# FEniCS Course

## Lecture 8: A posteriori error estimates and adaptivity

*Contributors*
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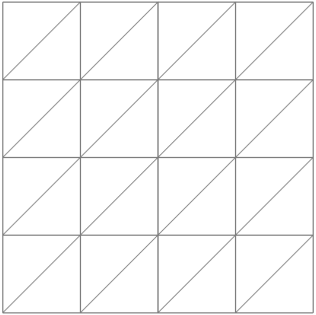
# *A priori* estimates

If $u \in H^{k+1}(\Omega)$ and $V_h = P^k(\mathcal{T}_h)$ then

$$|||u - u_h||| \leqslant Ch^k \|u\|_{\Omega,k+1}$$
$$\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^{k+1} \|u\|_{\Omega,k+1}$$



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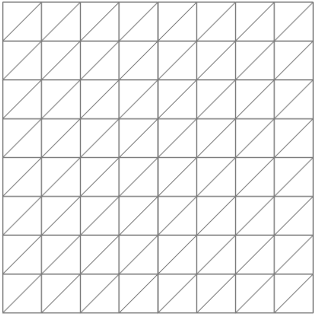

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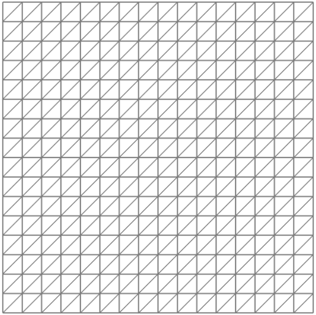

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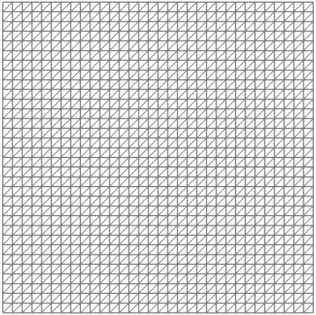

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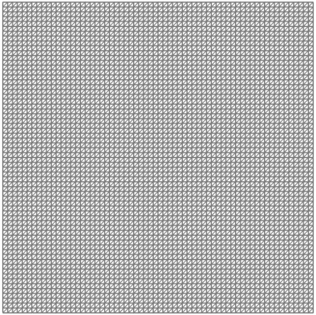

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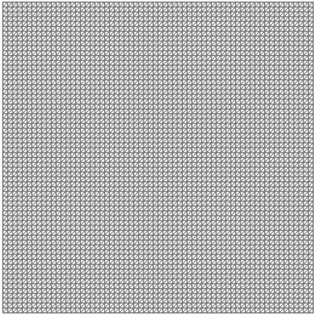

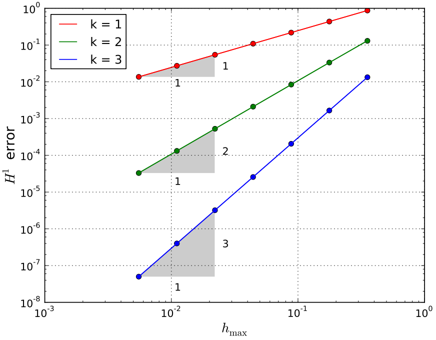

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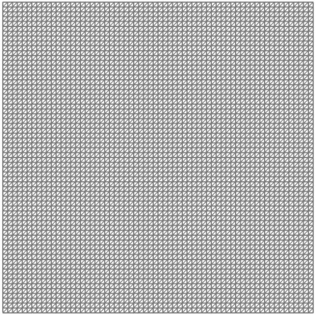
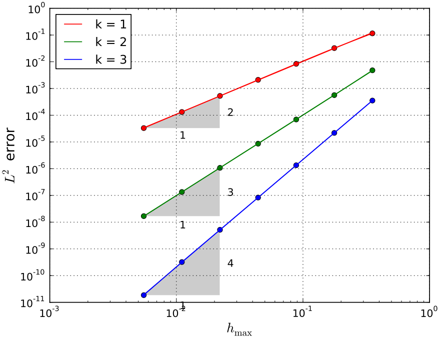
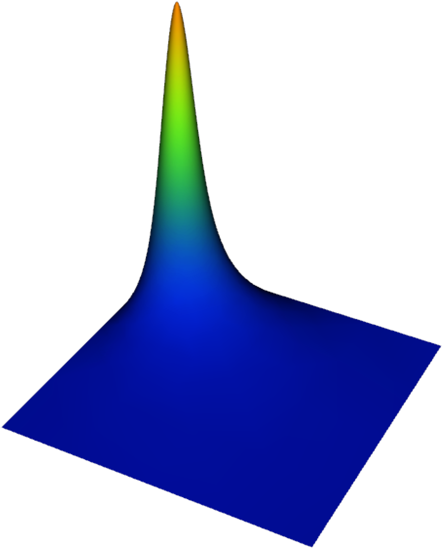




