

COMMON MIDPOINT VERSUS COMMON OFFSET ACQUISITION GEOMETRY IN SEISMIC IMAGING

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ABSTRACT. We compare and contrast the qualitative nature of backprojected images obtained in seismic imaging when *common offset* data are used versus when *common midpoint* data are used. Our results show that the image obtained using common midpoint data contains artifacts which are not present with common offset data. Although there are situations where one would still want to use common midpoint data, this result points out a shortcoming that should be kept in mind when interpreting the images.

1. Introduction. In seismic imaging, pressure waves generated on the surface of the earth travel into the subsurface, where they are scattered by heterogeneities. The scattered pressure waves returning to the surface are picked up by receivers and the inverse problem involves imaging the subsurface from these scattered waves. This problem is non-linear and as such is very difficult to solve. In this article, we model the heterogeneities in the subsurface as singular perturbations about a constant velocity field. We consider the corresponding linearized map F which maps the perturbation in velocity to the corresponding perturbation in the pressure waves. The goal is to recover, to the extent possible, the perturbation in the background wave speed from the perturbations in the pressure waves measured on the surface. In other words, we would like to study the invertibility of the linear operator F .

We compare two standard (see [33]) ways of collecting data in seismic experiments; *the common midpoint acquisition geometry* and *the common offset acquisition geometry* to determine which geometry shows more features of the subsurface and adds fewer artifacts. To do this, we will use microlocal analysis, which provides a precise characterization of how operators like F and its adjoint, F^* , map singularities.

Microlocal analysis has been used to analyze important problems in seismics (e.g., [2, 6, 3, 32, 38, 26, 39, 35, 27, 37, 36, 5, 4, 24, 10]), in the related field of synthetic aperture radar imaging (e.g., [28, 8, 9, 34, 1]), marine imaging (e.g., [7, 11, 31]), and in X-ray Tomography (e.g., [17, 19, 18, 14, 15, 13, 16, 30, 22, 12]). Our microlocal approach to the seismic imaging problem is influenced by the seminal work of Guillemin, who first specified the microlocal properties of Radon transforms [17, 20] and our characterization of the distribution class of the composition of F with F^* is motivated by the works of Greenleaf, Guillemin, Melrose and Uhlmann [25, 21, 14].

The nature of F and F^*F depends on the set of sources and receivers. In the case of a single source and receivers occupying a relatively open subset of the surface, Beylkin [2] proved that if caustics do not occur then F^*F is a pseudodifferential operator (Ψ DO). Rakesh then showed that F is a Fourier integral operator (FIO) [32] under the assumption of no caustics in the source ray field, but caustics can occur on the receiver side. Later, Nolan and Symes [26] studied a variety of acquisition geometries and gave sufficient conditions for F^*F to be a Ψ DO. Ten Kroode, Smit, and Verdel [39] considered relatively open sets of sources and receivers and investigated the microlocal properties of F in relation to the so-called travel time injectivity condition; see [26, Assumption 3, pp. 929] and [39]. Stolk [35] studied the microlocal analysis of F and F^*F even when the travel time injectivity condition fails.

To describe the data acquisition geometries, we introduce some notation. We denote the subsurface to be imaged by the half space,

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}. \quad (1)$$

Seismic sources and receivers will be parametrized by points in a bounded open set $\Omega \subset \mathbb{R}^2$ (where we identify \mathbb{R}^2 with the plane $x_3 = 0$). For each point $s = (s_1, s_2) \in \Omega$ we specify a source $S(s)$ and a receiver $R(s)$ on the surface of the earth, $x_3 = 0$. Data are taken for different times, $t > 0$, so the data space is

$$Y = \Omega \times (0, \infty).$$

In the *common midpoint data acquisition geometry*, the source and receiver move away from a common midpoint. For this case, $\Omega \subset \{(s_1, s_2) \in \mathbb{R}^2 : s_2 > 0\}$ and for each $s = (s_1, s_2) \in \Omega$,

$$S_{\text{cm}}(s) = (s_1, s_2, 0) \quad R_{\text{cm}}(s) = (s_1, -s_2, 0). \quad (2)$$

Hence the sources and the detectors move over the plane and are symmetric with respect to the x_1 axis.

In the *common offset data acquisition geometry*, the source and receiver collecting data on the surface of the earth are offset by a constant vector. Choose $\alpha > 0$ and for each $s = (s_1, s_2) \in \Omega$ let

$$S_{\text{co}}(s) = (s_1, s_2 + \alpha, 0), \quad R_{\text{co}}(s) = (s_1, s_2 - \alpha, 0). \quad (3)$$

So the sources and the detectors move so they are always $(0, 2\alpha, 0)$ apart.

These data acquisition geometries are realistic because seismic data can be synthesized to provide both common midpoint and common offset data [33]. Although we have made simplifying assumptions of constant background velocity, if one data acquisition method is shown to be better in this case, it is likely to be better at least for small perturbations away from it. This is because bicharacteristics depend smoothly on the background velocity as they involve solving an ODE and one can invoke smooth dependence of ODE solutions on parameters. This means that the wavefront relation also depends smoothly on the parameters. Therefore if artifacts appear for a constant background velocity, they would still be present when we make small perturbations away from the constant background velocity.

Under our assumptions (and with our data acquisition geometries), the forward operator can be described (see [26] for example) as an FIO of order m , where f models the singular velocity perturbations in the subsurface from the known constant background velocity:

$$Ff(s, t) = \int_{\omega \in \mathbb{R}} \int_{x \in X} e^{i\phi(s, t, x, \omega)} A(s, t, x, \omega) \varphi(s) f(x) dx d\omega \quad (4)$$

with phase function

$$\phi(s, t, x, \omega) = \omega (t - \|x - S(s)\| - \|x - R(s)\|) \quad (5)$$

and a smooth cutoff function $\varphi(s)$ that is supported in Ω and identically 1 in an arbitrary large compact proper subset of the support of φ . The canonical relation of F is given by

$$C = \{(s, t, \partial_s \phi, \partial_t \phi; x, -\partial_x \phi) : \partial_\omega \phi = 0\}$$

where ∂_s is the differential in the s variables, etc. [40]. The formal L^2 adjoint operator, F^* , is an FIO associated with C^t (which is C with $T^*(X)$ and $T^*(Y)$ coordinates flipped, if $C = \{x, \xi; y, \eta\}$ then $C^t = \{y, \eta; x, \xi\}$).

One standard reconstruction method in imaging is to apply F^* to the data. The result is the *normal operator* F^*F applied to f . Analyzing what this normal operator does to singularities of f will help determine whether it is a good reconstruction method and, if so, whether one data acquisition method is better than another.

Remark 1.1. In order to have a heuristic understanding of whether one acquisition geometry can be better than the other, consider the following special case. Assume the smooth component of the velocity in the subsurface is layered, i.e., only depends on x_3 . Reflectors in the subsurface can be detected in the data when there is an ellipsoidal surface (with foci at the source and receiver) which is tangent to the reflector. That is, the “dip” or normal to the reflector is also normal to the ellipsoid at the point of tangency. With this in mind, we see that the common midpoint data can only detect reflectors which are located on the plane $x_2 = 0$ and which also have a vertical dip. In contrast, the common offset geometry can detect reflectors which have a range of locations (including away from $x_2 = 0$) and also a range of dips.

Heuristically, this remark shows why imaging using a common offset geometry could be expected to be better than imaging using a common midpoint geometry. The authors thank one of the referees for pointing out this special case. However, it is also worth pointing out that the results of this paper (regarding the nature of the normal operator F^*F) do not directly follow from such heuristic arguments.

