}^d$ valued process y defined on the interval $[a, b]$,

$$\|y\|_{\mu,a,b} = \sup_{a \le c \le d \le b} \frac{\|y(c) - y(d)\|_2}{|c - d|^\mu}, \tag{8.2}$$

with the understanding that $\|y\|_{\mu,b} \triangleq \|y\|_{\mu,0,b}$, and that for $\mu = \infty$, this norm is defined as the sup norm.

8.1 Main technical result

In the build up to the main result of Part II of this thesis, which will be presented in Section 8.2, we shall state and prove a technical result, which will form the crux of this section.

Theorem 8.1.1 *Under the assumptions stated in (A1)−(A4), and for $\alpha = 1-H+\delta$, $\beta = H - \epsilon$, such that $(1 - H) < \alpha < 1/2$ and $\delta > \epsilon$, there exist a constant c and a random variable C_T, such that*

$$\begin{aligned} \sup_{r\in[0,T]} \|v_r\|_2 &\le \sup_{r\in[0,T]} \|v_r\|_1 \\ &\le c\, 2^{C_T\, T}, \end{aligned}$$

where $\|v_r\|_2$ and $\|v_r\|_1$ denote the l_2 and l_1 norms, respectively, of the vector v_r as an element in $\mathbb{R}^n$. The random variable C_T depends on α, β, n, I, and $\{\|B^H_\gamma\|_{\beta,T}, M_\gamma, M^{(1)}_\gamma, M^{(2)}_\gamma\}_{\gamma\in I}$, *with* $\|B^H_\gamma\|_{\beta,T}$ *the Hölder norm of the process* B^H_γ*. Furthermore,*

$$E[C_T]^\beta \leq C \cdot E[\|B^H\|_{\beta,T}],$$

where the constant C *depends only on* α*,* β*,* n*,* $|I|$ *and* $\{M_\gamma, M^{(1)}_\gamma, M^{(2)}_\gamma\}_{\gamma\in I}$*.*

Remark 8.1.1 *For a better understanding of the results of Theorem 8.1.1, we note that for the case* $|I| = 1$*, these results boil down to the following*

$$\sup_{r\in[0,T]} \|v_r\|_2 \quad \leq \quad c\, 2^{C\,T\,\|B^H\|^{1/\beta}_{\beta,T}},$$

for some constants c and C*, dependent only on the various uniform bounds and the Lipschitz coefficients corresponding to the vector field.*

Remark 8.1.2 *The results listed in this chapter hold true for any* $I \subset \mathbb{N}$ *as long as the cardinality of the set satisfies* $|I| < \infty$*. However, extensions of these results to the case* $I = \mathbb{N}$*, though possible, require unnatural conditions on the summability of the constants appearing in Assumptions* $(A1) - (A3)$*. For instance, extending Lemma 8.1.1 to the case* $I = \mathbb{N}$ *will require*

$$\frac{\sum_{\gamma\in\mathbb{N}} M^{(1)}_\gamma \|B^H_\gamma\|_{\beta,T}}{\sum_{\gamma\in\mathbb{N}} M^{(2)}_\gamma \|B^H_\gamma\|_{\beta,T}} < \infty.$$

A sufficient condition for which can be stated as

$$\sum_{\gamma\in\mathbb{N}} \frac{M^{(1)}_\gamma}{M^{(2)}_\gamma} < \infty,$$

which does not seem to have a straightforward meaning in terms of the vector fields $\{U_\gamma\}_{\gamma\in\mathbb{N}}$*.*

The idea of the proof of Theorem 8.1.1 is to break up the interval $[0, T]$ into smaller units Δ, on which reasonable estimates of $\|v_r\|_2$ are possible, and then to glue the intervals together to obtain the required result. However, in the process, we shall require estimates on the flow, which are presented in the following lemma, for which we rely on the results obtained in [29].

Lemma 8.1.1 *Let M, $M^{(1)}$ be constants as defined in Assumptions $(A1)-(A4)$, and $0 \leq s \leq t \leq T$ be such that*

$$(t-s)^{-\beta} > \frac{n\,\alpha(2\alpha+\beta-1)}{2(1-\alpha)(1-2\alpha)(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{\gamma \in I} M_{\gamma}^{(1)} \|B_{\gamma}^{H}\|_{\beta,T},$$

where $\alpha = 1-H+\delta$, $\beta = H-\epsilon$, such that $(1-H) < \alpha < 1/2$ and $\delta > \epsilon$. Then for x_t defined in (8.1) there exists a random variable $K_{s,t}^{*}$ such that

$$\int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1+\alpha}}\, dr \;\leq\; K_{s,t}^{*}(t-s)^{\beta-\alpha}. \tag{8.3}$$

Furthermore, $K_{s,t}^{*}$ can be bounded above by another random variable, independent of s and t, with finite moments of order greater than 1, as long as $(t-s)$ is chosen sufficiently small.

Remark 8.1.3 *Note that under the aforementioned conditions concerning α and β, we have $\alpha+\beta > 1$, and $\beta > \alpha$.*

Proof: Writing $U_{\gamma}^{i}(\cdot)$ for the i-th component of the vector $U_{\gamma}(\cdot)$ and choosing $\{e_i\}_{i=1}^{n}$ as the canonical basis of $\mathbb{R}^n$, we have

$$\langle (x_t - x_s), e_i \rangle = \sum_{\gamma \in I} \int_s^t U_{\gamma}^{i}(x_r)\, dB_{\gamma}^{H}(r),$$

which is true by linearity of the operation, and where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Hence for $\alpha \in (1-H, \frac{1}{2})$,

$$\begin{aligned}
|\langle (x_t - x_s), e_i \rangle| &= \Big| \sum_{\gamma \in I} \int_s^t U_{\gamma}^{i}(x_r)\, dB_{\gamma}^{H}(r) \Big| \\
&= \Big| \sum_{\gamma \in I} \int_s^t D_{s+}^{\alpha} U_{\gamma}^{i}(x_r) D_{t-}^{1-\alpha} B_{\gamma,t-}^{H}(r)\, dr \Big| \\
&\leq \sum_{\gamma \in I} \int_s^t |D_{s+}^{\alpha} U_{\gamma}^{i}(x_r)| \cdot |D_{t-}^{1-\alpha} B_{\gamma,t-}^{H}(r)|\, dr
\end{aligned}$$

To obtain a bound on the second term in the integrand, choose $\beta < H$, such that $\alpha + \beta > 1$, so that using (7.3), we have

$$\begin{aligned}
|D_{t-}^{1-\alpha} B_{\gamma,t-}^{H}(r)| &= \Big|\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\Big(\frac{B_\gamma^H(t)-B_\gamma^H(r)}{(t-r)^{1-\alpha}} + \alpha\int_r^t \frac{B_\gamma^H(u)-B_\gamma^H(r)}{(u-r)^{2-\alpha}}du\Big)\Big| \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\frac{|B_\gamma^H(t)-B_\gamma^H(r)|}{|t-r|^{1-\alpha}} + \alpha\int_r^t \frac{|B_\gamma^H(u)-B_\gamma^H(r)|}{(u-r)^{2-\alpha}}du\Big) \\
&= \frac{1}{\Gamma(\alpha)}\Big(\frac{|B_\gamma^H(t)-B_\gamma^H(r)|(t-r)^{\beta}}{(t-r)^{\beta}(t-r)^{1-\alpha}} \\
&\quad + \alpha\int_r^t \frac{|B_\gamma^H(u)-B_\gamma^H(r)|}{(u-r)^{\beta}}(u-r)^{\alpha+\beta-2}du\Big) \\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\|B_\gamma^H\|_{\beta,T}(t-r)^{\alpha+\beta-1} + \alpha\|B_\gamma^H\|_{\beta,T}\frac{(t-r)^{\alpha+\beta-1}}{\alpha+\beta-1}\Big) \\
&= k_1(\alpha,\beta)\|B_\gamma^H\|_{\beta,T}(t-r)^{\alpha+\beta-1}, \qquad (8.4)
\end{aligned}$$

where $k_1(\alpha,\beta) = \frac{(2\alpha+\beta-1)}{(\alpha+\beta-1)\Gamma(\alpha)}$.

