

Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index ^{*}

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Abstract

We study the asymptotic behavior for radial solutions on an annulus of a semilinear elliptic equation with homogeneous Dirichlet boundary condition and polynomial nonlinearity. We are able to show that Morse index informations on such solutions provide a complete description of the blow up behavior. In particular, in many situations we obtain that the radial ground state solution blows up and concentrates at the inner boundary of the annulus.

Keywords: Blow up analysis, Morse index, Mountain Pass Theorem.

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1 Introduction

In this paper we study the asymptotic behavior of the radial solutions of the problem,

$$(P_\lambda) \begin{cases} -\Delta u(x) + \lambda V(x)u(x) = u^p(x) & \text{in } A \\ u(x) > 0 & \text{in } A \\ u(x) = 0 & \text{on } \partial A \end{cases}$$

where $\lambda > 0$, $p > 1$, $A = \{x \in \mathbb{R}^N : 1 < |x| < 2\}$ and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is smooth with $V(x) = V(|x|) \geq c_0 > 0 \forall x \in \mathbb{R}^N$. The main result we prove is,

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Theorem 4.2. Let λu be a radial solution of (P_λ) with Morse Index one in the radial space $H_{0,r}^1(A)$. Let $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ with a_n being the unique point of maximum of $\lambda_n u$. Let a_∞ be the limit of a subsequence of a_n . Then

$$\begin{cases} \dot{M}(a_\infty) \geq 0 & \text{if } a_\infty = 1 \\ \dot{M}(a_\infty) = 0 & \text{if } a_\infty \in (1, 2) \\ \dot{M}(a_\infty) \leq 0 & \text{if } a_\infty = 2 \end{cases}$$

holds, where $M(r) = r^{n-1}V^\theta(r)$ where $\theta = \frac{p+1}{p-1} - \frac{1}{2}$ where $\dot{\cdot}$ denotes the derivative with respect to r .

Remark 1. That λu has a unique maximum, under the hypothesis of Morse Index one is in the content of corollary 3.4.

Note that if

$$\begin{cases} \dot{M}(1) < 0 < \dot{M}(2) & \text{implies interior blow-up} \\ \dot{M}(r) > 0 & r \in [1, 2] \text{ implies blow-up at 1} \\ \dot{M}(r) < 0 & r \in [1, 2] \text{ implies blow-up at 2} \end{cases}$$

The behavior has been studied earlier in [3] and [4]. The result we have is more general and also the use of Morse Index, which in some sense seems more natural, to prove the main result in a different approach.

Remark 2. Consider the problem (P_λ) , then it is clear that the solutions of (P_λ) in the space $H_{0,r}^1(A)$ are critical points of the functional $J_\lambda : H_{0,r}^1(A) \rightarrow \mathbb{R}$, defined by,

$$J_\lambda(u) = \frac{1}{2} \int_A |\nabla u|^2 - \frac{\lambda}{2} \int_A V(r)u^2 - \frac{1}{p+1} \int_A (u^+)^{p+1}$$

It is easy to see that J_λ satisfies the hypothesis of Mountain Pass Theorem and that the Palais Smale condition holds. Hence there exist's a Mountain Pass level c_λ . At this level J_λ has a critical point with Morse Index less than or equal to one follows from Theorem 10.2 of [8] (See page 222 and Theorem 5.1 of [6] as well as page 1598 of [7]). In context of the problem we are looking at, it follows that there exists a solution of Morse Index one. Note any nontrivial solution of P_λ has Morse Index atleast one.

Throughout what follows we assume that we have a solution of Morse Index one of P_λ in the space $H_{0,r}^1(A)$ which is a consequence of Remark 2. It is this solution we will be considering.

Before we consider the general problem, we consider the case $V(x) \equiv 1$ in section 2 and give a simple proof of $a_\lambda \rightarrow 1$, where a_λ is the point of unique maximum of λu solution of (P_λ) and indicate how the ideas of the proof lead naturally to some ideas built in section 3 and 4 while dealing with the general case.

2 The case $V(x) = 1$.

In this section we consider the problem

$$(P'_\lambda) \begin{cases} -\Delta u(x) + \lambda u(x) = u^p(x) & \text{in } A \\ u(x) > 0 & \text{in } A \\ u(x) = 0 & \text{on } \partial A \end{cases}$$

and prove,

Theorem 2.1. *If λu is a radial solution of (P'_λ) with Morse Index one, in $H_{0,r}^1(A)$, then if a_λ denotes the maximum of λu , $a_\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$.*

Before we prove the Theorem we will prove the following.

Lemma 2.2. *If λu is as in Theorem 2.1, then λu has a unique maximum.*

We will prove the Lemma in two different ways, the reason being they serve as a good motivation in terms of ideas to be used in sections 3 and 4 dealing with the general case.

Proof 1: Dropping the suffix λ , we see that

$$-\Delta u_r = (pu^{p-1} - \lambda)u_r - (N-1)\frac{u_r}{r^2} \quad (1)$$

holds. Now if u has more than one maximum, then denoting by a and b the maximum and c the minimum between the two maxima a and b , we see that u_r satisfies

$$\langle -\Delta u_r - (pu^{p-1} - \lambda)u_r, u_r \rangle < 0$$

in each of the intervals (a, c) and (c, b) , which leads to a contradiction to the fact that Morse Index of u in $H_{0,r}^1(A)$ is one.

Proof 2: From,

$$\begin{cases} -u_{rr} - (N-1)\frac{u_r}{r} = u^p - \lambda u & \text{in } (1, 2) \\ u > 0 & \text{in } (1, 2) \\ u(1) = u(2) = 0 \end{cases} \quad (2)$$

We have, on multiplying (2) by u_r on both sides and integrating from 1 to 2,

$$\frac{u_r^2(1)}{2} - \frac{u_r^2(2)}{2} - (N-1) \int_1^2 \frac{u_r^2}{r} dr = 0 \quad (3)$$

(3) implies

$$\frac{u_r^2(1)}{2} > (N-1) \int_1^2 \frac{u_r^2}{r} dr \quad (4)$$

Suppose c is a local minimum, on multiplying (2) by u_r and integrating from 1 to c , we have

$$\frac{u_r^2(1)}{2} - (N-1) \int_1^c \frac{u_r^2}{r} = \frac{u^{p+1}(c)}{p+1} - \frac{\lambda}{2} u^2(c) \quad (5)$$

Note that by (4), L.H.S is positive. This implies since $p > 1$, $(u^p - \lambda u)(c) > 0$, a contradiction to c being a local minima. Hence the Lemma.

Remark 3. Now we will switch back to a_λ being the unique maximum in the context of Theorem 2.1. Notice that in $(1, a_\lambda)$, we have $\lambda u_r \geq 0$ implies $\lambda u \uparrow$ in $(1, a_\lambda)$. Since λu is a positive solution, there exists $d_\lambda \in (1, a_\lambda)$ such that $\lambda u^{p-1}(d_\lambda) = \lambda$. Also since in $(1, d_\lambda)$ we have $\lambda u^p - \lambda \lambda u \leq 0$, from

$$-(r^{N-1} \lambda u_r)_r = (\lambda u^p - \lambda \lambda u) r^{N-1}$$

We have by integration from 1 to r_0 , $r_0 \in (1, d_\lambda]$

$$\lambda u_r(1) - r_0^{N-1} \lambda u_r(r_0) \leq 0$$

which implies,

$$\lambda u_r(r_0) \geq C \lambda u_r(1) \quad (6)$$

where $C > 0$ is a constant independent of λ .

