

FEniCS Course

Lecture 3: Basic post-processing and code validation

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FENICS
PROJECT

Post-processing

Function evaluation

Expression and Function objects `f` can be evaluated at some Point x by

```
# 1D
x = 0.5
f(x)
# 2D
x = (0.5,0.3) # tuple
# x = [0.5,0.3] is also valid
f(x)
# 3D
x = (0.5,0.2,1.0) # tuple
# x = [0.5,0.2,1.0] is also valid
f(x)
print f(x)
```

Short-hand

```
f(0.5,0.5)
```

Exercise: Try it out! Use the code from the second Poisson example and compute f , u and $\sin(u_h(x))$ for some points.

Function evaluation vs. Function representation

Question: What about plotting $\sin \circ u_h$? And ∇u_h and $|\nabla u_h|$?

Experiment: Try it out! Use

```
sqrt(inner(grad(u), grad(u)))
```

for $|\nabla u|$. What happens if you plot these function? Have a closer look at the terminal output. Anything suspicious?

Question: What happened now? Why is there a
> Object cannot be plotted directly, projecting to
piecewise linears.

Answer:

- $\sin(u_h(x))$ is the evaluation of the built-in function \sin at a *given* value $u_h(x)$, which in turn results from a FEM function evaluation.
- $\sin \circ u_h$ is a composition of the built-in function \sin and a FEM function u_h . The composition is a UFL (Unified Form Language) expression.

Building FE representations via L^2 projection

Define $f = \sin \circ u_h$ and choose a FEM function space $\tilde{V}_h \subset L^2(\Omega)$ which is “suitable” for your post-process. Solve:

Find $w_h \in \tilde{V}_h \subset L^2(\Omega)$ such that for all $v_h \in \tilde{V}_h$

$$\underbrace{\int_{\Omega} w_h v \, dx}_{a(u,v)} = \underbrace{\int_{\Omega} f v \, dx}_{L(v)}$$

Exercise: Compute $|\nabla(u)|$ for the Poisson example 3 using this approach. Start with adding

```
abs_grad_V = FunctionSpace(mesh, "DG", 0)
f = sqrt(inner(grad(u), grad(u)))
```

to your original Python script.

A hack to plot $\nabla(u)$ only on $\partial\Omega$

```
V_ag = FunctionSpace(mesh, "CG", 1)
#V_ag = FunctionSpace(mesh, "DG", 0)
f = sqrt(inner(grad(u), grad(u)))

#Do the Projection only on the boundary
u_ag = TrialFunction(V_ag)
v = TestFunction(V_ag)
a = u_ag * v * ds
L = f*v*ds
A = assemble(a)
b = assemble(L)

# Set dofs not located on the boundary to
# zero by adding ones in the diagonal of A
A.ident_zeros()
u_ag = Function(V_ag)
solve(A, u_ag.vector(), b)

plot(u_ag, title="|grad(u)| on boundary",
      interactive=True)
```

Simple code validation

Theory can help you to validate your implementation!

A priori estimates for the Poisson problem

If

- $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$
- $V_h = \{v_h \in C(\Omega) : v_h \in P^k(T) \forall T \in \mathcal{T}\}$

then

$$E_1(h) := \|u - u_h\|_{1,\Omega} \leq Ch^k \|u\|_{k+1,\Omega}$$

$$E_0(h) := \|u - u_h\|_{0,\Omega} \leq Ch^{k+1} \|u\|_{k+1,\Omega}$$

where $\|\cdot\|_{l,\Omega} = \|\cdot\|_{H^l(\Omega)}$ for $l = 0, 1, k + 1$.

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Taking log on each side

$$\log(E_1(h)) \leq \log(Ch^k \|u\|_{k+1,\Omega}) = k \log(h) + \log(C \|u\|_{k+1,\Omega})$$

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Taking log on each side

$$\underbrace{\log(E_1(h))}_y \leq \log(Ch^k \|u\|_{k+1,\Omega}) = \underbrace{k \log(h)}_x + \underbrace{\log(C \|u\|_{k+1,\Omega})}_c$$

Method of manufactured solutions

Recipe

- 1 Take a suitable function u
- 2 Compute $-\Delta u$ to obtain f
- 3 Compute boundary values (trivial if only Dirichlet boundary conditions are used)
- 4 Solve the corresponding variational problem

$$a(u_h, v) = L(v)$$

for a sequence of meshes \mathcal{T}_h and compute the error

$$E_i(h) = \|u - u_h\|_{i, \Omega_i} \text{ for } i = 0, 1$$

- 5 Plot $\log(E_i(h))$ against $\log(h)$ and determine k

Homework

Try this by taking $u = \sin(2\pi x) \sin(2\pi y)$ on the unit square. Solve the problem for $N = 2, 4, 8, 16, 64, 128$ and compute both the L^2 and H^1 errors for $P1$, $P2$ and $P3$ elements as a function of h . Can you determine the convergence rate?