This pattern, that common offset can be better in some ways than common midpoint, also occurs in synthetic aperture radar (SAR), in which the two-dimensional topography of the earth is imaged. In the common offset SAR geometry it was shown that the normal operator introduces one artifact singularity into the reconstructed image in addition to each true singularity, whereas in the common midpoint case, *three* artifact singularities are introduced into the reconstructed image in addition to each true one. A more precise description of the normal operators in these cases are given in [23, 1].

We now present the main results of this article and then interpret them.

Theorem 1.2 (Microlocal Properties for Common Midpoint). *Let F_{cm} be the common midpoint forward operator defined in (2), (4), and (5), of order m then $F_{\text{cm}}^*F_{\text{cm}}$ is a singular FIO in the class $I^{2m,0}(\Delta, \tilde{C})$ where Δ is the diagonal in $(T^*(X) \setminus 0) \times (T^*(X) \setminus 0)$ and \tilde{C} is the graph of the reflection*

$$T^*(X) \ni (x, \xi) \mapsto ((x_1, -x_2, x_3), (\xi_1, -\xi_2, \xi_3)).$$

We will define $I^{p,l}$ classes in section 2 but this theorem has the following practical implication. Microlocalized away from \tilde{C} , $F_{\text{cm}}^*F_{\text{cm}}$ is a standard Ψ DO of order $2m$. So singularities of f can be visible in the image $F_{\text{cm}}^*F_{\text{cm}}(f)$. Microlocalized away from Δ , $F_{\text{cm}}^*F_{\text{cm}}$ is a FIO associated to \tilde{C} of order $2m$. So a singularity of f at (x, ξ) can produce an artifact in the image which is *of the same strength* at $((x_1, -x_2, x_3), (\xi_1, -\xi_2, \xi_3))$.

On the other hand, the common offset normal operator has the following properties.

Theorem 1.3 (Microlocal Properties for Common Offset). *Let F_{co} be the common offset forward operator defined in (3), (4), and (5), of order m then $F_{\text{co}}^*F_{\text{co}}$ is a standard Ψ DO of order $2m$.*

This theorem means that singularities of f can be visible in $F_{\text{co}}^*F_{\text{co}}$ and artifacts are not added to the reconstruction. Our theorems show that common offset can be better than common midpoint data acquisition since singularities are not added to the reconstruction using the normal operator. We remark that despite this advantage of common offset data acquisition compared to common midpoint data, different (and perhaps more desirable) singularities might be visible with common midpoint acquisition.

Our proof rests on a fundamental assumption of Guillemin and Sternberg [20, 17]: the Bolker Assumption, i.e., π_L is an injective immersion (see section 3.2). We show that F_{co} satisfies this Assumption and this implies that $F_{\text{co}}^* F_{\text{co}}$ is a Ψ DO, thus the singularities of f can be reconstructed from the singularities of F_{co} . We would like to point out that [3] provides another way to obtain the reconstruction of the singularities. It uses the Beylkin Determinant for common offset, which is nonzero (see pp. 247-249, (5.2.30)) and an explicit inversion formula for the ground reflectivity function (5.1.56). Using microlocal techniques we obtain the same nonzero determinant of $d\pi_L$ (see (22) and subsequent calculations in section 3.2) which implies that this projection is an immersion. This is a part of the Bolker Assumption.

The proofs in Section 3 use similar techniques as the ones done in [8, 1] and the pertinent details will be given.

2. Microlocal analysis and $I^{p,l}$ classes. We present some basic definitions in microlocal analysis.

Definition 2.1 (Wavefront Set). Let $x_0 \in \mathbb{R}^n$ and $\xi_0 \in \mathbb{R}^n$, $\xi_0 \neq 0$. The function f is in C^∞ at x_0 in direction ξ_0 if \exists a cut-off function φ at x_0 such that the Fourier transform

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{x \in \mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) f(x) dx$$

is rapidly decreasing at ∞ in some open cone from the origin, V , containing ξ_0 .

On the other hand, $(x_0, \xi_0) \in \text{WF}(f)$, that is, f has a singularity at x_0 in the direction ξ_0 if f is not rapidly decreasing at x_0 in direction ξ_0 .

Now we define the types of geometric singularities exhibited by the canonical relation of our operators.

Definition 2.2 ([18, p.109-111]). Let M and N be manifolds of dimension n and let $f : M \rightarrow N$ be C^∞ . Let Ω be a non-vanishing volume form on N and define $\Sigma = \{\sigma \in M : f^* \Omega(\sigma) = 0\}$, that is, Σ is the set of critical points of f . Note that, equivalently, Σ is defined by the vanishing of the determinant of the Jacobian of f .

- (a) If for all $\sigma \in \Sigma$, we have (i) the corank of f at σ is 1, (ii) $\ker(df_\sigma) \cap T_\sigma \Sigma = \{0\}$, (iii) $f^* \Omega$ vanishes exactly to first order on Σ , then we say that f has a *fold* singularity along Σ .
- (b) If for all $\sigma \in \Sigma$, we have (i) the rank of f is constant; let us call this constant k , (ii) $\ker(df_\sigma) \subset T_\sigma \Sigma$, (iii) $f^* \Omega$ vanishes exactly to order $n - k$ on Σ , then we say that f has a *blowdown* singularity along Σ .

The fundamental mathematics for $I^{p,l}$ classes is in [21, 25] and the techniques we use appeared initially in [15] and we follow [9]. They were used in the context of radar imaging in [29, 8, 9]. First, consider the following example.

Example 2.3. Let $\tilde{\Lambda}_0 = \Delta_{T^* \mathbb{R}^n} = \{(x, \xi; x, \xi) | x \in \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus \{0\}\}$ be the diagonal in $T^* \mathbb{R}^n \times T^* \mathbb{R}^n$ and let $\tilde{\Lambda}_1 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) | x' \in \mathbb{R}^{n-1}, \xi' \in \mathbb{R}^{n-1} \setminus \{0\}\}$. Then, $\tilde{\Lambda}_0$ intersects $\tilde{\Lambda}_1$ cleanly in codimension 1.

The class of product-type symbols $S^{p,l}(m, n, k)$ is defined as follows.

Definition 2.4. [21] $S^{p,l}(m, n, k)$ is the set of all functions $a(z, \xi, \sigma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k)$ such that for every $K \subset \mathbb{R}^m$ and every $\alpha \in \mathbb{Z}_+^m, \beta \in \mathbb{Z}_+^n, \gamma \in \mathbb{Z}_+^k$ there is $c_{K, \alpha, \beta, \gamma}$ such that

$$|\partial_z^\alpha \partial_\xi^\beta \partial_\sigma^\gamma a(z, \xi, \sigma)| \leq c_{K, \alpha, \beta, \gamma} (1 + \|\xi\|)^{p - |\beta|} (1 + |\sigma|)^{l - |\gamma|}, \forall (z, \xi, \tau) \in K \times \mathbb{R}^n \times \mathbb{R}^k.$$

Since any two sets of cleanly intersecting Lagrangians are equivalent via a canonical transformation [21], we first define $I^{p,l}$ classes for the case in Example 2.3.

Definition 2.5. [21] Let $I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_0^\infty$ and

$$u_2(x, y) = \int e^{i((x'-y') \cdot \xi' + (x_n - y_n - s) \cdot \xi_n + s \cdot \sigma)} a(x, y, s; \xi, \sigma) d\xi d\sigma ds$$

with $a \in S^{p',l'}$ where $p' = p - \frac{n}{2} + \frac{1}{2}$ and $l' = l - \frac{1}{2}$.