Proof of Theorem 2.1. From

$$-\lambda u_{rr} - \frac{(N-1)}{r} \lambda u_r = \lambda u^p - \lambda \lambda u$$

On multiplication by λu_r and integrating from 1 to d_λ , where d_λ is as in Remark 3, we have

$$\frac{\lambda u_r^2(d_\lambda)}{2} - \frac{\lambda u_r^2(1)}{2} + (N-1) \int_1^{d_\lambda} \frac{\lambda u_r^2}{r} = \frac{p-1}{2(p+1)} \lambda^{\frac{p+1}{p-1}} \quad (7)$$

Using (4) in (7), we have

$$\lambda u_r(d_\lambda) \geq C_0 \lambda^{\frac{p+1}{2(p-1)}} = C_0 \lambda^{\frac{1}{2} + \frac{1}{p-1}} \quad (8)$$

By Pohozaev Identity, we have

$$\lambda u_r^2(1) = 2\lambda u_r^2(2) + 2(N-2) \int_1^2 \lambda u_r^2 + \frac{n-1}{2} \int_1^2 \frac{\lambda u_r^2}{r^2} + \frac{p-1}{p-1} \int_1^2 \lambda u^{p+1} \quad (9)$$

and hence

$$\lambda u_r^2(1) \geq C_2 \int_1^2 \lambda u^{p+1} \quad (10)$$

Now consider

$$-(r^{N-1} \lambda u_r)_r = (\lambda u^p - \lambda \lambda u) r^{N-1}$$

We have on integration from d_λ to a_λ ,

$$d_\lambda^{N-1} \lambda u_r(d_\lambda) = \int_{d_\lambda}^{a_\lambda} \lambda u^p r^{N-1} - \lambda \int_{d_\lambda}^{a_\lambda} \lambda u r^{N-1} \quad (11)$$

which implies,

$$\begin{aligned} \int_{d_\lambda}^{a_\lambda} \lambda u^p r^{N-1} &\geq d_\lambda^{N-1} \lambda u_r(d_\lambda) \\ \int_{d_\lambda}^{a_\lambda} \lambda u^p r^{N-1} &\geq C_2 \lambda^{\frac{1}{2} + \frac{1}{p-1}} \end{aligned} \quad (12)$$

by (8). Since $1 < r < 2$, we have

$$\int_{d_\lambda}^{a_\lambda} \lambda u^p \geq C_3 \lambda^{\frac{1}{2} + \frac{1}{p-1}} \quad (13)$$

Combining (10) and (13), we have

$$\lambda u_r^2(1) \geq C_4 \lambda^{\frac{1}{p-1}} \lambda^{\frac{1}{2} + \frac{1}{p-1}} \quad (14)$$

which implies,

$$\lambda u_r(1) \geq C_4 \lambda^{\frac{1}{4} + \frac{1}{p-1}} \quad (15)$$

we have used the fact that $\lambda u \geq \lambda^{\frac{1}{p-1}}$ in (d_λ, a_λ) in (10) to arrive at (14).

Now to complete the proof, note that it is enough to show $d_\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$. Since if $d_\lambda \rightarrow 1$ and $a_\lambda \not\rightarrow 1$, then if $a_\lambda \rightarrow a_\infty \neq 1$ for some subsequence then in $(1 - \varepsilon, a_\infty)$, $\lambda u \geq \lambda^{\frac{1}{p-1}}$ leads to a contradiction to the fact that Morse Index of λu is one.

Note that the linearized eigenvalue problem on $\Omega_{(1-\varepsilon, a_\infty)}$ which is an annulus with inner radius $1 - \varepsilon$ and outer radius a_∞ is

$$\begin{cases} -\Delta w = \mu(\lambda u^{p-1} - \lambda)w & \text{in } \Omega_{(1-\varepsilon, a_\infty)} \\ w(1 - \varepsilon) = w(a_\infty) = 0 \end{cases} \quad (16)$$

admits many radial eigenvalues $\mu < 1$, contradicting λu is of Morse Index one on $H_{0,r}^1(A)$. To prove $d_\lambda \rightarrow 1$, we have by Mean Value Theorem,

$$\lambda u(d_\lambda) = \lambda u_r(\xi)(d_\lambda - 1) \quad (17)$$

for some $\xi \in (d_\lambda, 1)$, i.e.

$$\lambda^{p-1} = \lambda u_r(\xi)(d_\lambda - 1) \quad (18)$$

But by (6) and (15) we have $\lambda u_r(\xi) \geq C_1 \lambda^{\frac{1}{4} + \frac{1}{p-1}}$. Using this in (18) we see that $d_\lambda \rightarrow 1$ as $\lambda \rightarrow \infty$. Hence the theorem.

3 Blow up profile

In this and the following section we study the general case. We first start with a blow up analysis which will be used to prove the main result in section 4.

Let u_n be a radial solution of

$$\begin{cases} -\Delta u_n + \lambda_n V u_n = u_n^p & \text{in } A, \\ u_n > 0 & \text{in } A, \\ u_n = 0 & \text{on } \partial A \end{cases} \quad (19)$$

for a sequence $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$, where $A = \{x \in \mathbb{R}^N : 1 < |x| < 2\}$, $N \geq 2$, $p > 1$ and V is a smooth radial potential. We are interested in the description of the blow up behavior of u_n and to this purpose we will assume:

$$\max_{1 < |x| < 2} u_n(x) \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

Further, assume on the potential V that

$$V(x) \geq c_0 \quad \text{in } A, \quad (20)$$

for some positive constant c_0 . Let $a_n \in (1, 2)$ be the absolute maximum of $u_n(r)$, $r = |x|$: $u_n(a_n) = \max_{1 < r < 2} u_n(r)$. We rewrite (19) in the form:

$$\begin{cases} -\ddot{u}_n - \frac{N-1}{r} \dot{u}_n = u_n^p - \lambda_n V(r) u_n & \text{in } (1, 2), \\ u_n > 0 & \text{in } (1, 2), \\ u_n(1) = u_n(2) = 0. \end{cases} \quad (21)$$

Let us introduce $\varepsilon_n = u_n(a_n)^{-\frac{p-1}{2}}$ and $U_n(r) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n r + a_n)$ for $r \in I_n = (-\frac{a_n-1}{\varepsilon_n}, \frac{2-a_n}{\varepsilon_n})$. Then, $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$ and U_n satisfies:

$$\begin{cases} -\ddot{U}_n(r) - (N-1) \frac{\varepsilon_n}{\varepsilon_n r + a_n} \dot{U}_n(r) = U_n^p(r) - \lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) U_n(r) & \text{for } r \in I_n, \\ 0 < U_n(r) \leq U_n(0) = 1 & \text{for } r \in I_n, \\ U_n|_{\partial I_n} = 0. \end{cases} \quad (22)$$