To bound the first term we use (7.2) and assumptions $(A1)-(A2)$, to see that

$$\begin{aligned}
|D_{s+}^{\alpha} U_\gamma^i(x_r)| &= \frac{1}{\Gamma(1-\alpha)}\Big|\frac{U_\gamma^i(x_r)}{(r-s)^{\alpha}} + \alpha\int_s^r \frac{(U_\gamma^i(x_r)-U_\gamma^i(x_\theta))}{(r-\theta)^{1+\alpha}}\, d\theta\Big| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big(\frac{|U_\gamma^i(x_r)|}{(r-s)^{\alpha}} + \alpha\int_s^r \frac{|U_\gamma^i(x_r)-U_\gamma^i(x_\theta)|}{(r-\theta)^{1+\alpha}}\, d\theta\Big) \\
&\leq c_\alpha\Big(\frac{M_\gamma}{(r-s)^{\alpha}} + \alpha\int_s^r \frac{M_\gamma^{(1)}\|x_r-x_\theta\|_2}{(r-\theta)^{1+\alpha}}\, d\theta\Big) \\
&\leq c_\alpha\Big(M_\gamma(r-s)^{-\alpha} + M_{\gamma,\alpha}^{(1)}\|x\|_{s,r,1-\alpha}(r-s)^{1-2\alpha}\Big), \qquad (8.5)
\end{aligned}$$

where $c_\alpha = \Gamma(\alpha)^{-1}$ and $M_{\gamma,\alpha}^{(1)} = \frac{\alpha M_\gamma^{(1)}}{(1-2\alpha)}$.

Therefore, combining the above two estimates , we get

$$\begin{aligned}
|\langle(x_t - x_s), e_i\rangle| &\leq c_\alpha k_1(\alpha,\beta)\sum_{\gamma\in I}\|B_\gamma^H\|_{\beta,T}\int_s^t \Big(M_\gamma(r-s)^{-\alpha}(t-r)^{\alpha+\beta-1} \\
&\quad + M_{\gamma,\alpha}^{(1)}\|x\|_{s,r,1-\alpha}(r-s)^{1-2\alpha}(t-r)^{\alpha+\beta-1}\Big)\, dr \\
&\leq c_\alpha k_1(\alpha,\beta)\sum_{\gamma\in I}\|B_\gamma^H\|_{\beta,T}(t-s)^{\alpha+\beta-1}\int_s^t \Big(M_\gamma(r-s)^{-\alpha} \\
&\quad + M_{\gamma,\alpha}^{(1)}\|x\|_{s,r,1-\alpha}(r-s)^{1-2\alpha}\Big)\, dr \\
&= c_\alpha k_1(\alpha,\beta)\sum_{\gamma\in I}\|B_\gamma^H\|_{\beta,T}\Big(M_\gamma(t-s)^{\beta}(1-\alpha)^{-1} \\
&\quad + \tilde{M}_{\gamma,\alpha}^{(1)}\|x\|_{s,t,1-\alpha}(t-s)^{1-\alpha+\beta}(2-2\alpha)^{-1}\Big).
\end{aligned}$$

Let

$$M_\alpha = (1-\alpha)^{-1}\sum_{\gamma\in I} M_\gamma\|B_\gamma^H\|_{\beta,T}, \tag{8.6}$$

and

$$\tilde{M}_\alpha^{(1)} = (2-2\alpha)^{-1}\sum_{\gamma\in I} M_{\gamma,\alpha}^{(1)}\|B_\gamma^H\|_{\beta,T}. \tag{8.7}$$

Then

$$\begin{aligned}
\|(x_t - x_s)\|_1 &= \sum_{i=1}^n |\langle(x_t - x_s), e_i\rangle| \\
&\leq c_\alpha n k_1(\alpha,\beta)\Big(M_\alpha(t-s)^\beta + \tilde{M}_\alpha^{(1)}\|x\|_{s,t,1-\alpha}(t-s)^{1-\alpha+\beta}\Big).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\frac{\|(x_t - x_s)\|_1}{(t-s)^{1-\alpha}} &\leq c_\alpha n k_1(\alpha,\beta)\Big(M_\alpha(t-s)^{\alpha+\beta-1} \\
&\quad + \tilde{M}_\alpha^{(1)}\|x\|_{s,t,1-\alpha}(t-s)^\beta\Big).
\end{aligned} \tag{8.8}$$

(Recall that $\alpha+\beta > 1$.)

Now using the above estimate together with (8.2), and the fact that $\|\cdot\|_2$ is bounded above by $\|\cdot\|_1$, we have

$$\begin{aligned}
\|x\|_{s,t,1-\alpha} &= \sup_{s\le u\le v\le t} \frac{\|(x_v - x_u)\|_2}{(v-u)^{1-\alpha}} \\
&\le \sup_{s\le u\le v\le t} \frac{\|(x_v - x_u)\|_1}{(v-u)^{1-\alpha}} \\
&\le \sup_{s\le u\le v\le t} c_\alpha n k_1(\alpha,\beta)\Big(M_\alpha (v-u)^{\alpha+\beta-1} \\
&\quad + \tilde{M}_\alpha^{(1)} \|x\|_{u,v,1-\alpha}(v-u)^\beta\Big) \\
&\le c_\alpha n k_1(\alpha,\beta)\Big(M_\alpha (t-s)^{\alpha+\beta-1} \\
&\quad + \tilde{M}_\alpha^{(1)} \|x\|_{s,t,1-\alpha}(t-s)^\beta\Big). \qquad (8.9)
\end{aligned}$$

Now choosing s, t such that

$$(t-s)^{-\beta} > c_\alpha n k_1(\alpha,\beta)\tilde{M}_\alpha^{(1)}, \qquad (8.10)$$

(8.9) can be rewritten as

$$\begin{aligned}
\|x\|_{s,t,1-\alpha} &\le \frac{c_\alpha n k_1(\alpha,\beta) M_\alpha (t-s)^{\alpha+\beta-1}}{1 - c_\alpha n k_1(\alpha,\beta)\tilde{M}_\alpha^{(1)}(t-s)^\beta} \\
&= K_{s,t}(t-s)^{\alpha+\beta-1}. \qquad (8.11)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1+\alpha}}\, dr &= \int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1-\alpha}}(t-r)^{-2\alpha}\, dr \\
&\le \|x\|_{s,t,1-\alpha} \int_s^t (t-r)^{-2\alpha}\, dr \\
&\le K_{s,t} \frac{(t-s)^{\beta-\alpha}}{(1-2\alpha)} \\
&= K^*_{s,t}(t-s)^{\beta-\alpha},
\end{aligned}$$

where $K^*_{s,t} = \frac{K_{s,t}}{(1-2\alpha)}$, thus establishing (8.3). The final claim, that $K^*_{s,t}$ can be bounded by a random variable independent of s and t will be proven later. □

Proof of Theorem 8.1.1: Taking the space derivative of (8.1), the existence of which is ensured by Theorem 3.2 in [29], we have

$$D\Phi_t(x) = I + \sum_{\gamma\in I} \int_0^t W_\gamma(\Phi_s(x)) D\Phi_s(x)\, dB_\gamma^H(s),$$

where the matrix $W_\gamma(\cdot) = (W^i_{\gamma,j}(\cdot))_{i,j}$ denotes the spatial derivative of the vector field U.