Let (Λ_0, Λ_1) be a pair of cleanly intersection Lagrangians in codimension 1 and let χ be a canonical transformation which maps (Λ_0, Λ_1) into $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ and maps $\Lambda_0 \cap \Lambda_1$ to $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$, where $\tilde{\Lambda}_j$ are from Example 2.3. Next we define the $I^{p,l}(\Lambda_0, \Lambda_1)$.

Definition 2.6 ([21]). Let $I^{p,l}(\Lambda_0, \Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2 + \sum v_i$ where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i = Aw_i$ where A is a zero order FIO associated to χ^{-1} , the canonical transformation from above, and $w_i \in I^{p,l}(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

If u is the Schwartz kernel of the linear operator F , then we say $F \in I^{p,l}(\Lambda_0, \Lambda_1)$.

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair (Λ_0, Λ_1) to itself and the intersection $\Lambda_0 \cap \Lambda_1$ to itself. If $F \in I^{p,l}(\Lambda_0, \Lambda_1)$ then $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ and $F \in I^p(\Lambda_1 \setminus \Lambda_0)$ [21]. Here by $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, we mean that the Schwartz kernel of F belongs to $I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ microlocally away from Λ_1 .

One way to show that a distribution belongs to an $I^{p,l}$ class is by using the iterated regularity property:

Proposition 2.7. [15, Prop. 1.35] *If $u \in \mathcal{D}'(X \times Y)$ then $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ if there is an $s_0 \in \mathbb{R}$ such that for all first order pseudodifferential operators P_i with principal symbols vanishing on $\Lambda_0 \cup \Lambda_1$, we have $P_1 P_2 \dots P_r u \in H_{loc}^{s_0}$.*

3. Proofs. We first calculate the canonical relation C for each operator. Then we analyze the geometry of the right projection $\pi_R : C \rightarrow T^*(X) \setminus \{\mathbf{0}\}$ and the left projection $\pi_L : C \rightarrow T^*(Y) \setminus \{\mathbf{0}\}$.

For the common midpoint case, we show that π_L and π_R both drop rank by 1 on the submanifold Σ of C_{cm} that is above $x_2 = 0$. We show π_L has a fold singularity on Σ and π_R has a blowdown singularity (similar to common midpoint SAR). To find the microlocal properties of $F_{\text{cm}}^* F_{\text{cm}}$, we prove

$$C_{\text{cm}}^t \circ C_{\text{cm}} \subset \Delta \cup \tilde{C}$$

where \tilde{C} is the graph of the reflection

$$\chi(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = (x_1, -x_2, x_3, \xi_1, -\xi_2, \xi_3).$$

We then show that $F_{\text{cm}}^* F_{\text{cm}} \in I^{2m,0}(\Delta, G)$ using arguments similar to those in [1] for common midpoint SAR that use the machinery in [15, 9]. The normal operator $F_{\text{cm}}^* F_{\text{cm}}$ is simpler than in common midpoint SAR [1] because there is only one geometric symmetry (across $x_2 = 0$), whereas in common midpoint SAR [1], there is symmetry about the $x_1 = 0$ and $x_2 = 0$ axes.

Common offset is easier and, we show that $\pi_L : C_{\text{co}} \rightarrow T^*(Y) \setminus \{\mathbf{0}\}$ is an injective immersion so the Bolker condition holds and $F_{\text{co}}^* F_{\text{co}}$ is a standard pseudodifferential operator.

3.1. Proof of Theorem 1.2. In this case we consider $S_{\text{cm}}(s_1, s_2) = (s_1, s_2, 0)$ and $R_{\text{cm}}(s_1, s_2) = (s_1, -s_2, 0)$, $(s_1, s_2) \in \Omega \subset \{(a, b) \in \mathbb{R}^2 : b > 0\}$. Using (5)

$$\begin{aligned} \phi_{\text{cm}}(s_1, s_2, t, \omega, x_1, x_2, x_3) = & \omega \left(t - \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2 + x_3^2} \right. \\ & \left. - \sqrt{(x_1 - s_1)^2 + (x_2 + s_2)^2 + x_3^2} \right) \end{aligned} \quad (6)$$

We use the notation:

$$A_{\text{cm}} = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2 + x_3^2}, \quad B_{\text{cm}} = \sqrt{(x_1 - s_1)^2 + (x_2 + s_2)^2 + x_3^2}.$$

The canonical relation $C_{\text{cm}} \subset (T^*(Y) \setminus \{\mathbf{0}\}) \times (T^*(X) \setminus \{\mathbf{0}\})$ becomes:

$$\begin{aligned} C_{\text{cm}} = & \left\{ \left(s_1, s_2, t, (x_1 - s_1) \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega, \left(\frac{x_2 - s_2}{A_{\text{cm}}} - \frac{x_2 + s_2}{B_{\text{cm}}} \right) \omega, \omega; \right. \right. \\ & x_1, x_2, x_3, (x_1 - s_1) \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega, \left(\frac{x_2 - s_2}{A_{\text{cm}}} + \frac{x_2 + s_2}{B_{\text{cm}}} \right) \omega, \\ & \left. \left. x_3 \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega \right) : t = A_{\text{cm}} + B_{\text{cm}} \right\}. \end{aligned}$$

Notice that the canonical coordinates on C_{cm} are $(s_1, s_2, \omega, x_1, x_2, x_3)$. We assume that $x_3 \neq 0$, that is, that scatterers are contained in the open subsurface $x_3 > 0$. Next, we consider the projections π_L and π_R . We have

$$\begin{aligned} \pi_L(s_1, s_2, \omega, x_1, x_2, x_3) = & \left(s_1, s_2, \omega, A_{\text{cm}} + B_{\text{cm}}, (x_1 - s_1) \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega, \right. \\ & \left. \left(\frac{x_2 - s_2}{A_{\text{cm}}} - \frac{x_2 + s_2}{B_{\text{cm}}} \right) \omega \right). \end{aligned}$$

Note that, in the description of π_L , we reordered the components to obtain the identity in the first variables. This ordering will help to find the determinant of $d\pi_L$. We get $d\pi_L$ to be

$$d\pi_L = \begin{pmatrix} I & 0 \\ * & C \end{pmatrix}$$

where C is the 3×3 matrix

$$\begin{pmatrix} (x_1 - s_1) \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) & \frac{x_2 - s_2}{A_{\text{cm}}} + \frac{x_2 + s_2}{B_{\text{cm}}} & x_3 \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \\ \frac{(x_2 - s_2)^2 + x_3^2}{A_{\text{cm}}^3} + \frac{(x_2 + s_2)^2 + x_3^2}{B_{\text{cm}}^3} \omega & -(x_1 - s_1) \left(\frac{x_2 - s_2}{A_{\text{cm}}^3} + \frac{x_2 + s_2}{B_{\text{cm}}^3} \right) \omega & -(x_1 - s_1) x_3 \left(\frac{1}{A_{\text{cm}}^3} + \frac{1}{B_{\text{cm}}^3} \right) \omega \\ -(x_1 - s_1) \left(\frac{x_2 - s_2}{A_{\text{cm}}^3} - \frac{x_2 + s_2}{B_{\text{cm}}^3} \right) \omega & ((x_1 - s_1)^2 + x_3^2) \left(\frac{1}{A_{\text{cm}}^3} - \frac{1}{B_{\text{cm}}^3} \right) \omega & -x_3 \left(\frac{x_2 - s_2}{A_{\text{cm}}^3} - \frac{x_2 + s_2}{B_{\text{cm}}^3} \right) \omega \end{pmatrix}.$$