Since a_n is a maximum point of u_n , we have that $u_n^p(a_n) - \lambda_n V(a_n) u_n(a_n) = -\Delta u_n(a_n) \geq 0$ and therefore,

$$1 \geq \frac{\lambda_n V(a_n) u_n(a_n)}{u_n^p(a_n)} = \frac{\lambda_n V(a_n)}{u_n^{p-1}(a_n)} = \lambda_n \varepsilon_n^2 V(a_n) \geq 0.$$

Up to a subsequence, we can assume that:

$$\lambda_n \varepsilon_n^2 V(a_n) \rightarrow \mu \in [0, 1] \text{ as } n \rightarrow +\infty, \quad (23)$$

and

$$\frac{a_n - 1}{\varepsilon_n} \rightarrow L_0 \in [0, +\infty], \quad \frac{2 - a_n}{\varepsilon_n} \rightarrow L_1 \in [0, +\infty] \text{ as } n \rightarrow +\infty.$$

Let I be some bounded interval so that $0 \in \bar{I}$ and let $r \in I$ be fixed. Then, by (22) and $U_n \leq 1$

$$\begin{aligned} |\dot{U}_n(r)| &= |\dot{U}_n(r) - \dot{U}_n(0)| \leq |r| \int_0^1 |\ddot{U}_n(tr)| dt \\ &\leq \left((N-1)\varepsilon_n \sup_{s \in I} |\dot{U}_n(s)| + 1 + \lambda_n \varepsilon_n^2 \sup_{s \in I} V(s) \right) |r| \\ &\leq C \left(\varepsilon_n \sup_{s \in I} |\dot{U}_n(s)| + 1 \right) |r| \end{aligned}$$

for some $C > 0$, since $\lambda_n \varepsilon_n^2 = O(1)$ in view of (20) and (23). Hence,

$$\sup_{r \in I} |\dot{U}_n(r)| \leq C|I| \left(\varepsilon_n \sup_{s \in I} |\dot{U}_n(s)| + 1 \right)$$

and, since $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$,

$$\sup_{r \in I} |\dot{U}_n(r)| \leq C|I| \quad (24)$$

for some $C > 0$ and n large (depending on $|I|$). Moreover, for all $r, s \in I$:

$$|\dot{U}_n(r) - \dot{U}_n(s)| \leq |r - s| \int_0^1 |\ddot{U}_n(s + t(r - s))| dt \leq C(\varepsilon_n \sup_{t \in I} |\dot{U}_n(t)| + 1)|r - s| \leq C|I||r - s| \quad (25)$$

for some $C > 0$. Since by (24) we get that, if $L_0 < +\infty$,

$$1 = U_n(0) - U_n\left(-\frac{a_n - 1}{\varepsilon_n}\right) = \int_{-\frac{a_n - 1}{\varepsilon_n}}^0 \dot{U}_n(t) dt \leq C\left(\frac{a_n - 1}{\varepsilon_n}\right)^2$$

and, if $L_1 < +\infty$,

$$1 = U_n(0) - U_n\left(\frac{2 - a_n}{\varepsilon_n}\right) = \int_{\frac{2 - a_n}{\varepsilon_n}}^0 \dot{U}_n(t) dt \leq C\left(\frac{2 - a_n}{\varepsilon_n}\right)^2,$$

we obtain that $L_0 > 0$ and $L_1 > 0$. Since

$$L_0 + L_1 = \lim_{n \rightarrow +\infty} \left(\frac{a_n - 1}{\varepsilon_n} + \frac{2 - a_n}{\varepsilon_n} \right) = \lim_{n \rightarrow +\infty} \frac{1}{\varepsilon_n} = +\infty,$$

we have that both L_0, L_1 cannot be bounded. Now we show that:

Lemma 3.1. *Up to a subsequence, $L_0 = L_1 = +\infty$.*

Proof By contradiction, assume for example $L_0 < +\infty$ and $L_1 = +\infty$. Since U_n is uniformly bounded in $C_{\text{loc}}^{1,1}\left[-\frac{a_n - 1}{\varepsilon_n}, \frac{2 - a_n}{\varepsilon_n}\right]$ in view of (24)-(25), up to a subsequence we assume that $U_n \rightarrow U$ in $C_{\text{loc}}^1(-L_0, +\infty)$, where $U \in C^1[-L_0, +\infty)$ satisfies

$$\begin{cases} -\ddot{U} + \mu U = U^p & \text{in } (-L_0, +\infty) \\ 0 \leq U(r) \leq U(0) = 1 & \text{in } (-L_0, +\infty) \\ U(-L_0) = 0. \end{cases}$$

We have that $U(r) > 0$ for any $r > -L_0$. Otherwise, there exists $r_0 > -L_0$ so that $U(r_0) = \dot{U}(r_0) = 0$. Hence, by uniqueness for the Cauchy problem, $U \equiv 0$ contradicting $U(0) = 1$.

Let us remark that $\mu > 0$. Otherwise, if $\mu = 0$ we have that $\ddot{U} < 0$ in $(-L_0, +\infty)$ and therefore, \dot{U} is strictly decreasing in $(-L_0, +\infty)$ and admits limit as $r \rightarrow +\infty$: $\lim_{r \rightarrow +\infty} \dot{U}(r) = U'_\infty$. Since $\dot{U}(0) = 0$, we get that $U'_\infty < 0$ and therefore, $\lim_{r \rightarrow +\infty} U(r) = -\infty$ contradicting $U \geq 0$.

Further, we have the following conservation law associated to U :

$$E(r) := \frac{1}{2} \dot{U}^2(r) + G(U(r)) \quad (26)$$

is constant for $r \in (-L_0, +\infty)$, where $G(s) = \frac{1}{p+1} s^{p+1} - \frac{\mu}{2} s^2$. Let us observe that $G(s)$ has the following shape:

$$\begin{aligned} G(s) &\leq 0 \text{ for } s \in [0, (\mu \frac{p+1}{2})^{\frac{1}{p-1}}] \text{ and } G(s) = 0 \text{ iff } s = 0, (\mu \frac{p+1}{2})^{\frac{1}{p-1}} \\ \dot{G}(s) &< 0 \text{ for } s \in (0, \mu^{\frac{1}{p-1}}) \text{ and } \dot{G}(s) > 0 \text{ for } s \in (\mu^{\frac{1}{p-1}}, (\mu \frac{p+1}{2})^{\frac{1}{p-1}}). \end{aligned} \quad (27)$$

Let $E_0 = \frac{1}{p+1} - \frac{\mu}{2}$. By (26) we get that $E(r) = E_0$ for any $r > -L_0$ and, since Hopf's Lemma provides $\dot{U}(-L_0) > 0$, $\frac{1}{p+1} - \frac{\mu}{2} = E_0 = E(-L_0) > 0$. So, $\dot{U}(0) = 0$, $\ddot{U}(0) = \mu U(0) - U(0)^p = \mu - 1 < -\frac{p-1}{p+1}$ and $G(s) < G(1) = E_0$ for $0 \leq s < 1$ imply that $\dot{U}(r) < 0$ for $r > 0$. Hence, $U(r)$ possesses limit as $r \rightarrow +\infty$ and $0 \leq \lim_{r \rightarrow +\infty} U(r) = U_\infty < 1$. Since $G(U_\infty) < E_0$, we have that $\lim_{r \rightarrow +\infty} \dot{U}(r) < 0$ contradicting $U \geq 0$. The proof is complete. \square