Now using the definition of the pushforward of a vector, we can write the evolution equation of the tangent vector as follows

$$v_t = v + \sum_{\gamma\in I}\int_0^t W_\gamma(x_s)v_s dB^H_\gamma(s).$$

Recall that $\|v_t\|_1 = \sum_{i=1}^n |\langle v_t, e_i\rangle|$, where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product, and $\{e_i\}_{i=1}^n$ denotes the canonical basis of $\mathbb{R}^n$. Since,

$$\langle v_t, e_i\rangle = x + \sum_{\gamma\in I}\int_0^t \langle W_\gamma(x_r)v_r, e_i\rangle dB^H_\gamma(r),$$

we have

$$\begin{aligned} |\langle v_t, e_i\rangle - \langle v_s, e_i\rangle| &= \Big|\sum_{\gamma\in I}\int_s^t \langle W_\gamma(x_r)v_r, e_i\rangle dB^H_\gamma(r)\Big| \\ &= \Big|\sum_{\gamma\in I}\int_s^t D^\alpha_{s+}\langle W_\gamma(x_r)v_r, e_i\rangle D^{1-\alpha}_{t-}B^H_{\gamma,t-}(r)dr\Big| \\ &\leq \sum_{\gamma\in I}\int_s^t |D^\alpha_{s+}\langle W_\gamma(x_r)v_r, e_i\rangle|\cdot|D^{1-\alpha}_{t-}B^H_{\gamma,t-}(r)|dr. \end{aligned}$$

The above inequality holds for any choice of s and t, but we are interested in pairs for which $(t-s)$ is sufficiently small. To this end, note first that from (8.4) we can bound the second integrand by

$$|D^{1-\alpha}_{t-}B^H_\gamma(r)| = k_1(\alpha,\beta)\|B^H_\gamma\|_{\beta,T}(t-r)^{\alpha+\beta-1}.$$

Now using (7.2) and Assumptions $(A2)$-$(A4)$, the first integrand can be bounded as

$$
\begin{aligned}
|D^{\alpha}_{s+}\langle W_\gamma(x_r)v_r, e_i\rangle| &\leq \frac{1}{\Gamma(1-\alpha)}\Big[\frac{|\langle W_\gamma(x_r)v_r, e_i\rangle|}{(r-s)^\alpha} + \alpha\int_s^r \frac{|\langle W_\gamma(x_r)v_r, e_i\rangle - \langle W_\gamma(x_\theta)v_\theta, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\sum_{j=1}^n |W^i_{\gamma,j}(x_r)\langle v_r, e_j\rangle|}{(r-s)^\alpha} + \alpha\int_s^r \frac{|\langle W_\gamma(x_r)v_r, e_i\rangle - \langle W_\gamma(x_\theta)v_\theta, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big[M^{(1)}_\gamma \sum_{j=1}^n \frac{|\langle v_r, e_j\rangle|}{(r-s)^\alpha} + \alpha\int_s^r \frac{|\langle W_\gamma(x_r)v_r, e_i\rangle - \langle W_\gamma(x_\theta)v_r, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta \\
&\quad + \alpha\int_s^r \frac{|\langle W_\gamma(x_\theta)v_r, e_i\rangle - \langle W_\gamma(x_\theta)v_\theta, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&= \frac{1}{\Gamma(1-\alpha)}\Big[M^{(1)}_\gamma \sum_{j=1}^n \frac{|\langle v_r, e_j\rangle|}{(r-s)^\alpha} \\
&\quad + \alpha\int_s^r \frac{|\sum_{j=1}^n (W^i_{\gamma,j}(x_r)\langle v_r, e_j\rangle - W^i_{\gamma,j}(x_\theta)\langle v_r, e_j\rangle)|}{(r-\theta)^{1+\alpha}}d\theta \\
&\quad + \alpha\int_s^r \frac{|\sum_{j=1}^n (W^i_{\gamma,j}(x_\theta)\langle v_r, e_j\rangle - W^i_{\gamma,j}(x_\theta)\langle v_\theta, e_i\rangle)|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big[M^{(1)}_\gamma \sum_{j=1}^n \frac{|\langle v_r, e_j\rangle|}{(r-s)^\alpha} \\
&\quad + \alpha\int_s^r \frac{\sum_{j=1}^n |W^i_{\gamma,j}(x_r) - W^i_{\gamma,j}(x_\theta)|\cdot|\langle v_r, e_j\rangle|}{(r-\theta)^{1+\alpha}}d\theta \\
&\quad + \alpha\int_s^r \frac{\sum_{j=1}^n |W^i_{\gamma,j}(x_\theta)|\cdot|\langle v_r, e_j\rangle - \langle v_\theta, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big[M^{(1)}_\gamma \sum_{j=1}^n \frac{|\langle v_r, e_j\rangle|}{(r-s)^\alpha} + \alpha M^{(2)}_\gamma \sum_{j=1}^n |\langle v_r, e_j\rangle| \int_s^r \frac{\|x_r - x_\theta\|_2}{(r-\theta)^{1+\alpha}}d\theta \\
&\quad + \alpha M^{(1)}_\gamma \sum_{j=1}^n \int_s^r \frac{|\langle v_r, e_j\rangle - \langle v_\theta, e_i\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big].
\end{aligned}
$$

Now using the result proven in Lemma 8.1.1, for r such that $s < r < t$, with $(t-s)$ satisfying the condition (8.10), we have

$$\int_s^r \frac{\|x_r - x_\theta\|_2}{(r-\theta)^{1+\alpha}}\, d\theta \leq K^*_{s,r}(r-s)^{\beta-\alpha}.$$

Hence,

$$
\begin{aligned}
|D^{\alpha}_{s+}\langle W_\gamma(x_r)v_r, e_i\rangle| &= \sum_{j=1}^{n}\Big[\frac{|\langle v_r, e_j\rangle|}{(r-s)^{\alpha}}\Big(\frac{M^{(1)}_\gamma + \alpha M^{(2)}_\gamma K^*_{s,r}(r-s)^{\beta}}{\Gamma(1-\alpha)}\Big) \\
&\quad + \frac{\alpha M^{(1)}_\gamma}{\Gamma(1-\alpha)}\int_s^r \frac{|\langle v_r, e_j\rangle - \langle v_\theta, e_j\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&= \sum_{j=1}^{n}\Big[a_{\gamma,1}\frac{|\langle v_r, e_j\rangle|}{(r-s)^{\alpha}} + \frac{\alpha M^{(1)}_\gamma}{\Gamma(1-\alpha)}\int_s^r \frac{|\langle v_r, e_j\rangle - \langle v_\theta, e_j\rangle|}{(r-\theta)^{1+\alpha}}d\theta\Big] \\
&\leq \sum_{j=1}^{n}\Big[a_{\gamma,1}|\langle v_r, e_j\rangle|(r-s)^{-\alpha} + b_{\gamma,1}\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(r-s)^{\beta-\alpha}\Big],
\end{aligned}
$$

where

$$
a_{\gamma,s,r,1} = \frac{M^{(1)}_\gamma + \alpha M^{(2)}_\gamma K^*_{s,r}(r-s)^{\beta}}{\Gamma(1-\alpha)}, \tag{8.12}
$$

and

$$
b_{\gamma,1} = \frac{\alpha M^{(1)}_\gamma}{(\beta-\alpha)\Gamma(1-\alpha)}. \tag{8.13}
$$

Note that $a_{\gamma,s,r,1} \leq a_{\gamma,s,t,1}$, for $s \leq r \leq t$.