The determinant of $d\pi_L$ is

$$4x_2s_2x_3\omega^2 \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \left(1 + \frac{x - S_{\text{cm}}(s)}{A_{\text{cm}}} \cdot \frac{x - R_{\text{cm}}(s)}{B_{\text{cm}}} \right).$$

Hence π_L drops rank by 1 on $\Sigma = \{(s, \omega, x) \in C_{\text{cm}} | x_2 = 0\}$ since we assume $x_3, s_2 \neq 0$ and since $\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \neq 0$ and $1 + \frac{x - S_{\text{cm}}(s)}{A_{\text{cm}}} \cdot \frac{x - R_{\text{cm}}(s)}{B_{\text{cm}}} \neq 0$. The last term is nonzero since the unit vectors $\frac{x - S_{\text{cm}}(s)}{A_{\text{cm}}}$ and $\frac{x - R_{\text{cm}}(s)}{B_{\text{cm}}}$ do not point in the opposite directions since $x_3 > 0$, that is, there is no direct scattering.

Remark 3.1. As already mentioned, we assume that scatterers are contained in the open subsurface $x_3 > 0$. Furthermore, for applicability of our results to the case when the background velocity is a slight perturbation of the constant background

velocity, we also make the assumption that no direct scattering takes place. In practice, such direct data are usually filtered out. Therefore this is a reasonable and valid assumption.

Next we classify the singularities of π_L on Σ . Notice that on Σ , $A_{\text{cm}} = B_{\text{cm}}$ and the second column of the matrix C above is 0. We have that $v \in \text{Ker } d\pi_L$, if $v = (0, 0, 0, \delta x_1, \delta x_2, \delta x_3) = \delta x_1 \partial_{x_1} + \delta x_2 \partial_{x_2} + \delta x_3 \partial_{x_3}$, with

$$\begin{aligned} (x_1 - s_1)\delta x_1 + x_3\delta x_3 &= 0 \\ (s_2^2 + x_3^2)\delta x_1 - (x_1 - s_1)x_3\delta x_3 &= 0 \end{aligned}$$

from the first and the second rows of the above matrix. Notice that the last row does not give us new information about the kernel. The 2×2 system we considered has only the zero solution. Thus $\text{Ker } d\pi_L = \text{span} \frac{\partial}{\partial x_2}$. We have that $\text{Ker } d\pi_L \not\subseteq T\Sigma$, hence π_L has a fold singularity.

Next we consider

$$\begin{aligned} \pi_R(x_1, x_2, x_3, s_1, s_2, \omega) &= \left(x_1, x_2, x_3, (x_1 - s_1) \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega, \right. \\ &\quad \left. \left(\frac{x_2 - s_2}{A_{\text{cm}}} + \frac{x_2 + s_2}{B_{\text{cm}}} \right) \omega, x_3 \left(\frac{1}{A_{\text{cm}}} + \frac{1}{B_{\text{cm}}} \right) \omega \right). \end{aligned}$$

We have that π_R drops rank by 1 also on Σ . Let $v \in \text{Ker } d\pi_R$, with $v = (0, 0, 0, \delta s_1, \delta s_2, \delta \omega) = \delta s_1 \partial_{s_1} + \delta s_2 \partial_{s_2} + \delta \omega \partial_\omega$. Since $\text{Ker } d\pi_R$ is a linear combination of $\partial_{s_1}, \partial_{s_2}, \partial_\omega$ we get that $\text{Ker } d\pi_R \subset T\Sigma$, thus π_R has a blowdown singularity along Σ .

To summarize, we have shown that $F \in I^m(C_{\text{cm}})$ with C_{cm} having π_L a map with fold singularities and π_R a map with blowdown singularities. Now, we consider the normal operator F^*F and the corresponding canonical relation $C_{\text{cm}}^t \circ C_{\text{cm}}$. Using the prolate spheroidal coordinates, as done in [1] for common midpoint SAR, we get: $C_{\text{cm}}^t \circ C_{\text{cm}} = \Delta \cup \tilde{C}$ where $\tilde{C} = \text{Gr}(\chi)$ with $\chi(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = (x_1, -x_2, x_3, \xi_1, -\xi_2, \xi_3)$. Also, Δ and \tilde{C} intersect cleanly in codimension 2.

We now show that $F_{\text{cm}}^* F_{\text{cm}} \in I^{2m,0}(\Delta, \tilde{C})$ where F is given by (4) with the phase function $\phi = \phi_{\text{cm}}$ given by (6). Our strategy of proof is to use the iterated regularity property characterizing $I^{p,l}$ classes due to Melrose and Greenleaf-Uhlmann (Proposition 2.7) and is similar to the ones given in [8, 1]. We have that the Schwartz kernel of $F_{\text{cm}}^* F_{\text{cm}}$ is given by

$$K(x, y) = \int e^{i\Phi(x, y, s_1, s_2, \omega)} \tilde{a}(x, y, s_1, s_2, \omega) ds_1 ds_2 d\omega, \quad (7)$$

where

$$\begin{aligned} \Phi(x, y, s_1, s_2, \omega) &= \\ &\omega \left(\sqrt{(y_1 - s_1)^2 + (y_2 - s_2)^2 + y_3^2} + \sqrt{(y_1 - s_1)^2 + (y_2 + s_2)^2 + y_3^2} \right. \end{aligned} \quad (8)$$

$$\left. - \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2 + x_3^2} - \sqrt{(x_1 - s_1)^2 + (x_2 + s_2)^2 + x_3^2} \right). \quad (9)$$

The ideal of functions vanishing on $\Delta \cup \widetilde{C}$ is generated by

$$\begin{aligned} x_1 - y_1, \quad x_2^2 - y_2^2, \quad \xi_1 - \eta_1, \quad (x_2 + y_2)(\xi_2 - \eta_2), \\ x_3 - y_3, \quad \xi_2^2 - \eta_2^2, \quad \xi_3 - \eta_3, \quad (x_2 - y_2)(\xi_2 + \eta_2). \end{aligned}$$

We will now write in terms of derivatives of Φ and smooth functions, the generators $x_1 - y_1$ and $x_3 - y_3$. Such expressions for the other generators are similar.

3.1.1. *Expression for $x_1 - y_1$.* Let us write $x_1 - y_1$ in the form

$$x_1 - y_1 = \frac{f(x, y, s)}{\omega} \partial_{s_1} \Phi + \frac{g(x, y, s)}{\omega} \partial_{s_2} \Phi + h(x, y, s) \partial_\omega \Phi,$$

where f, g, h are smooth functions and Φ is the phase function (9) for the Schwartz kernel (7).

Let us use the following prolate spheroidal coordinate system:

$$\begin{aligned} x_1 &= s_1 + s_2 \sinh \rho \sin \phi \cos \theta \\ x_2 &= s_2 \cosh \rho \cos \phi \\ x_3 &= s_2 \sinh \rho \sin \phi \sin \theta. \end{aligned} \tag{10}$$

For $y = (y_1, y_2, y_3)$, we use the above coordinates but with (ρ, ϕ, θ) replaced by (ρ', ϕ', θ') .

In this coordinate system, we have

$$\begin{aligned} A_{\text{cm}} &= \sqrt{(x_1 - s_1)^2 + (x_2 - s_2)^2 + x_3^2} = s_2(\cosh \rho - \cos \phi), \\ B_{\text{cm}} &= \sqrt{(x_1 - s_1)^2 + (x_2 + s_2)^2 + x_3^2} = s_2(\cosh \rho + \cos \phi). \end{aligned} \tag{11}$$

A'_{cm} and B'_{cm} are similarly defined with (ρ, ϕ) replaced by (ρ', ϕ') .