Since $L_0 = L_1 = +\infty$, up to a subsequence $U_n \rightarrow U$ in $C^1_{\text{loc}}(\mathbb{R})$, where $U \in C^1(\mathbb{R})$ satisfies

$$\begin{cases} -\ddot{U} + \mu U = U^p \text{ in } \mathbb{R} \\ 0 < U(r) \leq U(0) = 1 \text{ in } \mathbb{R}. \end{cases} \quad (28)$$

To go further in our analysis, let us assume that

$$u_n \text{ has Morse index 1 in the radial space } H^1_{0,r}(A). \quad (29)$$

We obtain the following easy result:

Lemma 3.2. *Assume (29). Then $\mu = \frac{2}{p+1}$ and there holds:*

$$\lim_{r \rightarrow \pm\infty} U(r) = \lim_{r \rightarrow \pm\infty} \dot{U}(r) = 0. \quad (30)$$

Further,

$$\frac{2}{p+1} \int_{\mathbb{R}} U^2 = \left(\frac{1}{2} + \frac{1}{p+1}\right) \int_{\mathbb{R}} U^{p+1}.$$

Proof First of all, we want to show that $\mu < 1$. Otherwise, if $\mu = 1$, by the uniqueness for the Cauchy problem we have $U \equiv 1$ in \mathbb{R} , and we can easily construct cut-off functions $\phi_i(r) \in C_0^\infty(\mathbb{R})$, $i = 1, 2$, with disjoint supports so that

$$\int_{\mathbb{R}} \left(\dot{\phi}_i^2(r) + \phi_i^2(r) - pU^{p-1}(r)\phi_i^2(r) \right) dr < 0 \quad (31)$$

for any $i = 1, 2$. Define now $\psi_{i,n}(x) = \sqrt{\varepsilon_n} \phi_i\left(\frac{|x| - a_n}{\varepsilon_n}\right)$, $i = 1, 2$. Hence, for $i = 1, 2$ we get that as $n \rightarrow +\infty$:

$$\begin{aligned} \int_A |\nabla \psi_{i,n}|^2 &= \omega_N \int_{\mathbb{R}} \dot{\phi}_i^2(r) (\varepsilon_n r + a_n)^{N-1} dr \rightarrow \omega_N a_\infty^{N-1} \int_{\mathbb{R}} \dot{\phi}_i^2(r) dr, \\ \int_A \lambda_n V \psi_{i,n}^2 &= \omega_N \int_{\mathbb{R}} \lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) \phi_i^2(r) (\varepsilon_n r + a_n)^{N-1} dr \rightarrow \omega_N a_\infty^{N-1} \int_{\mathbb{R}} \phi_i^2(r) dr, \\ p \int_A u_n^{p-1} \psi_{i,n}^2 &= \omega_N p \int_{\mathbb{R}} U_n^{p-1}(r) \phi_i^2(r) (\varepsilon_n r + a_n)^{N-1} dr \rightarrow \omega_N a_\infty^{N-1} p \int_{\mathbb{R}} U^{p-1}(r) \phi_i^2(r) dr, \end{aligned}$$

where $a_\infty = \lim_{n \rightarrow +\infty} a_n$ (up to a subsequence). Hence, we get that

$$\int_A (|\nabla \psi_{i,n}|^2 + \lambda_n V \psi_{i,n}^2 - p u_n^{p-1} \psi_{i,n}^2) \rightarrow \omega_N a_\infty^{N-1} \int_{\mathbb{R}} \left(\dot{\phi}_i^2(r) + \phi_i^2(r) - p U^{p-1}(r) \phi_i^2(r) \right) dr < 0$$

as $n \rightarrow +\infty$. Since $\psi_{i,n} \in C_0^\infty(A)$ are radial functions with disjoint supports, we get that the Morse index of u_n in the radial space $H_{0,r}^1(A)$ is at least 2 contradicting (29).

By (26) we get that $E(r) = E_0 = \frac{1}{p+1} - \frac{\mu}{2}$ for any $r \in \mathbb{R}$, where $E(r) = \frac{1}{2} \dot{U}^2(r) + G(U(r))$ and $G(s) = \frac{1}{p+1} s^{p+1} - \frac{\mu}{2} s^2$.

If $\frac{2}{p+1} < \mu < 1$, $E_0 < 0$ and by (27) we have the existence of an unique $s_0 \in (0, 1)$ solution of $G(s_0) = E_0$. The solution U is a periodic function of period $R = 2 \int_{s_0}^1 \frac{ds}{\sqrt{2(E_0 - G(s))}}$ which lies

in the region $s_0 \leq U \leq 1$. Since $s_0 < \mu^{\frac{1}{p-1}}$, by (27) we obtain that for the function $h(s) = s^p - \mu s_0 - p s^{p-1}(s - s_0)$ there hold: $h(s_0) = s_0^p - \mu s_0 = \dot{G}(s_0) < 0$, $\dot{h}(s) = -p(p-1)s^{p-2}(s - s_0) \leq 0$ for $s \geq s_0$. Hence, $h(s) < 0$ for $s_0 \leq s \leq 1$ and

$$-(U - s_0)'' + \mu(U - s_0) - pU^{p-1}(U - s_0) = U^p - \mu s_0 - pU^{p-1}(U - s_0) = h(U) < 0 \quad (32)$$

in view of $s_0 \leq U \leq 1$. Let $r_0 \in (0, R)$ be such that $U(r_0) = s_0$. Since $U(r) - s_0 = 0$ for any $r = r_0 + mR$, $m \in \mathbb{Z}$, we can consider $\phi_m \in C_0(\mathbb{R}) \cap H^1(\mathbb{R})$ defined as: $\phi_m(r) = U(r) - s_0$ if $r \in (mR + r_0, (m+1)R + r_0)$ and 0 otherwise. The functions ϕ_m , $m \in \mathbb{Z}$, have all disjoint supports and satisfy (31) in view of (32). Arguing as before, it is possible to show that the associated $\psi_{m,n} \in H_{0,r}^1(A)$, $m \in \mathbb{Z}$, give that the Morse index of u_n is not finite contradicting (29).