# Mesh adaptation can yield more accurate results with less computational resources



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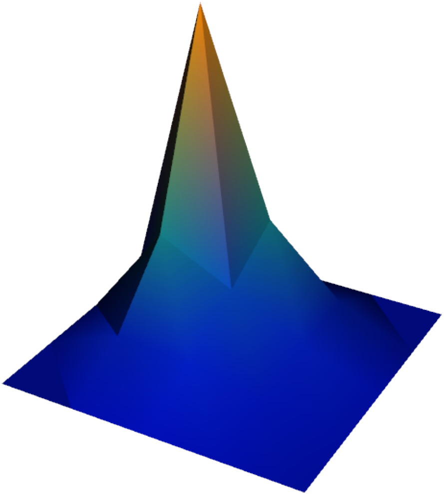

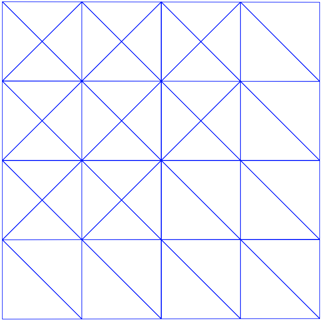

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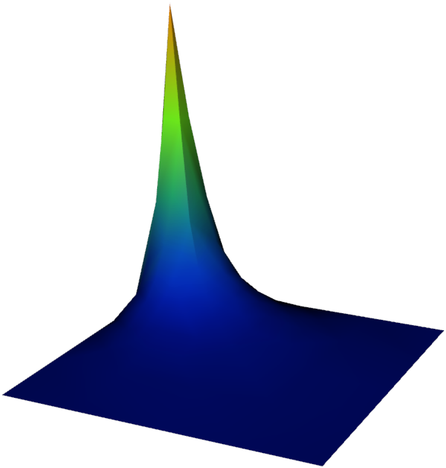

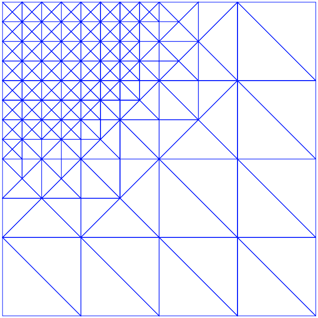

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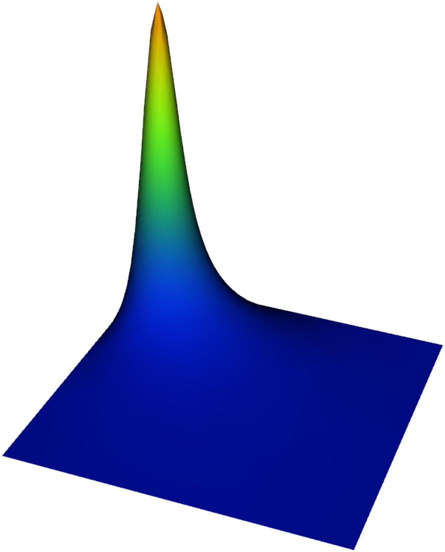
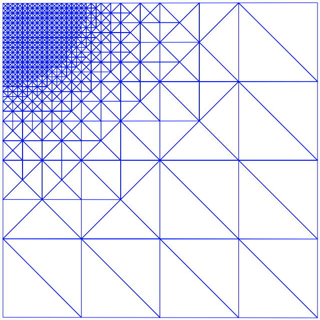
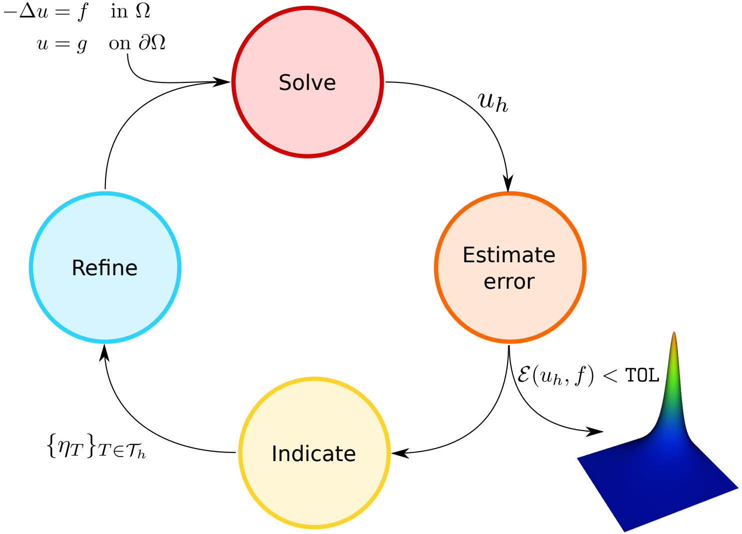