Furthermore,

$$\partial_\omega \Phi = 2s_2(\cosh \rho' - \cosh \rho) \tag{12}$$

$$\partial_{s_1} \Phi = 2\omega \left(\frac{\cosh \rho \sinh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} - \frac{\cosh \rho' \sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} \right) \tag{13}$$

$$\partial_{s_2} \Phi = 2\omega \left(\frac{\cosh \rho' \sin^2 \phi'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\cosh \rho \sin^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \right). \tag{14}$$

Now

$$x_1 - y_1 = s_2 (\sinh \rho \sin \phi \cos \theta - \sinh \rho' \sin \phi' \cos \theta').$$

From (11) we obtain,

$$-\frac{1}{2\omega} \partial_{s_1} \Phi = \frac{\cosh \rho' \sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\cosh \rho \sinh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} \tag{15}$$

Adding and subtracting $\cosh \rho$ in the first term on the right hand side of Equation (15) (the equation above), and rearranging, we get

$$\begin{aligned} -\frac{\partial_{s_1} \Phi}{2\omega} &= \frac{(\cosh \rho' - \cosh \rho + \cosh \rho) \sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\cosh \rho \sinh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} \\ &= (\cosh \rho' - \cosh \rho) \frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} \\ &\quad + \cosh \rho \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\sinh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} \right). \end{aligned}$$

Now using the expression for $\cosh \rho' - \cosh \rho$, adding and subtracting

$$\frac{\cosh \rho \sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho - \cos^2 \phi}$$

and simplifying, we get

$$\begin{aligned} -\frac{\partial_{s_1} \Phi}{2\omega} &= \frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')} \\ &\quad + \cosh \rho \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho - \cos^2 \phi} \right) \\ &\quad + \cosh \rho \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh^2 \rho - \cos^2 \phi} - \frac{\sinh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} \right) \\ &= -\frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')} \left(\frac{\cosh \rho \cosh \rho' + \cos^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \right) - \frac{(x_1 - y_1) \cosh \rho}{s_2 (\cosh^2 \rho - \cos^2 \phi)} \\ &\quad + \cosh \rho \sinh \rho' \sin \phi' \cos \theta' \left(\frac{\cos^2 \phi' - \cos^2 \phi}{(\cosh^2 \rho - \cos^2 \phi)(\cosh^2 \rho' - \cos^2 \phi')} \right) \end{aligned} \quad (16)$$

We now obtain an expression involving $\partial_{s_2} \Phi$ for

$$\frac{\cos^2 \phi' - \cos^2 \phi}{(\cosh^2 \rho - \cos^2 \phi)(\cosh^2 \rho' - \cos^2 \phi')}.$$

From (14), we have, $\frac{\partial_{s_2} \Phi}{2\omega} = \left(\frac{\cosh \rho' \sin^2 \phi'}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\cosh \rho \sin^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \right)$. Simplifying this, we get,

$$\begin{aligned} \frac{\partial_{s_2} \Phi}{2\omega} &= \frac{\cosh \rho' (\sin^2 \phi' - \sin^2 \phi)}{\cosh^2 \rho' - \cos^2 \phi'} + \frac{(\cosh \rho' - \cosh \rho) \sin^2 \phi}{\cosh^2 \rho' - \cos^2 \phi'} \\ &\quad + \frac{\cosh \rho \sin^2 \phi}{\cosh^2 \rho' - \cos^2 \phi'} - \frac{\cosh \rho \sin^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \\ &= \frac{\cosh \rho' (\cos^2 \phi - \cos^2 \phi')}{\cosh^2 \rho' - \cos^2 \phi'} + \frac{\partial_\omega \Phi}{2s_2} \left(\frac{\sin^2 \phi}{\cosh^2 \rho' - \cos^2 \phi'} \right) \\ &\quad + \frac{\cosh \rho \sin^2 \phi (\cosh^2 \rho - \cosh^2 \rho' + \cos^2 \phi' - \cos^2 \phi)}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \\ &= \frac{(\cos^2 \phi - \cos^2 \phi') (\cosh \rho (\cosh \rho' \cosh \rho - 1) + \cos^2 \phi (\cosh \rho - \cosh \rho'))}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \\ &\quad - \frac{\partial_\omega \Phi}{2s_2} \frac{\sin^2 \phi (\cosh \rho \cosh \rho' + \cos^2 \phi)}{(\cosh^2 \rho - \cos^2 \phi)(\cosh^2 \rho' - \cos^2 \phi)} \\ &= \frac{(\cos^2 \phi - \cos^2 \phi')}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} (\cosh \rho (\cosh \rho' \cosh \rho - 1)) \\ &\quad - \frac{\partial_\omega \Phi}{2s_2} \left(\frac{\cos^2 \phi (\cos^2 \phi - \cos^2 \phi') + \sin^2 \phi (\cosh \rho \cosh \rho' + \cos^2 \phi)}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \right) \\ &= \frac{(\cos^2 \phi - \cos^2 \phi') \cosh \rho (\cosh \rho' \cosh \rho - 1)}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \\ &\quad - \frac{\partial_\omega \Phi}{2s_2} \frac{(\sin^2 \phi \cosh \rho \cosh \rho' + \cos^2 \phi \sin^2 \phi')}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)}. \end{aligned}$$

Hence

$$\begin{aligned}
& \frac{\cos^2 \phi - \cos^2 \phi'}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \\
&= \frac{\partial_\omega \Phi}{2s_2} \frac{(\sin^2 \phi \cosh \rho \cosh \rho' + \cos^2 \phi \sin^2 \phi')}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)(\cosh \rho(\cosh \rho \cosh \rho' - 1))} \\
& \quad + \frac{\partial_{s_2} \Phi}{2\omega} \frac{1}{\cosh \rho(\cosh \rho \cosh \rho' - 1)}. \tag{17}
\end{aligned}$$

Now we have from (16)

$$\begin{aligned}
\frac{-(x_1 - y_1) \cosh \rho}{s_2(\cosh^2 \rho - \cos^2 \phi)} &= -\frac{\partial_{s_1} \Phi}{2\omega} + \frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')} \left(\frac{\cosh \rho \cosh \rho' + \cos^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \right) \\
& \quad + \cosh \rho \sinh \rho' \sin \phi' \cos \theta' \left(\frac{\cos^2 \phi - \cos^2 \phi'}{(\cosh^2 \rho - \cos^2 \phi)(\cosh^2 \rho' - \cos^2 \phi')} \right).
\end{aligned}$$

Then from(17), we have

$$\begin{aligned}
\frac{-(x_1 - y_1) \cosh \rho}{s_2(\cosh^2 \rho - \cos^2 \phi)} &= -\frac{\partial_{s_1} \Phi}{2\omega} + \frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')} \left(\frac{\cosh \rho \cosh \rho' + \cos^2 \phi}{\cosh^2 \rho - \cos^2 \phi} \right) \\
& \quad + \cosh \rho \sinh \rho' \sin \phi' \cos \theta' \left(\frac{\partial_{s_2} \Phi}{2\omega} \frac{1}{\cosh \rho(\cosh \rho \cosh \rho' - 1)} \right. \\
& \quad \left. + \frac{\partial_\omega \Phi}{2s_2} \frac{(\sin^2 \phi \cosh \rho \cosh \rho' + \cos^2 \phi \sin^2 \phi')}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)(\cosh \rho(\cosh \rho \cosh \rho' - 1))} \right).
\end{aligned}$$