If $\mu < \frac{2}{p+1}$, $\dot{U}(0) = 0$, $\ddot{U}(0) = \mu U(0) - U(0)^p = \mu - 1 < 0$ and $G(s) < E_0$ for $0 \leq s < 1$ imply that $\dot{U}(r) < 0$ for $r > 0$. Similarly, if $\mu = \frac{2}{p+1}$ we obtain that $E_0 = 0$ and $\dot{U}(r) < 0$ for $0 < r < R$, where R is the first zero of $U(r)$. If R is finite, since $E_0 = 0$ we have also that $\dot{U}(R) = 0$ and, by

uniqueness of the Cauchy problem, $U \equiv 0$ contradicting $U(0) = 1$. Hence, for $\mu \leq \frac{2}{p+1}$ we have that $\dot{U}(r) < 0$ for $r > 0$ and $U(r)$ possesses limit as $r \rightarrow +\infty$: $0 \leq \lim_{r \rightarrow +\infty} U(r) = U_\infty < 1$. We must have that $G(U_\infty) = E_0$ since $G(U_\infty) < E_0$ gives that $\lim_{r \rightarrow +\infty} \dot{U}(r) < 0$, contradicting $U \geq 0$. Property $G(U_\infty) = E_0$ ensures that $\mu = \frac{2}{p+1}$, $E_0 = 0$ and $\lim_{r \rightarrow +\infty} U(r) = U_\infty = 0$. Further, since $E(r) \equiv E_0 = 0$ the limit $\lim_{r \rightarrow +\infty} \dot{U}(r)$ exists and vanishes. By uniqueness of the Cauchy problem, $U(r) = U(-r)$ for any $r \in \mathbb{R}$. Hence, there holds also: $\lim_{r \rightarrow -\infty} U(r) = \lim_{r \rightarrow -\infty} \dot{U}(r) = 0$. Since $E(r) \equiv 0$, by integration on \mathbb{R} we get that

$$\frac{1}{2} \int_{\mathbb{R}} \dot{U}^2 = \frac{1}{p+1} \left(\int_{\mathbb{R}} U^2 - \int_{\mathbb{R}} U^{p+1} \right),$$

and, multiplying (28) by U and integrating on \mathbb{R} , by means of an integration by parts and (30) we obtain that

$$\frac{1}{2} \int_{\mathbb{R}} \dot{U}^2 = -\frac{1}{p+1} \int_{\mathbb{R}} U^2 + \frac{1}{2} \int_{\mathbb{R}} U^{p+1}.$$

Taking the difference of these two relations, finally we get that

$$\frac{2}{p+1} \int_{\mathbb{R}} U^2 = \left(\frac{1}{2} + \frac{1}{p+1} \right) \int_{\mathbb{R}} U^{p+1}$$

and the proof is complete. \square

We are now in position to show a decay property for U_n which will be crucial in the sequel:

Proposition 3.3. *Assume (29). Then*

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \max_{r \in I_n, |r| \geq R} U_n(r) = 0.$$

Proof We perform the following construction. Since $U_n \rightarrow U$ in $C_{\text{loc}}^1(\mathbb{R})$ and by Lemma 3.2 $\lim_{r \rightarrow \pm\infty} U(r) = 0$, for any fixed $\eta > 0$ small there exist bounded sequences $r_{1,n} < 0$, $r_{2,n} > 0$ so that

$$U_n(r) > \eta \text{ for } r_{1,n} < r < r_{2,n}, \quad U_n(r_{1,n}) = U_n(r_{2,n}) = \eta.$$

Consider the test function:

$$\psi_n(r) = \begin{cases} u_n(r) - \eta \varepsilon_n^{-\frac{2}{p-1}} & \text{for } c_n := \varepsilon_n r_{1,n} + a_n \leq r \leq d_n := \varepsilon_n r_{2,n} + a_n \\ 0 & \text{for } r \in [1, 2] \setminus [c_n, d_n] \end{cases}$$

for the linearized operator at u_n : $-\Delta + \lambda_n V - p u_n^{p-1}$. We have that, for $r \in [c_n, d_n]$, there holds:

$$\begin{aligned} -\Delta \psi_n + (\lambda_n V - p u_n^{p-1}) \psi_n &= -\Delta u_n + (\lambda_n V - p u_n^{p-1}) \left(u_n - \eta \varepsilon_n^{-\frac{2}{p-1}} \right) \\ &= -(p-1) u_n^p - \eta \varepsilon_n^{-\frac{2}{p-1}} (\lambda_n V - p u_n^{p-1}). \end{aligned}$$

By rescaling, we get that

$$\begin{aligned}
& \int_A |\nabla \psi_n|^2 + (\lambda_n V - p u_n^{p-1}) \psi_n^2 \\
&= \int_{c_n \leq |x| \leq d_n} \left(-(p-1) u_n^p - \eta \varepsilon_n^{-\frac{2}{p-1}} (\lambda_n V - p u_n^{p-1}) \right) (u_n - \eta \varepsilon_n^{-\frac{2}{p-1}}) \\
&= \omega_N \varepsilon_n^{-\frac{p+3}{p-1}} \int_{r_{1,n}}^{r_{2,n}} \left(-(p-1) U_n^p - \eta (\lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) - p U_n^{p-1}) \right) (U_n - \eta) (\varepsilon_n r + a_n)^{N-1} dr.
\end{aligned}$$

Let $r_\infty > 0$ be the unique solution of $U(r_\infty) = U(-r_\infty) = \eta$. Since any converging subsequence of $r_{1,n}/r_{2,n}$ converges to $-r_\infty/r_\infty$, we get that $\lim_{n \rightarrow +\infty} r_{1,n} = -r_\infty / \lim_{n \rightarrow +\infty} r_{2,n} = r_\infty$. Hence, since $\mu = \frac{2}{p+1}$ we have that

$$\begin{aligned}
& \int_A |\nabla \psi_n|^2 + (\lambda_n V - p u_n^{p-1}) \psi_n^2 \\
&= \omega_N \varepsilon_n^{-\frac{p+3}{p-1}} \left(a_n^{N-1} \int_{-r_\infty}^{r_\infty} \left(-(p-1) U^p - \eta \left(\frac{2}{p+1} - p U^{p-1} \right) \right) (U - \eta) + o_n(1) \right).
\end{aligned}$$

Since

$$\int_{-r_\infty}^{r_\infty} \left(-(p-1) U^p - \eta \left(\frac{2}{p+1} - p U^{p-1} \right) \right) (U - \eta) \rightarrow -(p-1) \int_{\mathbb{R}} U^{p+1} < 0 \text{ as } \eta \rightarrow 0,$$

we can fix $\eta > 0$ small so that

$$\int_{-r_\infty}^{r_\infty} \left(-(p-1) U^p - \eta \left(\frac{2}{p+1} - p U^{p-1} \right) \right) (U - \eta) \leq -\frac{p-1}{2} \int_{\mathbb{R}} U^{p+1}.$$

With this choice of η , since $1 \leq a_n \leq 2$ for n large ψ_n is a radial test function so that: $\text{supp } \psi_n = [c_n, d_n]$ and

$$\int_A |\nabla \psi_n|^2 + (\lambda_n V - p u_n^{p-1}) \psi_n^2 < 0.$$

After this construction, it is easy to show the validity of the Proposition. By contradiction, assume that

$$\limsup_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \max_{r \in I_n, |r| \geq R} U_n(r) > 0.$$