# A posteriori error estimation can be used to steer the mesh adaptivity process



# Desired properties of error estimators

An error estimator $\mathcal{E} \sim \|u - u_h\|$ has to be

$$\text{computable} \quad \mathcal{E} = \mathcal{E}(u_h, f)$$

and should be

$$\text{reliable} \quad |||u - u_h||| \leqslant C\mathcal{E}(u_h, f)$$

$$\text{efficient} \quad c\mathcal{E}(u_h, f) \leqslant |||u - u_h|||$$

$$\text{local} \quad \mathcal{E}(u_h, f)^2 = \sum_{T \in \mathcal{T}_h} \rho_T(u_h, f)^2$$

The quality of $\mathcal{E}(u_h, f)$ is measured by the efficiency index $\eta$

$$\eta(\mathcal{E}(u_h, f)) = \frac{\|u - u_h\|}{\mathcal{E}(u_h, f)}$$

If $\eta(\mathcal{E}(u_h, f)) \to 1$ as $h \to 0$, the error estimator is asymptotically exact.

# Types of error estimators

- Explicit residual-based error estimators
- Implicit error estimator based on local problems
- Gradient recovery estimators
- Hierarchic error estimators
- Goal-oriented error estimators

# Explicit residual based error estimators

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Residual equation

$$R(u_h, f; v) = (\nabla(u - u_h), \nabla v) = (f, \nabla v) - (\nabla u_h, \nabla v) \quad \forall\, v \in H_0^1(\Omega)$$

Recall Galerkin orthogonality

$$R(u_h, f; v_h) = 0 \quad \forall\, v_h \in \widehat{V}_h$$

Interpolation operator $\pi_h : V \to V_h$

$$\begin{aligned} R(v) &= (f, v - \pi_h v) - (\nabla u_h, \nabla(v - \pi_h v)) \\ &= \sum_{T \in \mathcal{T}_h} (f + \Delta u_h, v - \pi_h v)_T - \sum_{T \in \mathcal{T}_h} (\nabla u_h \cdot n_T, v - \pi_h)_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (f + \Delta u_h, v - \pi_h v)_T - \sum_{F \in \partial \mathcal{F}^i} ([\nabla u_h \cdot n_T], v - \pi_h)_F \end{aligned}$$

# Explicit residual based error estimators

Starting from

$$R(v) = \sum_{T \in \mathcal{T}_h} (f + \Delta u_h, v - \pi_h v)_T - \sum_{F \in \partial \mathcal{F}^i} ([\nabla u_h \cdot n_T], v - \pi_h)_F$$

and using the quasi-interpolent by Clement, which satisfies

$$\|v - \pi_h\|_{0,T} \leq C_1 h_T \|v\|_{\omega(T)}$$
$$\|v - \pi_h\|_{0,F} \leq C_2 h_F^{1/2} \|v\|_{\omega(F)},$$

one obtains

$$|R(v)| \leqslant C \|v\|_1 \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \|f + \Delta u_h\|^2 + \sum_{F \in \mathcal{F}^i} h_E \|[\nabla u_h] \cdot n\|_F^2 \right\}^{1/2}$$

# Explicit, residual based error estimators

Define

| | |
|---|---|
| Element residual | $r_T := f + \Delta\|_T$ |
| Facet residual | $r_F := [\nabla u_h \cdot n]\|_T$ |
| Error indicators | $\rho_T^2 := h_T^2 \|r_T\|_T^2 + \frac{1}{2} \sum_{F \in \partial T} h_F \|r_F\|_T^2$ |