Simplifying, we obtain,

$$\begin{aligned}
-\frac{(x_1 - y_1) \cosh \rho}{s_2(\cosh^2 \rho - \cos^2 \phi)} &= -\frac{\partial_{s_1} \Phi}{2\omega} + \frac{\partial_{s_2} \Phi}{2\omega} \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh \rho \cosh \rho' - 1} \right) \\
& \quad + \frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \times \\
& \quad \times \left((\cosh \rho \cosh \rho' + \cos^2 \phi) + \frac{\sin^2 \phi \cosh \rho \cosh \rho' + \cos^2 \phi \sin^2 \phi'}{\cosh \rho \cosh \rho' - 1} \right) \\
&= -\frac{\partial_{s_1} \Phi}{2\omega} + \frac{\partial_{s_2} \Phi}{2\omega} \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh \rho \cosh \rho' - 1} \right) \\
& \quad + \frac{\partial_\omega \Phi}{2s_2} \frac{\sinh \rho' \sin \phi' \cos \theta'}{(\cosh^2 \rho' - \cos^2 \phi')(\cosh^2 \rho - \cos^2 \phi)} \left(\frac{\cosh^2 \rho \cosh^2 \rho' - \cos^2 \phi \cos^2 \phi'}{\cosh \rho \cosh \rho' - 1} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
x_1 - y_1 &= \frac{s_2}{2\omega} \left(\frac{\cosh^2 \rho - \cos^2 \phi}{\cosh \rho} \right) \partial_{s_1} \Phi \\
& \quad - \frac{s_2}{2\omega} \left(\frac{\cosh^2 \rho - \cos^2 \phi}{\cosh \rho} \right) \left(\frac{\sinh \rho' \sin \phi' \cos \theta'}{\cosh \rho \cosh \rho' - 1} \right) \partial_{s_2} \Phi \\
& \quad - \frac{\sinh \rho' \sin \phi' \cos \theta'}{2(\cosh^2 \rho' - \cos^2 \phi')} \left(\frac{\cosh^2 \rho \cosh^2 \rho' - \cos^2 \phi \cos^2 \phi'}{(\cosh \rho \cosh \rho' - 1) \cosh \rho} \right) \partial_\omega \Phi. \tag{18}
\end{aligned}$$

We now write this expression in Cartesian coordinates using the notation from (11). We have the following expressions:

$$\begin{aligned}\cosh^2 \rho - \cos^2 \phi &= \frac{A_{\text{cm}} B_{\text{cm}}}{s_2^2}, & \cosh^2 \rho' - \cos^2 \phi' &= \frac{A'_{\text{cm}} B'_{\text{cm}}}{s_2^2}, \\ \sinh \rho' \sin \phi' \cos \theta' &= \frac{2(y_1 - s_1)}{s_2}, & \cosh \rho &= \frac{A_{\text{cm}} + B_{\text{cm}}}{2s_2}, & \cosh \rho' &= \frac{A'_{\text{cm}} + B'_{\text{cm}}}{2s_2}, \\ \cos \phi &= \frac{B_{\text{cm}} - A_{\text{cm}}}{2s_2}, & \cos \phi' &= \frac{B'_{\text{cm}} - A'_{\text{cm}}}{2s_2}.\end{aligned}$$

Thus $x_1 - y_1$ becomes

$$\begin{aligned}x_1 - y_1 &= \frac{1}{\omega} \left(\frac{A_{\text{cm}} B_{\text{cm}}}{A_{\text{cm}} + B_{\text{cm}}} \right) \partial_{s_1} \Phi \\ &\quad - \frac{8s_2}{\omega} \left(\frac{A_{\text{cm}} B_{\text{cm}}}{A_{\text{cm}} + B_{\text{cm}}} \right) \left(\frac{y_1 - s_1}{(A_{\text{cm}} + B_{\text{cm}})(A'_{\text{cm}} + B'_{\text{cm}}) - 4s_2^2} \right) \partial_{s_2} \Phi \\ &\quad - \left[\left(\frac{(A_{\text{cm}} + B_{\text{cm}})^2 (A'_{\text{cm}} + B'_{\text{cm}})^2 - (A_{\text{cm}} - B_{\text{cm}})^2 (A'_{\text{cm}} - B'_{\text{cm}})^2}{(A_{\text{cm}} + B_{\text{cm}})(A'_{\text{cm}} + B'_{\text{cm}}) - 4s_2^2} \right) \right. \\ &\quad \left. \times \left(\frac{y_1 - s_1}{2A'_{\text{cm}} B'_{\text{cm}} (A_{\text{cm}} + B_{\text{cm}})} \right) \right] \partial_{\omega} \Phi.\end{aligned}\tag{19}$$

3.1.2. *Expression for $x_3 - y_3$.* Next we consider $x_3 - y_3$. We have

$$\begin{aligned}x_3^2 - y_3^2 &= s_2^2 (\sinh^2 \rho \sin^2 \phi \sin^2 \theta - \sinh^2 \rho' \sin^2 \phi' \sin^2 \theta') \\ &= s_2^2 (\sinh^2 \rho \sin^2 \phi - \sinh^2 \rho \sin^2 \phi \cos^2 \theta - \sinh^2 \rho' \sin^2 \phi' + \sinh^2 \rho' \sin^2 \phi' \cos^2 \theta') \\ &= (y_1 - s_1)^2 - (x_1 - s_1)^2 + s_2^2 ((1 - \cosh^2 \rho)(1 - \cos^2 \phi) - (1 - \cosh^2 \rho')(1 - \cos^2 \phi')) \\ &= (y_1 - x_1)(y_1 + x_1 - 2s_1) + s_2^2 (\cosh^2 \rho' - \cosh^2 \rho + \cos^2 \phi' - \cos^2 \phi) \\ &\quad + s_2^2 (\cosh^2 \rho \cos^2 \phi - \cosh^2 \rho' \cos^2 \phi') \\ &= (y_1 - x_1)(y_1 + x_1 - 2s_1) + s_2^2 ((\cosh^2 \rho' - \cosh^2 \rho) \sin^2 \phi + \sinh^2 \rho' (\cos^2 \phi - \cos^2 \phi')).\end{aligned}\tag{20}$$

We already have an expression for $(x_1 - y_1)$ from (19), the second expression in the right hand side of (20) can be written in terms of $\partial_{\omega} \Phi$ (see (12)) and for the third expression in (20), we use (17). Now since $x_3 - y_3 = \frac{x_3^2 - y_3^2}{x_3 + y_3}$ and due to the assumption that the support of the function lies above the x_1 - x_2 plane, $x_3 + y_3$ will never be zero; see Remark 3.1. Therefore, we have an expression in terms of the derivatives (12), (13) and (14) for $x_3 - y_3$.