Then, up to a subsequence, there exists a sequence $r_n \in I_n$, $|r_n| \rightarrow +\infty$ as $n \rightarrow +\infty$, so that $U_n(r_n) \geq \delta$, for some $\delta > 0$. Assume for example that $r_n > 0$ (up to a subsequence). Since $U_n \rightarrow U$ in $C_{\text{loc}}^1(\mathbb{R})$ and $\lim_{r \rightarrow +\infty} U(r) = 0$, we can find some $R > 0$ large so that $U_n(R) < \delta$ and therefore, we can assume that r_n has the following maximality property:

$$U_n(r_n) = \max_{R \leq r \leq \frac{2-a_n}{\varepsilon_n}} U_n(r).$$

Let $b_n = \varepsilon_n r_n + a_n$. Clearly, we have that $b_n \in (1, 2)$, $u_n(b_n) = \max_{\varepsilon_n R + a_n \leq |x| \leq 2} u_n(x)$ and, for $\mu_n = u_n(b_n)^{-\frac{p-1}{2}}$,

$$\delta \leq U_n(r_n) = \left(\frac{\varepsilon_n}{\mu_n} \right)^{\frac{2}{p-1}} \leq 1. \tag{33}$$

In particular, $\mu_n \rightarrow 0$ as $n \rightarrow +\infty$. Let us define $\tilde{U}_n(r) = \mu_n^{\frac{2}{p-1}} u_n(\mu_n r + b_n)$ for $r \in \tilde{I}_n := (-\frac{r_n - R}{\mu_n} \varepsilon_n, \frac{2 - b_n}{\mu_n})$ in order to get

$$0 < \tilde{U}_n(r) \leq \tilde{U}_n(0) = 1 \quad \text{for } r \in \tilde{I}_n.$$

Repeating what we did for U_n , we are able to show for n large the existence of a radial function $\tilde{\psi}_n$ so that: $\text{supp } \tilde{\psi}_n = [\tilde{c}_n, \tilde{d}_n]$ and

$$\int_A |\nabla \tilde{\psi}_n|^2 + (\lambda_n V - p u_n^{p-1}) \tilde{\psi}_n^2 < 0,$$

where $\tilde{c}_n = \mu_n \tilde{r}_{1,n} + b_n$, $\tilde{d}_n = \mu_n \tilde{r}_{2,n} + b_n$ for some suitable bounded sequences $\tilde{r}_{1,n} < 0$, $\tilde{r}_{2,n} > 0$. Since by (33)

$$\tilde{c}_n - d_n = \mu_n \tilde{r}_{1,n} + b_n - \varepsilon_n r_{2,n} - a_n = \varepsilon_n \left(\frac{\mu_n}{\varepsilon_n} \tilde{r}_{1,n} + r_n - r_{2,n} \right) > 0$$

for n large in view of $r_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we obtain that $\text{supp } \psi_n \cap \text{supp } \tilde{\psi}_n = \emptyset$ contradicting (29). \square

As a by-product, we obtain that solutions with Morse index 1 have for n large exactly one maximum point:

Corollary 3.4. *Assume (29). Then, for n large a_n is the unique maximum point of u_n .*

Proof By Proposition 3.3 we can fix some $R > 0$ large so that

$$\max_{r \in I_n, |r| \geq R} U_n(r) \leq \left(\frac{c_0}{(p+1) \max V} \right)^{\frac{1}{p-1}}$$

for n large. Then, for $|r| \geq R$ and n large:

$$U_n^p(r) - \lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) U_n(r) \leq U_n(r) \left(\frac{1}{p+1} - \lambda \varepsilon_n^2 V(a_n) \right) \frac{c_0}{\max V} < 0$$

since $\lambda_n \varepsilon_n^2 V(a_n) \rightarrow \frac{2}{p+1}$ as $n \rightarrow +\infty$. Hence, for n large all the maximum points of U_n stay in the fixed region $|r| \leq R$. The proof of the Corollary is reduced to show that 0 is the only maximum point of U_n in $[-R, R]$, for n large.

By contradiction, assume that, up to a subsequence, there exist a sequence $s_n \neq 0$ so that s_n is a local maximum point of U_n . Since 0 is the only maximum point of the limit function U , by the convergence $U_n \rightarrow U$ in $C_{\text{loc}}^1(\mathbb{R})$ we get that $s_n \rightarrow 0$ as $n \rightarrow +\infty$. Let $t_n \neq 0$ be some minimum point of U_n between s_n and 0. Then, we have that $t_n \rightarrow 0$, $U_n(t_n) \rightarrow U(0) = 1$ and

$$-\ddot{U}_n(t_n) = U_n^p(t_n) - \lambda_n \varepsilon_n^2 V(\varepsilon_n t_n + a_n) U_n(t_n) \rightarrow 1 - \frac{2}{p+1} > 0$$

as $n \rightarrow +\infty$, contradicting that t_n is a minimum point of U_n .

Hence, u_n has an unique maximum point in $[1, 2]$, say a_n , for n large and the proof is complete. \square

To conclude this Section, by the uniform decay of U_n at infinity contained in Proposition 3.3 we can strengthen the result in order to obtain an uniform exponential decay:

Remark 4. The contents of Lemma 2 are well known, however we have included the proof for the sake of completeness.

Remark 5. A comparison with Lemma 2.2 and the two proofs given there show how naturally the ideas involved extend to the general case.

Proposition 3.5. *Assume (29). There exist $C > 0$ and $N \in \mathbb{N}$ so that:*

$$U_n(r) \leq C \varepsilon_n^{-\frac{2}{p-1}} e^{-\gamma \frac{|r-a_n|}{\varepsilon_n}} \quad \text{for } r \in I_n, \quad n \geq N,$$

for some $\gamma > 0$ small.

Proof By Proposition 3.3 we can fix some $R > 0$ large so that

$$\max_{r \in I_n, |r| \geq R} U_n(r) \leq \left(\frac{c_0}{(p+1) \max V} \right)^{\frac{1}{p-1}}$$

for n large. Define the linear operator:

$$L_n \phi = -\ddot{\phi} - (N-1) \frac{\varepsilon_n}{\varepsilon_n r + a_n} \dot{\phi} + (\lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) - U_n^{p-1}) \phi$$

for functions ϕ defined on I_n . Since U_n is a positive solution in I_n of $L_n U_n = 0$, we have that L_n satisfies the minimum principle in any subinterval of I_n (see [5]). Letting $\Phi(r) = e^{-\gamma(r-R)}$, $\gamma > 0$, we compute for $r \in I_n$, $r > R$:

$$L_n \phi(r) = \left(-\gamma^2 + (N-1) \frac{\varepsilon_n}{\varepsilon_n r + a_n} \gamma + \lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) - U_n^{p-1}(r) \right) e^{-\gamma r} > 0$$

if γ is small enough and n is large since

$$\lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) \geq \frac{c_0}{\max V} \lambda_n \varepsilon_n^2 V(a_n) \rightarrow \frac{2c_0}{(p+1) \max V} \text{ as } n \rightarrow +\infty$$

and $U_n^{p-1}(r) \leq \frac{c_0}{(p+1) \max V}$ in the region under consideration. Moreover, $\Phi(r) - U_n(r) |_{r=R} = 1 - U_n(r) \geq 0$ and $\Phi(r) - U_n(r) |_{r=\frac{2-a_n}{\varepsilon_n}} = \Phi\left(\frac{2-a_n}{\varepsilon_n}\right) > 0$. Since $L_n(\Phi - U_n) \geq 0$ in $r \in I_n$, $r > R$, by applying the minimum principle on subintervals of $(R, \frac{2-a_n}{\varepsilon_n})$ which exhaust the interval $(R, \frac{2-a_n}{\varepsilon_n})$ we obtain that $U_n(r) \leq e^{-\gamma(r-R)}$ for $R \leq r \leq \frac{2-a_n}{\varepsilon_n}$ and n large. Arguing in a symmetric

way, for some $\gamma > 0$ small we have that $U_n(r) \leq e^{\gamma(r+R)}$ for $-\frac{a_n-1}{\varepsilon_n} \leq r \leq -R$ and n large. Take $\gamma > 0$ small (in order to have the validity of previous estimates) and $C = e^{\gamma R}$. There exists $N \in \mathbb{N}$ large so that

$$U_n(r) \leq Ce^{-\gamma|r|} \quad \text{for } r \in I_n, \quad n \geq N \quad (34)$$

since this estimate is always satisfied in $|r| \leq R$. By the relation between U_n and u_n we get the desired result. \square