Writing $a_{s,r,1} = \sum_{\gamma\in I} a_{\gamma,s,r,1}\|B^H_\gamma\|_{\beta,T}$ and $b_1 = \sum_{\gamma\in I} b_{\gamma,1}\|B^H_\gamma\|_{\beta,T}$, and using the above estimates for the integrands, together with (8.4) and Remark 8.1.3, we have

$$\begin{aligned}
|\langle(v_t - v_s), e_i\rangle| &\le k_1(\alpha,\beta)\int_s^t \sum_{j=1}^n \Big(a_{s,r,1}|\langle v_r, e_j\rangle|(r-s)^{-\alpha}(t-r)^{\alpha+\beta-1} \\
&\quad + b_1\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\Big)dr \\
&\le k_1(\alpha,\beta)(t-s)^{\alpha+\beta-1}\int_s^t \sum_{j=1}^n \Big(a_{s,r,1}|\langle v_r, e_j\rangle|(r-s)^{-\alpha} \\
&\quad + b_1\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(r-s)^{\beta-\alpha}\Big)dr \\
&\le k_1(\alpha,\beta)(t-s)^{\alpha+\beta-1}\int_s^t \sum_{j=1}^n \Big(a_{s,r,1}\|\langle v_\cdot, e_j\rangle\|_{s,t,\infty}(r-s)^{-\alpha} \\
&\quad + b_1\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(r-s)^{\beta-\alpha}\Big)dr \\
&\le k_1(\alpha,\beta)(t-s)^{\alpha+\beta-1}\sum_{j=1}^n \Big(a_{s,t,1}\|\langle v_\cdot, e_j\rangle\|_{s,t,\infty}\frac{(t-s)^{1-\alpha}}{1-\alpha} \\
&\quad + b_1\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}\frac{(t-s)^{1+\beta-\alpha}}{1+\beta-\alpha}\Big) \\
&= k_1(\alpha,\beta)\sum_{j=1}^n \Big(a_{s,t,2}\|\langle v_\cdot, e_j\rangle\|_{s,t,\infty}(t-s)^{\beta} \\
&\quad + b_2\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(t-s)^{2\beta}\Big),
\end{aligned}$$

where $a_{s,t,2} = a_{s,t,1}(1-\alpha)^{-1}$ and $b_2 = b_1(1-\alpha+\beta)^{-1}$.

Therefore,

$$\begin{aligned}
\|\langle v_\cdot, e_i\rangle\|_{s,t,\beta} &= \sup_{s\le r\le\theta\le t}\frac{|\langle(v_\theta - v_r), e_i\rangle|}{(\theta-r)^\beta} \\
&\le k_1(\alpha,\beta)\sum_{j=1}^n \sup_{s\le r\le\theta\le t}\Big(a_{r,\theta,2}\|\langle v_\cdot, e_j\rangle\|_{r,\theta,\infty} \\
&\quad + b_2\|\langle v_\cdot, e_j\rangle\|_{r,\theta,\beta}(\theta-r)^\beta\Big) \\
&\le k_1(\alpha,\beta)\sum_{j=1}^n \Big(a_{s,t,2}\|\langle v_\cdot, e_j\rangle\|_{s,t,\infty} \\
&\quad + b_2\|\langle v_\cdot, e_j\rangle\|_{s,t,\beta}(t-s)^\beta\Big).
\end{aligned}$$

As a consequence of the above estimate we have

$$\begin{aligned}\sum_{i=1}^{n}\|\langle v.,e_i\rangle\|_{s,t,\beta} &\le n\,k_1(\alpha,\beta)\sum_{j=1}^{n}\Big(a_{s,t,2}\|\langle v.,e_j\rangle\|_{s,t,\infty} \\ &\quad + b_2\|\langle v.,e_j\rangle\|_{s,t,\beta}(t-s)^{\beta}\Big).\end{aligned} \tag{8.14}$$

For further analysis we shall require that

$$(t-s)^{-\beta} > nk_1(\alpha,\beta)b_2. \tag{8.15}$$

Thereby, for $(t-s)$ satisfying conditions (8.10) and (8.15), we can rewrite (8.14) as

$$\sum_{i=1}^{n}\|\langle v.,e_i\rangle\|_{s,t,\beta} \le n\,k_1(\alpha,\beta)\,a_{s,t,2}\,\sum_{i=1}^{n}\frac{\|\langle v.,e_i\rangle\|_{s,t,\infty}}{(1-n\,k_1(\alpha,\beta)\,b_2\,(t-s)^{\beta})}.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n}|\langle v_t,e_i\rangle| &\le \sum_{i=1}^{n}\Big(|\langle v_s,e_i\rangle| + |\langle v_t,e_i\rangle - \langle v_s,e_i\rangle|\Big) \\ &\le \sum_{i=1}^{n}\Big(|\langle v_s,e_i\rangle| + \|\langle v.,e_i\rangle\|_{s,t,\beta}(t-s)^{\beta}\Big) \\ &\le \sum_{i=1}^{n}\Big(|\langle v_s,e_i\rangle| + n\,k_1(\alpha,\beta)\,a_{s,t,2}\frac{\|\langle v.,e_i\rangle\|_{s,t,\infty}(t-s)^{\beta}}{(1-n\,k_1(\alpha,\beta)\,b_2\,(t-s)^{\beta})}\Big)\end{aligned}$$

Clearly, for any $r\in[s,t]$ we have

$$\sum_{i=1}^{n}|\langle v_r,e_i\rangle| \le \sum_{i=1}^{n}\Big(|\langle v_s,e_i\rangle| + n\,k_1(\alpha,\beta)\,a_{s,r,2}\frac{\|\langle v.,e_i\rangle\|_{s,r,\infty}(r-s)^{\beta}}{(1-n\,k_1(\alpha,\beta)\,b_2\,(r-s)^{\beta})}\Big).$$

Now using the fact that $s<r<t$, so that $\|\langle v.,e_i\rangle\|_{s,r,\infty} \le \|\langle v.,e_i\rangle\|_{s,t,\infty}$ and $a_{s,r,2}\le a_{s,t,2}$, we have

$$\begin{aligned}\sum_{i=1}^{n}\|\langle v.,e_i\rangle\|_{s,t.\infty} &\le \sum_{i=1}^{n}\Big(|\langle v_s,e_i\rangle| \\ &\quad + n\,k_1(\alpha,\beta)\,a_{s,t,2}\frac{\|\langle v.,e_i\rangle\|_{s,t,\infty}(t-s)^{\beta}}{(1-n\,k_1(\alpha,\beta)\,b_2\,(t-s)^{\beta})}\Big).\end{aligned} \tag{8.16}$$

Finally, we shall require $(t-s)$ to satisfy

$$(t-s)^{-\beta} > n\,k_1(\alpha,\beta)\,[a_{s,t,2}+b_2], \tag{8.17}$$

to allow us to rewrite (8.16) as

$$\sum_{i=1}^{n} \|\langle v_{\cdot}, e_i\rangle\|_{s,t,\infty} \Big[1 - \frac{n\, k_1(\alpha,\beta)\, a_{s,t,2}\,(t-s)^{\beta}}{(1 - n\, k_1(\alpha,\beta)\, b_2\,(t-s)^{\beta})}\Big] \leq \sum_{i=1}^{n} |\langle v_s, e_i\rangle|.$$

We shall note that for $(t-s)$ sufficiently small, the inequality (8.17) does hold true, as $a_{s,t,2}$ is a decreasing function of $(t-s)$.