We can find expressions similar to the one for $(x_1 - y_1)$ in (19) for the remaining terms in (10) by using (12)-(14) and (17) as in the above two calculations. Finally we proceed as in [8, 1] to show that $F_{\text{cm}}^* F_{\text{cm}} \in I^{2m,0}(\Delta, \tilde{C})$. \square

3.2. **Proof of Theorem 1.3.** In the common offset geometry, we consider, for $\alpha > 0$ fixed, $S_{\text{co}}(s_1, s_2) = (s_1, s_2 + \alpha, 0)$ and $R_{\text{co}}(s_1, s_2) = (s_1, s_2 - \alpha, 0)$. The phase function

$$\begin{aligned}\phi_{\text{co}}(s_1, s_2, t, \omega, x_1, x_2, x_3) &= \omega \left(t - \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 - \alpha)^2 + x_3^2} \right. \\ &\quad \left. - \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 + \alpha)^2 + x_3^2} \right).\end{aligned}\tag{21}$$

We let

$$A_{\text{co}} = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 - \alpha)^2 + x_3^2},$$

$$B_{\text{co}} = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 + \alpha)^2 + x_3^2}.$$

The canonical relation $C_{\text{co}} \subset T^*Y \setminus \{\mathbf{0}\} \times T^*X \setminus \{\mathbf{0}\}$ is

$$C_{\text{co}} = \left\{ \left(s_1, s_2, A_{\text{co}} + B_{\text{co}}, \omega(x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right), \omega \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}} \right), \omega, \right. \right.$$

$$x_1, x_2, x_3, \omega(x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right), \omega \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}} \right),$$

$$\left. \left. \omega x_3 \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right) \right) \right\}.$$

Note that $(s_1, s_2, \omega, x_1, x_2, x_3)$ is a parametrization of C_{co} . Using this parametrization, we compute the determinant of $d\pi_L$ where π_L is the left projection

$$\pi_L(s_1, s_2, \omega, x_1, x_2, x_3) = \left(s_1, s_2, \omega, A_{\text{co}} + B_{\text{co}}, \omega(x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right), \right.$$

$$\left. \omega \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}} \right) \right).$$

Notice that in the description of π_L , we reordered the components to obtain the identity in the first variables. This will help to find the determinant of $d\pi_L$. With this ordering, we have

$$d\pi_L = \begin{pmatrix} I & 0 \\ * & D \end{pmatrix} \quad (22)$$

where D is the matrix

$$\begin{pmatrix} (x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right) & \frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}} & x_3 \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right) \\ \omega \left(\frac{(x_2 - s_2 - \alpha)^2 + x_3^2}{A_{\text{co}}^3} + \frac{(x_2 - s_2 + \alpha)^2 + x_3^2}{B_{\text{co}}^3} \right) & -\omega(x_1 - s_1) \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}^3} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}^3} \right) & -\omega x_3 (x_1 - s_1) \left(\frac{1}{A_{\text{co}}^3} + \frac{1}{B_{\text{co}}^3} \right) \\ -\omega(x_1 - s_1) \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}^3} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}^3} \right) & \omega((x_1 - s_1)^2 + x_3^2) \left(\frac{1}{A_{\text{co}}^3} + \frac{1}{B_{\text{co}}^3} \right) & -\omega x_3 \left(\frac{x_2 - s_2 - \alpha}{A_{\text{co}}^3} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}^3} \right) \end{pmatrix}$$

and the determinant of $d\pi_L$ is

$$\omega^2 x_3 \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right) \left(1 + \frac{x - S_{\text{co}}(s)}{A_{\text{co}}} \cdot \frac{x - R_{\text{co}}(s)}{B_{\text{co}}} \right) \left(\frac{1}{A_{\text{co}}^2} + \frac{1}{B_{\text{co}}^2} \right).$$

We have $x_3 \neq 0$, $\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \neq 0$, $\frac{1}{A_{\text{co}}^2} + \frac{1}{B_{\text{co}}^2} \neq 0$ and $1 + \frac{x - S_{\text{co}}(s)}{A_{\text{co}}} \cdot \frac{x - R_{\text{co}}(s)}{B_{\text{co}}} \neq 0$.

The last term is nonzero since the unit vectors $\frac{x - S_{\text{co}}(s)}{A_{\text{co}}}$ and $\frac{x - R_{\text{co}}(s)}{B_{\text{co}}}$ do not point in opposite directions since $x_3 > 0$; see Remark 3.1.

Since every term in the determinant is nonzero we obtain that π_L is a local diffeomorphism. Thus π_R is also a local diffeomorphism and we get that C_{co} is a local canonical graph. Next we show that π_L is injective. We use the following prolate coordinates

$$x_1 = s_1 + \alpha \sinh \rho \sin \phi \cos \theta$$

$$x_2 = s_2 + \alpha \cosh \rho \cos \phi$$

$$x_3 = \alpha \sinh \rho \sin \phi \sin \theta$$

with $\rho > 0$, $0 < \phi < \pi$, $0 < \theta < \pi$ (since $x_3 > 0$). In these new coordinates we have

$$A_{\text{co}} = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 - \alpha)^2 + x_3^2} = \alpha(\cosh \rho - \cos \phi),$$

$$B_{\text{co}} = \sqrt{(x_1 - s_1)^2 + (x_2 - s_2 + \alpha)^2 + x_3^2} = \alpha(\cosh \rho + \cos \phi).$$

Thus

$$A_{\text{co}} + B_{\text{co}} = 2\alpha \cosh \rho \quad (23)$$

$$(x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right) = \frac{2 \sinh \rho \cosh \rho \sin \phi \cos \theta}{\cosh^2 \rho - \cos^2 \phi} \quad (24)$$

$$\frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}} = \frac{2 \sinh^2 \rho \cos \phi}{\cosh^2 \rho - \cos^2 \phi} \quad (25)$$

In order to prove injectivity, we show that we can uniquely and smoothly determine x_1, x_2, x_3 from

$$A_{\text{co}} + B_{\text{co}}, \quad (x_1 - s_1) \left(\frac{1}{A_{\text{co}}} + \frac{1}{B_{\text{co}}} \right), \quad \frac{x_2 - s_2 - \alpha}{A_{\text{co}}} + \frac{x_2 - s_2 + \alpha}{B_{\text{co}}}$$

or equivalently uniquely determine ρ, ϕ, θ from relations (23) to (25).

Notice that from (23) we can uniquely and smoothly determine ρ .

Using (25), we know $\frac{2 \sinh^2 \rho \cos \phi}{\cosh^2 \rho - \cos^2 \phi} := D$ (say) or $\frac{\cos \phi}{\cosh^2 \rho - \cos^2 \phi} = \frac{D}{2 \sinh^2 \rho} := D_1$.

Solving for $\cos \phi$ we obtain $\cos \phi = \frac{-1 + \sqrt{1 + 4D_1^2 \cosh^2 \rho}}{2D_1}$ which uniquely and smoothly determines ϕ .

From (24), knowing ρ and ϕ we can uniquely and smoothly determine θ . Thus π_L is an embedding, and the Bolker Assumption holds, so by [20, 17], we have $F_{\text{co}}^* F_{\text{co}} \in I^{2m}(\Delta)$ which means that $F_{\text{co}}^* F_{\text{co}}$ is a pseudodifferential operator and hence the normal operator does not introduce additional singularities. \square

Remark 3.2. Now that we have proven that the normal operator for common offset geometry is a Ψ DO but the normal operator for common midpoint is not, we would like to make some geometric observations about the associated canonical relations.

For the common midpoint wavefront relation, C_{cm} , we see that pairs of wavefront set elements $(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3)$ and $(x_1, -x_2, x_3; \xi_1, -\xi_2, \xi_3)$ are associated to the same wavefront element of the data. Note that this symmetry in the wavefront relation also essentially comes from an obvious reciprocity between the source and the receiver. This shows that the canonical relation C_{cm} is (at least) a 2:1 relation. Hence we intuitively expect artifacts and that $F_{\text{cm}}^* F_{\text{cm}}$ is not a Ψ DO; see Remark 1.1 as well. No obvious symmetry is present in the common offset geometry and it is not surprising that $F_{\text{co}}^* F_{\text{co}}$ is a Ψ DO.

