Euler equations in 1-D

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in/~praveen

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Euler equations in 1-D

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \qquad U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F(U) = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix}$$

$$\rho = \text{density}, \quad u = \text{velocity}, \quad p = \text{pressure}$$

$$E = \text{total energy per unit volume} = \rho e + \frac{1}{2}\rho u^2$$

$$\rho e = \text{internal energy per unit volume}$$

$$e = \text{internal energy per unit mass}$$

The pressure p is related to the internal energy e by the caloric equation of state $p = p(\rho, e)$; for a calorically ideal gas, $p = (\gamma - 1)\rho e$, so that

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right]$$

Flux Jacobian

The flux jacobian $A \in \mathbb{R}^{3\times 3}$ is defined as

$$A(U) := F'(U) = \frac{\partial F}{\partial U}$$

The jacobian can be computed by first expressing the flux vector in terms of the conserved variables

$$F(U) = \begin{bmatrix} U_2 \\ \frac{1}{2}(3-\gamma)\frac{U_2^2}{U_1} + (\gamma-1)U_3 \\ \gamma\frac{U_2 U_3}{U_1} - \frac{1}{2}(\gamma-1)\frac{U_2^2}{U_1^2} \end{bmatrix}$$

The jacobian is then given by

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(\gamma-3)\left(\frac{U_2}{U_1}\right)^2 & (3-\gamma)\frac{U_2}{U_1} & \gamma-1 \\ -\gamma\frac{U_2 U_3}{U_1^2} + (\gamma-1)\left(\frac{U_2}{U_1}\right)^3 & \gamma\frac{U_3}{U_1} - \frac{3}{2}(\gamma-1)\left(\frac{U_2}{U_1}\right)^2 & \gamma\frac{U_2}{U_1} \end{bmatrix}$$

Flux Jacobian

Defining the total specific enthalpy H

$$H = (E + p)/\rho = \frac{a^2}{\gamma - 1} + \frac{1}{2}u^2$$

The jacobian matrix can be written as

$$A(U) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (3 - \gamma)u & \gamma - 1 \\ u[\frac{1}{2}(\gamma - 1)u^2 - H] & H - (\gamma - 1)u^2 & \gamma u \end{array} \right]$$

Hyperbolicity

We can write Euler equations in quasi-linear form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0$$

The flux Jacobian A has eigenvalues $\lambda_1 = u - a$, $\lambda_2 = u$ and $\lambda_3 = u + a$. The corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} 1 \\ u - a \\ H - ua \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ u + a \\ H + ua \end{bmatrix}$$

which are linearly independent. Thus the time dependent Euler equations are hyperbolic. The flux Jacobian can be expressed in terms of the eigenvalues and eigenvectors by the following diagonal decomposition

$$A = R\Lambda R^{-1}$$

Hyperbolicity

where the matrix R has the eigenvectors on its columns and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. The rows of R^{-1} are the left eigenvectors of A; the left and right eigenvectors are mutually orthogonal. In fact, since

$$R^{-1} = \begin{bmatrix} \frac{\gamma-1}{4}\frac{u^2}{a^2} + \frac{u}{2a} & -\frac{\gamma-1}{a}\frac{u}{a^2} - \frac{1}{2a} & \frac{\gamma-1}{2a^2} \\ 1 - \frac{\gamma-1}{2}\frac{u^2}{a^2} & (\gamma-1)\frac{u}{a^2} & -\frac{\gamma-1}{a^2} \\ \frac{\gamma-1}{4}\frac{u^2}{a^2} - \frac{u}{2a} & -\frac{\gamma-1}{a}\frac{u}{a^2} + \frac{1}{2a} & \frac{\gamma-1}{2a^2} \end{bmatrix}$$

we have $l_i r_j = \delta_{ij}$.

Homogeneity property

If the equation of state satisfies

$$p(\alpha\rho, e) = \alpha p(\rho, e), \qquad \text{for every} \quad \alpha > 0$$

then it is easy to check[1] that the flux vector satisfies

$$F(\alpha U) = \alpha F(U) \quad \text{for every} \quad \alpha > 0$$

This implies that

$$F = \frac{\partial F}{\partial U} U = AU$$

which is called the homogeneity property. It can also be directly checked by computing the product AU. This special property of the Euler equations is used in the Steger-Warming flux splitting scheme and in the Beam-Warming scheme.