This, in turn implies,

$$\begin{aligned}
\sum_{i=1}^{n} \sup_{0\leq r\leq t} |\langle v_r, e_i\rangle| &= \sum_{i=1}^{n} \max\{\sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|, \|\langle v_{\cdot}, e_i\rangle\|_{s,t,\infty}\} \\
&\leq \sum_{i=1}^{n} \max\{\sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|, \frac{|\langle v_s, e_i\rangle|}{\Big[1 - \frac{nk_1(\alpha,\beta)a_{s,t,2}(t-s)^{\beta}}{(1-nk_1(\alpha,\beta)b_2(t-s)^{\beta})}\Big]}\} \\
&\leq \sum_{i=1}^{n} \max\{\sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|, \frac{\sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|}{\Big[1 - \frac{nk_1(\alpha,\beta)a_{s,t,2}(t-s)^{\beta}}{(1-nk_1(\alpha,\beta)b_2(t-s)^{\beta})}\Big]}\} \\
&= \sum_{i=1}^{n} \frac{\sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|}{\Big[1 - \frac{nk_1(\alpha,\beta)a_{s,t,2}(t-s)^{\beta}}{(1-nk_1(\alpha,\beta)b_2(t-s)^{\beta})}\Big]} \\
&= S \sum_{i=1}^{n} \sup_{0\leq r\leq s} |\langle v_r, e_i\rangle|, \qquad (8.18)
\end{aligned}$$

where $S = \Big[1 - \frac{nk_1(\alpha\beta)a_{s,t,2}(t-s)^{\beta}}{(1-nk_1(\alpha,\beta)b_2(t-s)^{\beta})}\Big]^{-1}$.

Next we divide the interval $[0,T]$ into p pieces of size $\Delta = (t-s)$, with Δ being small enough, so that none of the above estimates are negated, and we shall write $a_{\Delta,2}$ for $a_{s,t,2}$, as $a_{s,t,2}$ depends on s,t only through the difference $(t-s) = \Delta$.

More precisely, in view of (8.10), (8.15) and (8.17), we require Δ to satisfy the following condition

$$\begin{aligned}
\Delta^{-\beta} &> n\, k_1(\alpha,\beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)},\, b_2,\, (a_{\Delta,2} + b_2)] \\
&= n\, k_1(\alpha,\beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)},\, (a_{\Delta,2} + b_2)].
\end{aligned}$$

For example, we can choose

$$\Delta^{-\beta} = 3\, n\, k_1(\alpha,\beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)},\, (a_{\Delta,2} + b_2)], \qquad (8.19)$$

and thus, for this specific choice of Δ, we have $S \leq 2$.

To ensure the existence of such a Δ, we start with

$$\Delta_0^{-\beta} = 3\, n\, k_1(\alpha,\beta)\, c_\alpha\, \tilde{M}_\alpha^{(1)},$$

then, if

$$\Delta_0^{-\beta} \geq 3\, n\, k_1(\alpha,\beta)\, (a_{\Delta_0,2} + b_2), \tag{8.20}$$

we shall choose $\Delta = \Delta_0$, else, we solve the following equation

$$\Delta_0^{-\beta} \geq 3\, n\, k_1(\alpha,\beta)\, (a_{\Delta_0,2} + b_2),$$

in the range $\Delta \leq \Delta_0$. It is now easy see that the solution to this equation is ensured since the left side increases to infinity as $\Delta \to 0$, whereas the right side, which is larger than the left side at $\Delta = \Delta_0$, decreases as Δ decreases to zero.

Using the above notation, and repeatedly applying the technique used in (8.18), we can write

$$\begin{aligned}
\sup_{t\in[0,T]} |v_t| &= \sup_{t\in[0,\, p\Delta]} [\sum_{i=1}^{n} |\langle v_t, e_i\rangle|] \\
&\leq \sum_{i=1}^{n} \sup_{t\in[0,\, p\Delta]} |\langle v_t, e_i\rangle| \\
&\leq S^p \sum_{i=1}^{n} |\langle v, e_i\rangle|,
\end{aligned}$$

where

$$\begin{aligned}
p &= \frac{T}{\Delta} \\
&= T\Big(3nk_1(\alpha,\beta)\cdot \max[c_\alpha \tilde{M}_\alpha^{(1)},\ (a_{\Delta,2} + b_2)]\Big)^{1/\beta} \\
&= T\, C_T,
\end{aligned}$$

and

$$C_T = \Big(3n\, k_1(\alpha,\beta)\cdot [(c_\alpha \tilde{M}_\alpha^{(1)}) \vee (a_{\Delta,2} + b_2)]\Big)^{1/\beta}).$$

Proof of Lemma 8.1.1 continued: To prove the final claim of Lemma 8.1.1, note that for specific choice $(t-s) = \Delta$, together with (8.6) and (8.11) we have

$$\begin{aligned}
K^*_{s,t} &= \frac{K_{s,t}}{(1-2\alpha)} \\
&\leq \frac{3}{2(1-2\alpha)} \cdot c_\alpha\, n\, k_1(\alpha,\beta) M_\alpha \\
&= \frac{3}{2(1-2\alpha)} \cdot c_\alpha\, n\, k_1(\alpha,\beta) \sum_{\gamma\in I} \frac{M_\gamma \|B^H_\gamma\|_{\beta,T}}{1-\alpha}
\end{aligned}$$

and, so there exists a constant $K(\alpha,\beta)$, dependent only on α and β, such that

$$K^*_{s,t}(t-s)^\beta \leq K(\alpha,\beta)\frac{\sum_{\gamma\in I} M_\gamma \|B^H_\gamma\|_{\beta,T}}{2\sum_{\gamma\in I} M^{(1)}_\gamma \|B^H_\gamma\|_{\beta,T}} \leq K(\alpha,\beta)\sum_{\gamma\in I}\frac{M_\gamma}{M^{(1)}_\gamma}.$$

Consequently, $a_{\Delta,2}$ can also be bounded above by a constant a_2, hence we shall replace $a_{\Delta,2}$ by a_2, in the following discussion. □

Note that,

$$(C_T)^\beta \leq 3n\, c_\alpha\, k_1(\alpha,\beta)\sum_{\gamma\in I}(\tilde{M}^{(1)}_{\alpha,\gamma} + a_{2,\gamma} + b_{2,\gamma})\|B^H_\gamma\|_{\beta,T},$$

where $\tilde{M}^{(1)}_{\alpha,\gamma}$, $a_{2,\gamma}$, and $b_{2,\gamma}$ are the coefficients of $\|B^H_\gamma\|_{\beta,T}$ in the constants $\tilde{M}^{(1)}_\alpha$, a_2 and b_2, respectively.