Poincaré inequality gives $\|v\|_1 \sim \|\nabla v\|$ an thus

$$\|u - u_h\|_1 \leqslant \sup_{v \in V} \frac{(\nabla(u - u_h), \nabla v)}{\|v\|_1} = \sup_{v \in V} \frac{|R(v)|}{\|v\|_1}$$
$$\leqslant C\Big(\sum_{T \in \mathcal{T}_h} \rho_T^2\Big)^{1/2}$$

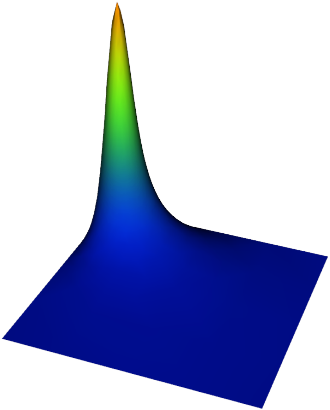
proving the reliability of the error estimator defined by $\{\rho_T\}_T$.

# FEniCS coding interlude

Again, we look at the Poisson problem

$$-\Delta u = e^{-100(x^2+y^2)} \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

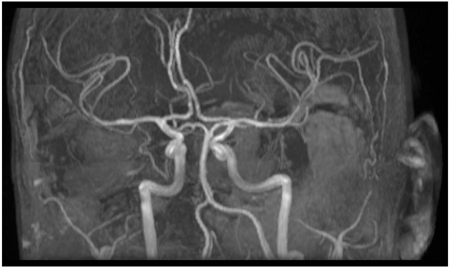
Our mission is to implement the classical, residual-based error estimator and to solve this problem adaptively for a given tolerance $TOL = 1.0e - 3$. As marking strategy we choose to mark those 30% of elements with the largest element residual.

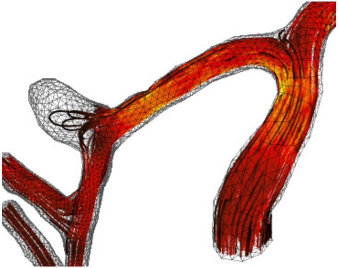


# Goal-oriented error control

# What is goal-oriented error control?

## The scientist's viewpoint





Shear stress at vessel wall?

[Valen-Sendstad '08]

# What is goal-oriented error control?

## The mathematician's viewpoint

### Input

- PDE: find $u \in V$ such that $a(u, v) = L(v) \quad \forall\, v \in V$
- Quantity of interest/Goal: $\mathcal{M} : V \to \mathbb{R}$
- Tolerance: $\epsilon > 0$

### Challenge

Find $V_h \subset V$ such that $|\mathcal{M}(u) - \mathcal{M}(u_h)| < \epsilon$ where $u_h \in V_h$ is determined by

$$a(u_h, v) = L(v) \quad \forall\, v \in V_h$$

# The error measured in the goal is the residual of the dual solution

1. Recall residual $R(v) := L(v) - a(u_h, v)$
2. Introduce dual problem

$$\text{Find } z \in \widetilde{V}: \quad a^*(z, v) = \mathcal{M}(v) \quad \forall\, v \in \widehat{V}$$

3. Dual solution + residual $\Longrightarrow$ error

$$\mathcal{M}(u) - \mathcal{M}(u_h) = \mathcal{M}(u - u_h) = a^*(z, u - u_h) = a(u - u_h, z)$$
$$= L(z) - a(u_h, z) = R(z) = R(z - z_h)$$

4. A good dual approximation $\tilde{z}_h$ gives computable error estimate $\eta_h = r(\tilde{z}_h)$
5. Error indicators ... ?

# The dual-weighted residual method for the Poisson problem

Start with representation

$$|\mathcal{M}(u) - \mathcal{M}(u_h)| = |R(u - u_h, z - z_h)|.$$

Integrate by parts as for the classical energy-norm estimate gives

$$|\mathcal{M}(u) - \mathcal{M}(u_h)| \leqslant \sum_{T \in \mathcal{T}_h} \rho_T \omega_T,$$