[1] See [**?**]

Primitive form

Primitive variables

$$V = [\rho,\ u,\ p]^\top$$

The transformation between U and V is given by

$$\begin{array}{l|l} U_1 = \rho & \rho = U_1 \\ U_2 = \rho u & u = U_2/U_1 \\ U_3 = p/(\gamma-1) + \rho u^2/2 & p = (\gamma-1)(U_3 - U_2^2/(2U_1)) \end{array}$$

Defining the jacobian $M := U'(V)$, the Euler equations can be transformed to the primitive form

$$\frac{\partial V}{\partial t} + \tilde{A}\frac{\partial V}{\partial x} = 0, \quad \tilde{A} = M^{-1}AM$$

The Jacobian of the transformation is

$$M = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{bmatrix}$$

Primitive form

This matrix is invertible since $det(M) = \rho/(\gamma - 1) > 0$. The matrix $\tilde{A}$ can be computed as

$$\tilde{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix}$$

whose eigenvalues are again $u - a$, u and $u + a$. This is obvious since A and $\tilde{A}$ are related by a similarity transformation.

The primitive form can also be derived by manipulating the conservation form in the following way. The continuity equation gives

$$\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0}$$

which is in the primitive form. The momentum equation can be written as

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u^2\frac{\partial \rho}{\partial x} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$

Primitive form

Using the continuity equation to eliminate the time derivative of ρ we have

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0}$$

Similarly, from the energy equation and eliminating ρ_t and u_t, we obtain

$$\boxed{\frac{\partial p}{\partial t} + \rho a^2\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} = 0}$$

Writing the three equations as a system, we have

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ u \\ p\end{bmatrix} + \begin{bmatrix}u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho \\ u \\ p\end{bmatrix} = 0$$

which immediately gives us the matrix $\tilde{A}$.

Entropy equation

Consider the quantity $s = p/\rho^{\gamma}$. Using the primitive form of the Euler equations, we can show that

$$\begin{aligned} \frac{\partial s}{\partial t} &= \frac{1}{\rho^{\gamma}}\left(\frac{\partial p}{\partial t} - a^2 \frac{\partial \rho}{\partial t}\right) \\ &= -u\frac{1}{\rho^{\gamma}}\left(\frac{\partial p}{\partial x} - a^2 \frac{\partial \rho}{\partial x}\right) \\ &= -u\frac{\partial s}{\partial x} \end{aligned}$$

which gives us an additional conservation law

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0$$

This equation tells us that the quantity s which is the entropy, is convected along with the fluid; the entropy of a fluid element remains

Entropy equation

constant. This is however not always true, e.g. when shocks are present. Using the continuity equation this can also be written in conservation form

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x}(\rho s u) = 0$$

We can replace s by any convex function $\eta(s)$ and derive a similar conservation law for η. In the presence of shocks, the equality must be replaced by an inequality and the equation interpreted in the weak sense. For smooth solutions, the entropy equation implies that $p = \mathrm{const}\rho^{\gamma}$ along a particle path. For 1-D problems, if the inflow is uniform in time then the entropy is constant everywhere.

Characteristic form

We can put the Euler equations in the form

$$\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} = 0$$

which leads to the characteristic equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

The entropy equation is already in this form, i.e.,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Combining the primitive form of the momentum and pressure equations, we have

$$\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x}\right) + a\left(\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}\right) = 0$$

Characteristic form

or

$$\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x} + \rho a \left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] = 0$$

which implies that

$$\frac{1}{\rho a}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}u}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + a$$

Integrating this equation we have

$$\int \left(\frac{\mathrm{d}p}{\rho a} + \mathrm{d}u\right) = C \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + a$$

The entropy condition implies that ρ, a can be written as functions of pressure so that the first integral can be evaluated to

$$\frac{a}{\gamma - 1} + \frac{u}{2} = \text{const}, \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u + a$$

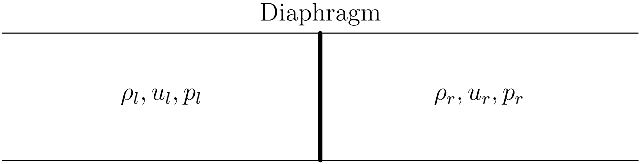
Characteristic form

Similarly we get

$$\frac{a}{\gamma - 1} - \frac{u}{2} = \mathrm{const}, \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = u - a$$