Our theorems make precise these intuitive observations. Additionally the fact that $F_{\text{cm}}^* F_{\text{cm}}$ is in an $I^{2m,0}$ class shows that artifacts are added by our reconstruction method and they are of the same strength as the original singularities that generate them.

REFERENCES

- [1] Gaik Ambartsoumian, Raluca Felea, Venkateswaran P. Krishnan, Clifford Nolan, and Eric Todd Quinto. A class of singular Fourier integral operators in synthetic aperture radar imaging. *J. Funct. Anal.*, 264(1):246–269, 2013.
- [2] G. Beylkin. Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform. *J. Math. Phys.*, 26(1):99–108, 1985.
- [3] N. Bleistein, J.K. Cohen, and Jr. J.W. Stockwell. *Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion*, volume 13 of *Interdisciplinary Applied Mathematics*. Springer, New York, 2001.

- [4] M. V. de Hoop, H. Smith, G. Uhlmann, and R. D. van der Hilst. Seismic imaging with the generalized Radon transform: a curvelet transform perspective. *Inverse Problems*, 25(2):025005, 21, 2009.
- [5] Maarten V. de Hoop. Microlocal analysis of seismic inverse scattering. In *Inside out: inverse problems and applications*, volume 47 of *Math. Sci. Res. Inst. Publ.*, pages 219–296. Cambridge Univ. Press, Cambridge, 2003.
- [6] A.J. Devaney. Geophysical Diffraction Tomography. *IEEE Transactions on Geoscience and Remote Sensing*, GE-22:3–13, 1984.
- [7] R. Felea and A. Greenleaf. An FIO calculus for marine seismic imaging: folds and crosscaps. *Communications in Partial Differential Equations*, 33:45–77, 2008.
- [8] Raluca Felea. Composition of Fourier integral operators with fold and blowdown singularities. *Comm. Partial Differential Equations*, 30(10-12):1717–1740, 2005.
- [9] Raluca Felea. Displacement of artefacts in inverse scattering. *Inverse Problems*, 23(4):1519–1531, 2007.
- [10] Raluca Felea and Allan Greenleaf. Fourier integral operators with open umbrellas and seismic inversion for cusp caustics. *Math. Res. Lett.*, 17(5):867–886, 2010.
- [11] Raluca Felea, Allan Greenleaf, and Malabika Pramanik. An FIO calculus for marine seismic imaging, II: Sobolev estimates. *Math. Ann.*, 352(2):293–337, 2012.
- [12] Jürgen Frikel and Eric Todd Quinto. Characterization and reduction of artifacts in limited angle tomography. *Inverse Problems*, 29(12):125007, December 2013.
- [13] A. Greenleaf and G. Uhlmann. Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms. *Ann. Inst. Fourier (Grenoble)*, 40(2):443–466, 1990.
- [14] Allan Greenleaf and Gunther Uhlmann. Non-local inversion formulas for the X-ray transform. *Duke Math. J.*, 58:205–240, 1989.
- [15] Allan Greenleaf and Gunther Uhlmann. Estimates for singular Radon transforms and pseudodifferential operators with singular symbols. *J. Funct. Anal.*, 89(1):202–232, 1990.
- [16] Allan Greenleaf and Gunther Uhlmann. Microlocal techniques in integral geometry. In *Integral geometry and tomography (Arcata, CA, 1989)*, volume 113 of *Contemp. Math.*, pages 121–135. Amer. Math. Soc., Providence, RI, 1990.
- [17] Victor Guillemin. Some remarks on integral geometry. Technical report, MIT, 1975.
- [18] Victor Guillemin. *Cosmology in (2+1)-dimensions, cyclic models, and deformations of $M_{2,1}$* , volume 121 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.
- [19] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.
- [20] Victor Guillemin and Shlomo Sternberg. *Geometric Asymptotics*. American Mathematical Society, Providence, RI, 1977.
- [21] Victor Guillemin and Gunther Uhlmann. Oscillatory integrals with singular symbols. *Duke Math. J.*, 48(1):251–267, 1981.
- [22] A. I. Katsevich. Local Tomography for the Limited-Angle Problem. *Journal of mathematical analysis and applications*, 213(1):160–182, September 1997.
- [23] Venkateswaran P. Krishnan and Eric Todd Quinto. Microlocal aspects of bistatic synthetic aperture radar imaging. *Inverse Problems and Imaging*, 5:659–674, 2011.
- [24] A.E. Malcolm, B. Ursin, and M.V. de Hoop. Seismic imaging and illumination with internal multiples. *Geophysical Journal International*, 176:847–864, 2009.
- [25] Richard B. Melrose and Gunther A. Uhlmann. Lagrangian intersection and the Cauchy problem. *Comm. Pure Appl. Math.*, 32(4):483–519, 1979.
- [26] C. J. Nolan and W. W. Symes. Global solution of a linearized inverse problem for the wave equation. *Comm. Partial Differential Equations*, 22(5-6):919–952, 1997.
- [27] Clifford J. Nolan. Scattering in the presence of fold caustics. *SIAM J. Appl. Math.*, 61(2):659–672, 2000.
- [28] Clifford J. Nolan and Margaret Cheney. Synthetic Aperture inversion. *Inverse Problems*, 18(1):221–235, 2002.
- [29] Clifford J. Nolan and Margaret Cheney. Microlocal analysis of synthetic aperture radar imaging. *J. Fourier Anal. Appl.*, 10(2):133–148, 2004.
- [30] Eric Todd Quinto. Singularities of the X-ray transform and limited data tomography in \mathbb{R}^2 and \mathbb{R}^3 . *SIAM J. Math. Anal.*, 24(5):1215–1225, 1993.
- [31] Eric Todd Quinto, Andreas Rieder, and Thomas Schuster. Local inversion of the sonar transform regularized by the approximate inverse. *Inverse Problems*, 27:035006 (18p), 2011.

- [32] Rakesh. A linearised inverse problem for the wave equation. *Comm. Partial Differential Equations*, 13(5):573–601, 1988.
- [33] Robert E. Sheriff. *Encyclopedic Dictionary of Applied Geophysics*. Society Of Exploration Geophysicists, 2002.
- [34] Plamen Stefanov and Gunther Uhlmann. Is a curved flight path in SAR better than a straight one? *SIAM J. Appl. Math.*, 73(4):1596–1612, 2013.
- [35] Christiaan C. Stolk. Microlocal analysis of a seismic linearized inverse problem. *Wave Motion*, 32(3):267–290, 2000.
- [36] Christiaan C. Stolk and Maarten V. de Hoop. Microlocal analysis of seismic inverse scattering in anisotropic elastic media. *Comm. Pure Appl. Math.*, 55(3):261–301, 2002.
- [37] Christiaan C. Stolk and Maarten V. de Hoop. Seismic inverse scattering in the downward continuation approach. *Wave Motion*, 43(7):579–598, 2006.
- [38] W. W. Symes. Mathematics of Reflection Seismology. Technical report, Department of Computational and Applied Mathematics, Rice University, Houston, Texas, 1990. Technical Report TR90-02.
- [39] A. P. E. ten Kroode, D.-J. Smit, and A. R. Verdel. A microlocal analysis of migration. *Wave Motion*, 28(2):149–172, 1998.
- [40] François Trèves. *Introduction to Pseudodifferential and Fourier Integral Operators*. Volume 2: Fourier Integral Operators. Plenum Press, New York and London, 1980.

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