Now using the bound on S available due to the specific choice of Δ, we get the desired result. □

8.2 The main result

The results obtained in the previous section, finally bring us to the main result of this part of the thesis.

The estimates in Theorem 8.1.1, in turn imply similar bounds on the Hausdorff measure of the m-dimensional manifold M_t, evolving under the flow Φ_t. More precisely, let $\{v^x_i\}_{i=1}^m$ be an orthonormal basis of the tangent space T_xM, at the point $x \in M$, then writing $\mathcal{L}_m(M_t)$ for m-th Lipschitz-Killing curvature of M_t, we have the following result.

Theorem 8.2.1 *Let M be a C^2, m dimensional manifold, evolving under the flow Φ_t defined in* (8.1)*. Then under the conditions* $(A1)-(A4)$*, and for $\alpha = 1-H+\delta$, $\beta = H-\epsilon$, such that $(1-H) < \alpha < 1/2$ and $\delta > \epsilon$, there exists a constant c_1, and* *a random variable $C_{1,T}$, such that*

$$\sup_{t\in[0,T]} \mathcal{L}_m(M_t) \ \leq \ c_1\, \mathcal{L}_m(M)\, 2^{C_{1,T}\,T},$$

where the function $C_{1,T}$ depends on α, β, n, I, and $\{\|B^H_\gamma\|_{\beta,T}, M_\gamma, M^{(1)}_\gamma, M^{(2)}_\gamma\}_{\gamma\in I}$, with $\|B^H_\gamma\|_{\beta,T}$ the Hölder norm of the process B^H_γ for $\beta < H$, such that

$$E[C_{1,T}]^\beta \leq C_1 \cdot E[\|B^H\|_{\beta,T}],$$

with constant C_1 dependent only on α, β, n, $|I|$ and $\{M_\gamma, M^{(1)}_\gamma, M^{(2)}_\gamma\}_{\gamma\in I}$.

Proof: Consider the pushforwards $\{v^x_{i,t}\}_{i=1}^m$ of tangent vectors $\{v^x_i\}_{i=1}^m$ under the flow Φ_t. Then by using a simple formula for change of variables on a manifold, as in (3.5), we get

$$\begin{aligned} \mathcal{L}(M_t) &= \int_{M_t} \mathcal{H}(dy) \\ &= \int_M \|\alpha^x(t)\| \mathcal{H}(dx), \end{aligned}$$

where $\|\alpha^x(t)\| = \sqrt{|\det(\langle v^x_{i,t}, v^x_{j,t}\rangle)|}$. By the Cauchy-Schwartz inequality we know that

$$\langle v^x_{i,t}, v^x_{j,t}\rangle \leq \|v^x_{i,t}\|_2\ \|v^x_{i,t}\|_2.$$

Therefore, using Theorem 8.1.1 and the above expression, we get

$$\begin{aligned} \sup_{t\in[0,T]} \|\alpha^x(t)\| &\leq m!(\sup_{t\in[0,T]} \|v^x_{i,t}\|)^m \\ &\leq c\, m!\, 2^{m\,C\,T\,\|B^H\|^{1/\beta}_{\beta,T}}, \end{aligned}$$

which proves the required result. □

We end this chapter with the following remark on the growth of the random variables C_T and $C_{1,T}$, appearing in the above results.

Remark 8.2.1 *The rate of growth of the β-th moment of the random variable $C_{1,T}$ appearing in Theorem 8.2.1, together with some rough calculations, implies that the magnitude of $C_{1,T}$ is of the order $T^{1+\epsilon_0}$, for some ϵ_0 small enough.*

Chapter 9

Future research and open problems

As can be gathered from the remark following Theorem 8.2.1, we do not believe that the bounds obtained in the previous chapter are sharp. In this chapter we shall present a few ideas aimed at improving the results obtained in the previous chapter, together with the some problems associated with implementing them.

9.1 On improving Theorem 8.2.1

Consider the case $|I| = 1$ in the definition (8.1) of a fractional Brownian flow. Then equation (8.19) can be restated as

$$\Delta^{-\beta} = 3\, n\, k_1(\alpha, \beta)\, \|B^H\|_{\beta,T} \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)},\, (a_{\Delta,2} + b_2)], \tag{9.1}$$

where $\tilde{M}_\alpha^{(1)}$, $a_{\Delta,2}$ and b_2 are constants.

Now a careful reading of the proofs of Theorem 8.1.1 and Lemma 8.1.1 yields that $\|B^H\|_{\beta,T}$ in (9.1) can be replaced by

$$Y(\beta, \Delta, B^H) \triangleq \max\{\|B^H\|_{\beta,i\Delta,(i+1)\Delta} :\ 0 \le i \le T/\Delta\},$$

where $\|B^H\|_{\beta,i\Delta,(i+1)\Delta}$ is the β Hölder norm of B^H in the interval $[i\Delta, (i+1)\Delta]$.

Clearly, not only is it true that

$$\|B^H\|_{\beta,T} \ge Y(\beta, \Delta, B^H),$$

but one expects that the right hand side, which is a local quantity is considerably smaller than the global quantity on the left hand side. Making this replacement should improve the bounds in Theorem 8.1.1, and hence in Theorem 8.2.1, but it seems that following this approach is not going to bear any fruit. For a start, after replacing $\|B^H\|_{\beta,T}$ by $Y(\beta, \Delta, B^H)$ in (9.1), a rough calculation yields that the solution to the new equation does not exist for each realization of the process B^H.

9.2 Ergodicity

Note that, even in the light of Lemma 7.0.1, it is inappropriate to compare the results of Theorem 8.2.1 with the ones obtained in Part I. For a start, Theorem 8.2.1 is a uniform, almost sure bound, whereas the results of Part I, in case of an isotropic Brownian flow, are results explaining the *average* behavior. Actually similar almost sure bounds can be obtained in the case of stochastic flows driven by standard Brownian motion by studying the Lyapunov exponents of the flow. However flows driven by fractional Brownian motions are not known to have an invariant measure, and hence are not known to exhibit ergodic behavior. Thus arguments concerning the Lyapunov exponents do not work in this case.

In [18], the ergodicity of stochastic differential equations driven by additive fractional noise is proven. In particular, let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, satisfying some regularity conditions, and define the process

$$x_t = x + \int_0^t f(x_s)\, ds + \sigma\, B^H(t), x \in \mathbb{R}^n, \qquad (9.2)$$

where B^H is an n-dimensional fractional Brownian motion with Hurst parameter H, and σ a constant, invertible $n \times n$ matrix. Then it is proven that the solution to the above stochastic differential equation converges to a unique stationary solution in an appropriate norm. It is noteworthy that the result is true for all $H \in (0, 1)$.

The idea of the proof is to build a stochastic dynamical system over an appropriate noise space, and then with some compactness arguments to show the existence of an invariant measure for the system. Finally, the result is achieved by a coupling construction.

Note that the stochastic differential equation (9.2), is driven by an additive noise, whereas the systems that we have studied are driven by a multiplicative noise. However, if it were possible to extend the ergodic properties of the solution of (9.2) to the case involving multiplicative noise, this would be a major step in the direction of improving the results obtained in Part II. But, as is noted by the author towards the end of [18], the case of multiplicative noise is considerably more involved than the additive case and requires better estimates.