4 Location of the blow up set

By previous Section, assuming (29) on u_n , we know that for n large $u_n(r)$ has an unique maximum point $a_n \in (1, 2)$. Up to a subsequence, let us assume that $a_n \rightarrow a_\infty \in [1, 2]$ as $n \rightarrow +\infty$. We want to determine the location of a_∞ .

To this purpose, let us introduce the modified potential

$$M(r) = r^{N-1}V^\theta(r), \quad r > 0, \quad (35)$$

where $\theta = \frac{p+1}{p-1} - \frac{1}{2}$. The role of the modified potential in concentration phenomena has been already pointed out in the papers of Ambrosetti, Malchiodi and Ni [1]-[3] in order to construct radial spike-layer solution for the same problem either in \mathbb{R}^N or in a ball/annulus in \mathbb{R}^N with homogeneous Dirichlet boundary condition. Here, we show in a rigorous way by an asymptotic approach the role of $M(r)$. We have to combine the results in the previous Section with the following Pohozaev-type identity:

Lemma 4.1. *Let $1 \leq a < b \leq 2$. There holds*

$$\begin{aligned} \frac{a}{2}\dot{u}_n^2(a) &= \frac{b}{2}\dot{u}_n^2(b) + \left(r \frac{u_n^{p+1}}{p+1} - \frac{\lambda_n}{2} r V u_n^2 + (N - \frac{3}{2}) \dot{u}_n u_n + (N - \frac{3}{2}) \frac{N-1}{2r} u_n^2 \right) \Big|_a^b \\ &+ (N - \frac{3}{2} - \frac{1}{p+1}) \int_a^b u_n^{p+1} + \lambda_n \int_a^b \left(\frac{r}{2} \dot{V} - (N-2)V \right) u_n^2 \\ &+ (N - \frac{3}{2}) \int_a^b \frac{N-1}{2r^2} u_n^2. \end{aligned} \quad (36)$$

Proof Multiply (21) by $r\dot{u}_n$ and integrate on $[a, b]$:

$$\int_a^b \left(-\ddot{u}_n - \frac{N-1}{r} \dot{u}_n \right) r \dot{u}_n = \int_a^b (u_n^p - \lambda_n V u_n) r \dot{u}_n.$$

By some integration by parts we get that the L.H.S. term takes the form:

$$\int_a^b \left(-\ddot{u}_n - \frac{N-1}{r} \dot{u}_n\right) r \dot{u}_n = -\frac{r}{2} \dot{u}_n^2 \Big|_a^b - \left(N - \frac{3}{2}\right) \int_a^b \dot{u}_n^2,$$

and the R.H.S. term becomes:

$$\int_a^b (u_n^p - \lambda_n V u_n) r \dot{u}_n = r \left(\frac{u_n^{p+1}}{p+1} - \frac{\lambda_n}{2} V u_n^2 \right) \Big|_a^b - \frac{1}{p+1} \int_a^b u_n^{p+1} + \frac{\lambda_n}{2} \int_a^b (V + r\dot{V}) u_n^2.$$

Hence, we have that:

$$\begin{aligned} \frac{a}{2} \dot{u}_n^2(a) &= \frac{b}{2} \dot{u}_n^2(b) + r \left(\frac{u_n^{p+1}}{p+1} - \frac{\lambda_n}{2} V u_n^2 \right) \Big|_a^b + \left(N - \frac{3}{2}\right) \int_a^b \dot{u}_n^2 \\ &\quad - \frac{1}{p+1} \int_a^b u_n^{p+1} + \frac{\lambda_n}{2} \int_a^b (V + r\dot{V}) u_n^2. \end{aligned} \quad (37)$$

Multiplying (21) by u_n and integrating on $[a, b]$, we get that:

$$\int_a^b \left(-\ddot{u}_n - \frac{N-1}{r} \dot{u}_n\right) u_n = \int_a^b (u_n^{p+1} - \lambda_n V u_n^2).$$

Integrating by parts the L.H.S. term, we obtain:

$$\int_a^b \left(-\ddot{u}_n - \frac{N-1}{r} \dot{u}_n\right) u_n = -\dot{u}_n u_n \Big|_a^b + \int_a^b \dot{u}_n^2 - \frac{N-1}{2r} u_n^2 \Big|_a^b - \int_a^b \frac{N-1}{2r^2} u_n^2,$$

and so, we obtain that

$$\int_a^b \dot{u}_n^2 = (\dot{u}_n u_n + \frac{N-1}{2r} u_n^2) \Big|_a^b + \int_a^b \frac{N-1}{2r^2} u_n^2 + \int_a^b (u_n^{p+1} - \lambda_n V u_n^2). \quad (38)$$

Inserting (38) in (37), finally we get that:

$$\begin{aligned} \frac{a}{2} \dot{u}_n^2(a) &= \frac{b}{2} \dot{u}_n^2(b) + \left(r \frac{u_n^{p+1}}{p+1} - \frac{\lambda_n}{2} r V u_n^2 + \left(N - \frac{3}{2}\right) \dot{u}_n u_n + \left(N - \frac{3}{2}\right) \frac{N-1}{2r} u_n^2 \right) \Big|_a^b \\ &\quad + \left(N - \frac{3}{2} - \frac{1}{p+1}\right) \int_a^b u_n^{p+1} + \lambda_n \int_a^b \left(\frac{r}{2} \dot{V} - (N-2)V \right) u_n^2 \\ &\quad + \left(N - \frac{3}{2}\right) \int_a^b \frac{N-1}{2r^2} u_n^2 \end{aligned}$$

and the proof is complete. \square

The asymptotic analysis combined with (36) leads to the following property:

Theorem 4.2. *Assume (29). Let a_∞ be the limit of some subsequence of a_n . Then,*

$$\frac{1}{2} \dot{u}_n^2(1) = \frac{1}{2} \dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \frac{1}{V^\theta(a_\infty) a_\infty^{N-2}} \left(\dot{M}(a_\infty) + o(1) \right) \int_{\mathbb{R}} U^2. \quad (39)$$