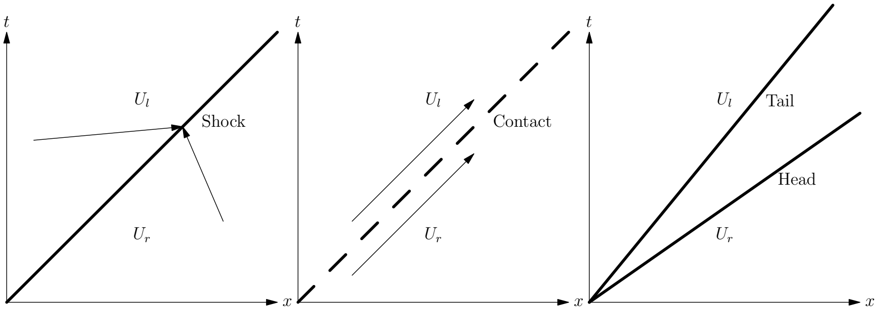
Riemann problem (Shock tube problem)

Two gases are separated by a diaphragm inside a tube. The gases on the two sides of the diaphragm are at different states; when the diaphragm is ruptured, a pattern of waves is set up in the tube which may travel along the length of the tube.



We have seen that for a linear system of n hyperbolic PDEs, the Riemann problem consists of n discontinuity waves propagating with speeds given by the eigenvalues. For the 1-D Euler equations, the Riemann problem has in general three waves known as *shock*, *contact* and *expansion* wave. What type of waves are actually present in the solution will depend on the initial conditions of the Riemann problem.

Riemann problem (Shock tube problem)



- A shock is a discontinuity across which all the flow variables density, velocity, pressure, are discontinuous. A shock is associated with the characteristic fields corresponding to the eigenvalues $\lambda_1 = u - a$ and $\lambda_3 = u + a$. The characteristics on either side of the shock intersect into the shock. Fluid particles can cross the shock; when this happens, their velocity decreases, and, density and pressure increase.

Riemann problem (Shock tube problem)

- A contact is a discontinuity across which density is discontinuous but pressure and velocity are continous. It is associated with the characteristic field corresponding to the eigenvalue $\lambda_2 = u$. The characteristics on either side of the contact are parallel to the contact line. Fluid particles do not cross a contact discontinuity.
- A rarefaction or expansion wave is a continous wave which consists of a *head* and a *tail*; all the flow quantities vary continuously through the wave and the entropy is constant. This wave is associated with the characteristic fields corresponding to the eigenvalues $\lambda_1 = u - a$ and $\lambda_3 = u + a$.

Riemann problem (Shock tube problem)

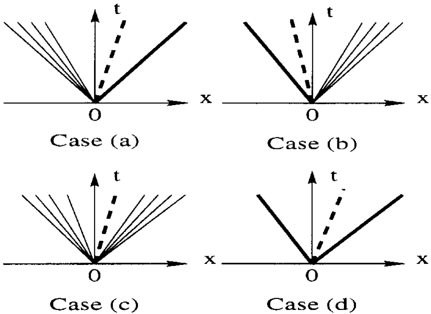


Fig. 4.2. Possible wave patterns in the solution of the Riemann problem: (a) left rarefaction, contact, right shock (b) left shock, contact, right rarefaction (c) left rarefaction, contact, right rarefaction (d) left shock, contact, right shock

Riemann problem (Shock tube problem)

Test	ρ_L	u_L	p_L	ρ_R	u_R	p_R
1	1.0	0.0	1.0	0.125	0.0	0.1
2	1.0	-2.0	0.4	1.0	2.0	0.4
3	1.0	0.0	1000.0	1.0	0.0	0.01
4	1.0	0.0	0.01	1.0	0.0	100.0
5	5.99924	19.5975	460.894	5.99242	-6.19633	46.0950

Table 4.1. Data for five Riemann problem tests

Riemann problem (Shock tube problem)