Therefore, one of the directions of future research in this field would be to try to prove ergodicity for the flow described in (8.1). A good starting point would be to take a compact state space, so as to stop the flow from blowing up. With appropriate conditions on the vector fields, the flow will traverse almost all the points in the space, and hence can be believed to exhibit ergodicity.

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פרק 1

תקציר

נושא הנחקר לאחרונה בתחום תהליכים סטוכסטיים הוא דינאמיקה של יריעות המתפתחות באופן אקראי תחת זרימה סטוכסטית. המחקר בתחום זה מתמקד בעיקר בזרימות סטוכסטיות בהן מתאפשרים חישובים מפורשים. לדוגמא, זרימות בראוניות איזוטרופיות נחקרו ב-*Baxendale and Harris* [6], *Le Jan* [20, 21], וזרימות איזוטרופיות השומרות נפח נחקרו ב-*Craston and Le Jan* [12].

Baxendale and Harris [6] חקרו את האפיון של זרימה בראונית בעזרת שימוש בפונקצית השונות משותפת שלה. כמו כן, הם יסדו תוצאות לגבי תנועת שתי נקודות של הזרימה, וחקרו מאפיינים שונים של דינאמיקה של וקטורים משיקים וזרימות משיקות. מתחילת המחקר של זרימות סטוכסטיות אנשי מחקר הסתברות וגיאומטריה ניסו למצוא פרמטרים מתאימים כדי לאפיין זרימות, ובפרט מאפייני דינאמיקה. בעשרים השנים האחרונות חוקרים הראו עניין נכר ב-*Lyapunov exponents* ככלי בשביל להסביר את ההתנהגות האסימפטוטית של כמה מהמאפיינים של הזרימה, לדוגמא במחקר של שווי משקל סטטיסטי ותנועת שתי נקודות של זרימות ב- *Baxendale* [5]. הצעד הראשון בקישור בין *Lyapunov exponents* לבין זרימות סטוכסטיות נעשה על ידי *Carverhill* [8, 9]. ב-[9] *Carverhill* הוכיח גרסא של משפט ארגודי כפלי של *Oseledec* [31] עבור זרימות בראוניות סטוכסטיות של דיפאומורפיזמים של יריעה קומפקטית חלקה M תוך כדי שהראה קיום של מספרים $\lambda_1 > ... > \lambda_k$, הנקראים *Lyapunov exponents*, ותתי-מרחב $\{V_i^{x,\omega}\}_{i=1}^k$ אקראים של המרחב המשיק T_xM עבור כל x ביריעה M, כך ש- $V_i^{x,\omega} \subset V_{i+1}^{x,\omega}$ עבור $1 \leq i \leq k$ וה- *Lyapunov exponent* ה-i-י מוגדר על ידי $v \in V_i^{x,\omega} \setminus V_{i-1}^{x,\omega}$ כלשהו. (ה-ω באה להדגיש את האקראיות של תתי-המרחבים).

בערך באותו הזמן *Le Jan* [20] הוכיח תוצאות דומות, כולל ביטוי מפורש עבור

של זרימה בראונית איזוטרופית המוגדרת על המרחב האויקלידי *Lyapunov exponents* .$\mathbb{R}^n$

הדעה המקובלת בתחום הייתה שעל ידי *Lyapunov exponents* ניתן להסביר את הנשנות של התבנית היסודית השניה בכל נקודה של יריעה המתפתחת באופן אקראי תחת זרימה סטוכסטית. למרות זאת, *Cranston* [10] הצביע על כך שההרחבה למקרה הרציף של התוצאות שנתקבלו ב- *Cranston and Le Jan* [11] מראה שהאינטואיציה שהובילה להסבר של הנשנות של התבנית היסודית השניה תוך שימוש ב-*Lyapunov exponents* הייתה לא נכונה. בהסתמך על הרשום לעיל ניתן להסיק שיותר הגיוני לבחון מאפיינים אחרים של הזרימה כדי לחקור את ההתנהגות בזמן סופי של ההתפתחות הגיאומטרית של הזרימה. הדבר הוביל את *Cranston and Le Jan* [12] לחקור את ההתפתחות של הפולינומים הסימטריים של עקמומיות עקרוניות, כולל עקמומית ממוצעת ועקמומית גאוסית של יריעה $(n-1)$-ממדית המשוכנת ב-$\mathbb{R}^n$, המתפתחת תחת זרימה איזוטרופית שומרת נפח ב- $\mathbb{R}^n$. הם קיבלו נוסחת *Itô* עבור הפולינומים הסימטריים של עקמומיות עקרוניות ומזה הסיקו שלמרות שהווקטור של כל הפולינומים הסימטריים של העקמומיות העקרוניות מהווה דיפוזיה, הדבר אינו נכון עבור כל תת קבוצה של הווקטור.

בחלק הראשון של הדיסרטאציה אנו מרחיבים תוצאה זו על ידי עיון בהתנהגות דינאמית של עקמומיות ה-$Lipschitz-Killing$ של יריעות המתפתחות אקראית תחת זרימה בראונית איזוטרופית ושומרת נפח על $\mathbb{R}^n$. ניתן להתיחס אל העקמומיות ה-$Lipschitz-Killing$, $\{\mathcal{L}_k(M)\}_{k=0}^{dim(M)}$, הידועות גם כמידות עקמומיות, בתור הרחבות של נפחים אינטרינסיים ולכן הן יכולות להיקרא גם נפחים מוכללים. זוהי הרחבה טבעית אך משמעותית של *Cranston and Le Jan* [12], כאשר ניתן לייצג את העקמומיות ה-$Lipschitz-Killing$ כממוצע של פולינומים סימטריים של עקמומיות עקרוניות מעל היריעה.

לעומת המידע המסופק על ידי פילטראציה אקראית המושגת על ידי לימוד של *Lyapunov exponents*, אשר נותנים בעקרון רק מידע מקומי, העקמומיות ה-$Lipschitz-Killing$ מתארות את הגיאומטריה הגלובלית של היריעות המתפתחות באופן אקראי. וזו בהחלט מוטיבציה נוספת לעבוד עם עקמומיות $Lipschitz-Killing$. אנו מנסים לפתור בעיה זו על ידי עיון בהתפתחות סטוכסטית של עקמומיות $Lipschitz-Killing$ של יריעה המתפתחת אקראית תחת זרימה בראונית איזוטרופית ושומרת נפח. התוצאה העיקרית שלנו בחלק הראשון של הדיסרטציה היא הנוסחה של התפתחות סטוכסטית עבור עקמומיות $Lipschitz-Killing$ של יריעה המתפתחת אקראית תחת זרימה בראונית איזוטרופית ושומרת נפח, וכתוצאה מכך ביטוי מפורש פשוט עבור התוחלות שלהן כפונקציה

של הזמן. במילים פשוטות, נניח ש-M_t היא התמונה של יריעה $(n-1)$-ממדית קומפקטית חלקה M המשוכנת ב-$\mathbb{R}^n$ תחת זרימה Φ_t. יתרה מזאת, נסמן ב-$\mathcal{L}_m(M_t)$ את העקמומית ה-$Lipschitz-Killing$ ה-m-ית של היריעה M_t, עבור $0 \leq m \leq (n-1)$. אזי אנו יכולים להוכיח את התוצאה הבאה.