In particular, there holds:

$$\begin{cases} \dot{M}(a_\infty) \geq 0 & \text{if } a_\infty = 1 \\ \dot{M}(a_\infty) = 0 & \text{if } a_\infty \in (1, 2) \\ \dot{M}(a_\infty) \leq 0 & \text{if } a_\infty = 2. \end{cases}$$

Proof We will restrict our attention to the subsequence along which a_n converges to a_∞ and we will label it always by n . Take $a = 1$, $b = 2$ in (36). Since $u_n(1) = u_n(2) = 0$, we get that

$$\frac{1}{2}\dot{u}_n^2(1) = \frac{1}{2}\dot{u}_n^2(2) + (N - \frac{3}{2} - \frac{1}{p+1}) \int_1^2 u_n^{p+1} + \lambda_n \int_1^2 \left(\frac{r}{2}\dot{V} - (N-2)V \right) u_n^2 + (N - \frac{3}{2}) \int_1^2 \frac{N-1}{2r^2} u_n^2$$

Since $U_n \rightarrow U$ a.e. and (34), by means of Lebesgue's Theorem we can expand all the terms in the expression above:

$$\int_1^2 u_n^{p+1} = \varepsilon_n^{-\frac{p+3}{p-1}} \int_{I_n} U_n^{p+1} = \varepsilon_n^{-\frac{p+3}{p-1}} \left(\int_{\mathbb{R}} U^{p+1} + o(1) \right),$$

$$\begin{aligned} \lambda_n \int_1^2 r \dot{V} u_n^2 &= \varepsilon_n^{-\frac{p+3}{p-1}} \left(\lambda_n \varepsilon_n^2 \int_{I_n} (\varepsilon_n r + a_n) \dot{V}(\varepsilon_n r + a_n) U_n^2 \right) \\ &= \frac{2}{p+1} a_\infty \frac{\dot{V}(a_\infty)}{V(a_\infty)} \varepsilon_n^{-\frac{p+3}{p-1}} \left(\int_{\mathbb{R}} U^2 + o(1) \right), \end{aligned}$$

$$\lambda_n \int_1^2 V u_n^2 = \varepsilon_n^{-\frac{p+3}{p-1}} \left(\lambda_n \varepsilon_n^2 \int_{I_n} V(\varepsilon_n r + a_n) U_n^2 \right) = \frac{2}{p+1} \varepsilon_n^{-\frac{p+3}{p-1}} \left(\int_{\mathbb{R}} U^2 + o(1) \right)$$

and

$$\int_1^2 \frac{u_n^2}{r^2} \leq \int_1^2 u_n^2 = \varepsilon_n^{-\frac{p+3}{p-1}} (\varepsilon_n^2 \int_{I_n} U_n^2 dr) = o(\varepsilon_n^{-\frac{p+3}{p-1}})$$

as $n \rightarrow +\infty$, since $\lambda_n \varepsilon_n^2 V(a_n) \rightarrow \frac{2}{p+1}$ as $n \rightarrow +\infty$. Hence, we get that

$$\begin{aligned} \frac{1}{2}\dot{u}_n^2(1) &= \frac{1}{2}\dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \left((N - \frac{3}{2} - \frac{1}{p+1}) \int_{\mathbb{R}} U^{p+1} - \frac{2(N-2)}{p+1} \int_{\mathbb{R}} U^2 \right. \\ &\quad \left. + \frac{a_\infty}{p+1} \frac{\dot{V}(a_\infty)}{V(a_\infty)} \int_{\mathbb{R}} U^2 + o(1) \right) = \dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \left((N-1) \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}} U^{p+1} \right. \\ &\quad \left. + (N-2) \left(\frac{1}{2} + \frac{1}{p+1} \right) \int_{\mathbb{R}} U^{p+1} - \frac{2(N-2)}{p+1} \int_{\mathbb{R}} U^2 + \frac{a_\infty}{p+1} \frac{\dot{V}(a_\infty)}{V(a_\infty)} \int_{\mathbb{R}} U^2 + o(1) \right). \end{aligned}$$

Since by Lemma 3.2 $\frac{2}{p+1} \int_{\mathbb{R}} U^2 = \left(\frac{1}{2} + \frac{1}{p+1} \right) \int_{\mathbb{R}} U^{p+1}$, we get that

$$\begin{aligned} \frac{1}{2}\dot{u}_n^2(1) &= \frac{1}{2}\dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \left((N-1) \left(\frac{1}{2} - \frac{1}{p+1} \right) \frac{4}{p+3} + \frac{a_\infty}{p+1} \frac{\dot{V}(a_\infty)}{V(a_\infty)} + o(1) \right) \int_{\mathbb{R}} U^2 \\ &= \frac{1}{2}\dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \left((N-1) + \theta a_\infty \frac{\dot{V}(a_\infty)}{V(a_\infty)} + o(1) \right) \int_{\mathbb{R}} U^2. \end{aligned}$$

Hence, by definition of $M(r)$:

$$\frac{1}{2}\dot{u}_n^2(1) = \frac{1}{2}\dot{u}_n^2(2) + \varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \frac{1}{V^\theta(a_\infty)a_\infty^{N-2}} \left(\dot{M}(a_\infty) + o(1) \right) \int_{\mathbb{R}} U^2,$$

and (39) is proved.

Finally, let us remark that by Proposition 3.5 $u_n^p - \lambda_n u_n = o(\varepsilon_n^{-\frac{p+3}{p-1}})$ in $C_{\text{loc}}^0(\bar{A} \setminus \{x : |x| = a_n\})$ and by elliptic regularity theory (see [5]) $u_n = o(\varepsilon_n^{-\frac{p+3}{p-1}})$ in $C_{\text{loc}}^1(\bar{A} \setminus \{x : |x| = a_n\})$. If $a_\infty = 1$, by $\dot{u}_n^2(2) = o(\varepsilon_n^{-\frac{p+3}{p-1}})$ combined with (39) we get that

$$0 \leq \frac{1}{2}\dot{u}_n^2(1) = \varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \frac{1}{V^\theta(a_\infty)a_\infty^{N-2}} \left(\dot{M}(a_\infty) + o(1) \right) \int_{\mathbb{R}} U^2$$

and therefore, $\dot{M}(a_\infty) \geq 0$. If $a_\infty \in (1, 2)$ we get that

$$\varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \frac{1}{V^\theta(a_\infty)a_\infty^{N-2}} \left(\dot{M}(a_\infty) + o(1) \right) \int_{\mathbb{R}} U^2 = 0$$

and therefore, $\dot{M}(a_\infty) = 0$. If $a_\infty = 2$, we get that

$$\varepsilon_n^{-\frac{p+3}{p-1}} \frac{2(p-1)}{(p+1)(p+3)} \frac{1}{V^\theta(a_\infty)a_\infty^{N-2}} \left(\dot{M}(a_\infty) + o(1) \right) \int_{\mathbb{R}} U^2 \leq 0$$

and therefore, $\dot{M}(a_\infty) \leq 0$. The proof is now complete. \square

Remark 6. It is not hard to check that as $\lambda \rightarrow \infty$, the solution under consideration will have Morse Index tending to infinity on the whole space $H_0^1(A)$ even if the Morse Index on the radial space $H_{0,r}^1(A)$ is one.

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