where $\rho_T$ is resembles the standard element residual

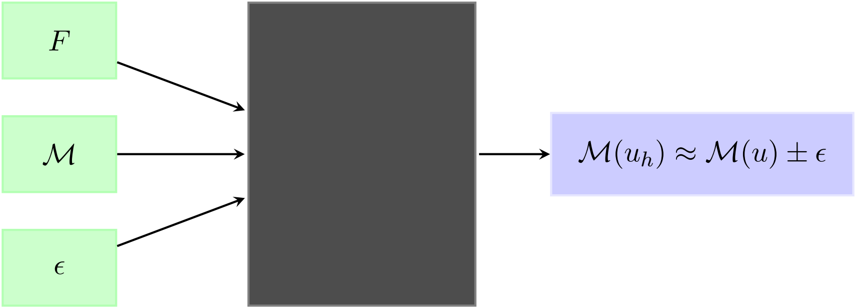
$$\rho_T = \|r_T\|_T + h_T^{1/2} \|r_F\|_{\partial T}$$

and $\omega_T$ is a weight

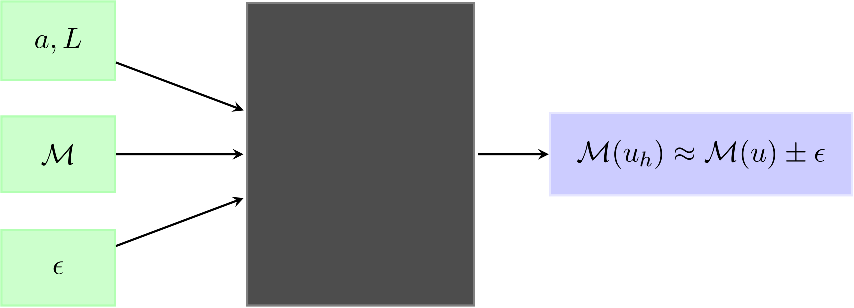
$$\omega_T = \|z - z_h\|_T + h_T^{1/2} \|z - z_h\|_{\partial T}$$

derived from the dual solution $z$.

# What is automated goal-oriented error control?



# What is automated goal-oriented error control?




**FEniCS/DOLFIN**

```python
pde = VariationalProblem(a, L, bc)
pde.solve(u, tol, M)
```

# Automated goal-oriented adaptivity – A complete example

```
from dolfin import *

# Create mesh and define function space
mesh = UnitSquare(8, 8)
V = FunctionSpace(mesh, "Lagrange", 1)

# Define boundary condition
u0 = Function(V)
bc = DirichletBC(V, u0, "x[0] < DOLFIN_EPS ||
    x[0] > 1.0 - DOLFIN_EPS")
```

# Automated goal-oriented adaptivity – A complete example

Define variational problem:

```
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("10*exp(-(pow(x[0] - 0.5, 2) +
    pow(x[1] - 0.5, 2)) / 0.02)",
               degree=1)
g = Expression("sin(5*x[0])", degree=1)
a = inner(grad(u), grad(v))*dx
L = f*v*dx + g*v*ds
```

# Automated goal-oriented adaptivity – A complete example

Define function for the solution:

```
u = Function(V)
```

Define goal functional (quantity of interest) and tolerance:

```
M = u*dx
tol = 1.e-5
```

# Automated goal-oriented adaptivity – A complete example

Solve equation a = L with respect to u and the given boundary conditions, such that the estimated error (measured in M) is less than tol

```
solver_parameters = {"error_control":
                        {"dual_variational_solver":
                         {"linear_solver": "cg"}}}
solve(a == L, u, bc, tol=tol, M=M,
    solver_parameters=solver_parameters)
```

# Automated goal-oriented adaptivity – A complete example

Alternative, more verbose version (+ illustrating how to set parameters)

```
problem = LinearVariationalProblem(a, L, u, bc)
solver = AdaptiveLinearVariationalSolver(problem)
solver.parameters["error_control"]["dual_variational_solver"]["linear_solver"]
    = "cg"
solver.solve(tol, M)
```

Extract solutions on coarsest and finest mesh:

```
plot(u.root_node(), title="Solution on initial
    mesh")
plot(u.leaf_node(), title="Solution on final
    mesh")
interactive()
```

# FEniCS coding exercise

Implement the problem described in the walk-through example! Afterwards modify it in such a way that you solve the same problem as in the coding exercise for the residual error estimator. As goal functional compute

$$\mathcal{M}(u) = \int_{\partial\Gamma} u \, \mathrm{d}S$$

where $\Gamma$ is given by

$$\Gamma = [0.5, 1] \times \{1\} \cup \{1\} \times [0.5, 1]$$

Choose different TOL starting from TOL=0.01. How do the generated meshes compare to the generated meshes in the first exercise?