משפט 1.0.1 תהיה Φ_t זרימה בראונית איזוטרופית ושומרת נפח של דיפאומורפיזמים C^2 של $\mathbb{R}^n$. אזי עבור $0 \leq m \leq (n-1)$ תוחלת קצב הצמיחה של העקמומיות ה-$Lipschitz-Killing$ נתונה על ידי

$$\mathbb{E}\{\mathcal{L}_m(M_t)\} = \mathcal{L}_m(M)\exp(Ct),$$

כאשר C קבוע ובלתי תלוי ב-t.

מאפיין חשוב של זרימות בראוניות המכריע עבור החלק הראשון הוא תכונה מרקובית של תנועת נקודה בודדת של הזרימה.

בחלק השני של הדיסרטציה אנו מנסים לבדוק מה יכול להיעשות במקרה יותר קשה, כאשר הזרימה אינה מרקובית ואינה מהווה דיפוזיה. כלומר, בחלק השני אנו בוחנים זרימות סטוכסטיות המונעות על ידי תנועת בראון פרקציונאלית עם פרמטר $Hurst$ $H > 1/2$. קיימות שתי סיבות מאחורי הבחירה של תנועת בראון פרקציונאלית. הסיבה הראשונה היא שזרימות כאלו מופיעות במספר יישומים (ראה $[15, 35]$). הסיבה השניה היא שלמרות הסרת תכונת המרקוביות, המבנה הגאוסי עדיין מאפשר לבצע כמה חישובים מפורשים.

תנועת בראון פרקציונאלית $\{B^H(t), t \geq 0\}$ עם פרמטר $Hurst$ $H \in (0,1)$ היא תהליך גאוסי עם ממוצע 0 ואינקרמנטים סטציונארים עם תכונת דמיון עצמי עם אינדקס H. כלומר:

$$(B^H(t) - B^H(s)) \stackrel{\mathcal{L}}{=} B^H(|t-s|),$$

$$B^H(t) \stackrel{\mathcal{L}}{=} t^H B^H(1),$$

עבור כל $t, s \geq 0$.

הודות לכך שזהו תהליך גאוסי הוא יכול להיות מאופיין על ידי פונקציית השונות המשותפת שלו, הנתונה על ידי:

$$E(B^H(s)B^H(t)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1.1}$$

נשים לב ש-$E(B^H(t)-B^H(s)) = |t-s|^{2H}$, ולכן לתהליך B^H אינקרמנטים סטציונארים. יתר על כך עבור מקרה $H = 1/2$, לתהליך אינקרמנטים בלתי תלויים וזה המקרה של

תנועה בראונית סטנדרטית, שהיא תהליך מרקוב וגם מרטינגייל. יישום פשוט של אי-שוויון $Garsia-Rodemich-Rumsey$ יחד עם (1.1) גורר שלתהליך B^H מסלולים רציפים $\alpha - Hölder$ עבור כל ה-$\alpha \in (0, H)$. (ראה [17] עבור האי-שוויון המקורי.) חשוב לציין שלעומת זאת במקרה של $H \neq 1/2$, B^H אינו תהליך מרקוב ואינו סמי-מרטינגייל.

בחלק השני של הדיסרטאציה אנו חוקרים את תנועת בראון הפרקציונאלית, מציגים דרכים שונות להגדרת אינטגראל סטוכסטי ביחס לתהליך הנ"ל ובוחנים זרימות המונעות על ידי תנועת בראון פרקציונאלית עם $H > 1/2$.

אחת מהעבודות הראשונות העוסקות במשוואות דיפרנציאליות/אינטגראליות המונעות על ידי תנועת בראון פרקציונאלית עם $H > 1/2$ נעשתה על ידי $Lyons$ [25] ב-1994. הרעיון העיקרי היה מבוסס על הניתוח של $Young$ העוסק באינטגראלים המונעים על ידי פונ-קציות לא חלקות. קיום ויחידות פתרון של משוואה אינטגראלית המונעת על ידי תנועת בראון פרקציונאלית עם $H > 1/2$ הוכח תוך שימוש בוריאציה ה-p-ית של אינטגראלים חוזרים ונשנים של התהליך B^H בשביל להגיע להתנהגות ליפשיציאנית עבור איטראציות בסכמת $Picard$. יותר מאוחר, ב-1998, $Lyons$ הכליל את הטיעון ב-[26] והרחיב את השי-טה למקרים רבים אחרים, כולל המקרה של $H = 1/2$. הדבר היה מורחב גם למקרה של $H > 1/4$ על ידי $Unterberger$ [37].

בערך באותו הזמן $Zähle$ [43, 44], הגדיר אינטגראלים מסלוליים מן הצורה $\int_0^t u_s \, dB^H(s)$, עבור $H > 1/2$, כאשר u ו-B^H הם אלמנטים של מרחב $Sobolev$ פרקציונאלי. בעקבות השיטה לעיל של שימוש בהגדרה מסלולית של אינטגראלים סטוכסטים המונעים על ידי תנועת בראון פרקציונאלית, $Nualart\ and\ Răscanu$ [30] הוכיחו את הקיום והיחידות של פתרונות של משוואות דיפרנציאליות סטוכסטיות רב-ממדיות המונעות על ידי תנועת בראון פרקציונאלית עם $H > 1/2$. האינטגראל ביחס ל-B^H מוגדר במובן אינטגראל $Riemann - Stieltjes$ מסלולי, כמו שמוגדר על ידי $Zähle$.

אנו נשתמש בשיטה המתוארת לעיל כדי לקבל את התוצאה העיקרית של החלק השני, שהיא הערכת צמיחת מידת $Hausdorff$ של יריעה המתפתחת אקראית תחת זרימה סטוכסטית המונעת על ידי תנועת בראון פרקציונאלית. במילים אחרות:

משפט 1.0.2 תהיה M_t תמונה של היריעה ה-m-ממדית חלקה M המשוכנת ב-$\mathbb{R}^n$ עבור $m < n$ כלשהו, תחת זרימה פרקציונאלית Φ_t. ותהיה $\mathcal{L}_m(M_t)$ מידת $Hausdorff$ m-ממדית של היריעה M_t. אזי קיימים קבועים c_1 ו-C_1 כך ש

$$\sup_{t \in [0,T]} \mathcal{L}_m(M_t) \leq c_1 \mathcal{L}_m(M) 2^{C_1 T \|B^H\|_{\beta,T}^{1/\beta}},$$

כאשר $\|B^H\|_{\beta,T}$ נורמת $\beta - Hölder$ של התהליך המניע B^H, ו-β הוא פרמטר המוגדר מטה.

המחקר נעשה בהנחייתו של פר' רוברט אדלר בפקולטה הנדסת תעשיה וניהול.

כל תקופת ההשתלמות היתה חוויה נהדרת עבורי ועל כך אני מודה לפר' רוברט אדלרץ

כמו כן ברצוני להודות לכל האנשים בפקולטה להנדסת תעשיה וניהול. היה לי מזל רב לפגוש את כולכם!

אני מודה לבית הספר ללימוד מוסמכים רוברט אדלר על התמיכה הכספית הנדיבה בהשתלמות.

על הדיפוזיה של הצורות

חיבור על מחקר

לשם מילוי חלקי של הרישות לקבלת תואר

דוקטור לפילוסופיה

סריקר ודלמני

הוגש לסנט הטכניון – מכון טכנולוגי לישראל
חשון תשס"ח   חיפה   נובמבר 2007