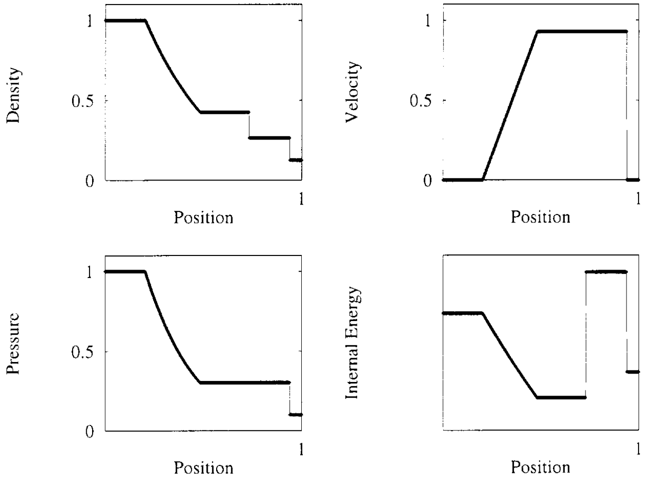


Fig. 4.7. Test 1: Exact solution for density, velocity, pressure and specific internal energy at time t = 0.25 units

Riemann problem (Shock tube problem)

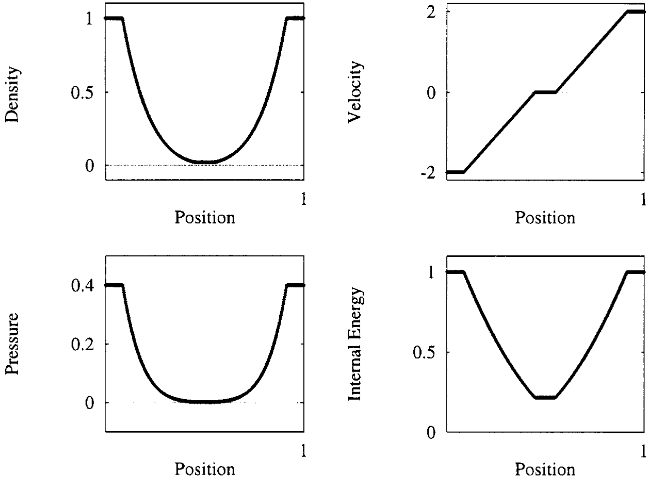


Fig. 4.8. Test 2: Exact solution for density, velocity, pressure and specific internal energy at time t = 0.15 units

Riemann problem (Shock tube problem)

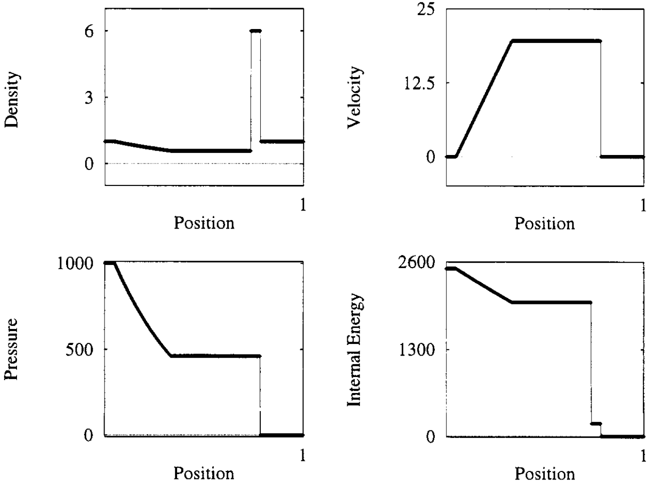


Fig. 4.9. Test 3: Exact solution for density, velocity, pressure and specific internal energy at time t = 0.012 units

Riemann problem (Shock tube problem)

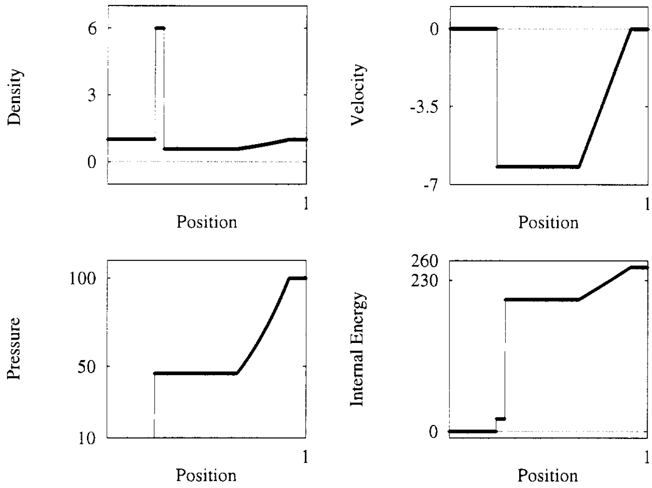


Fig. 4.10. Test 4: Exact solution for density, velocity, pressure and specific internal energy at time t = 0.035 units

Riemann problem (Shock tube problem)

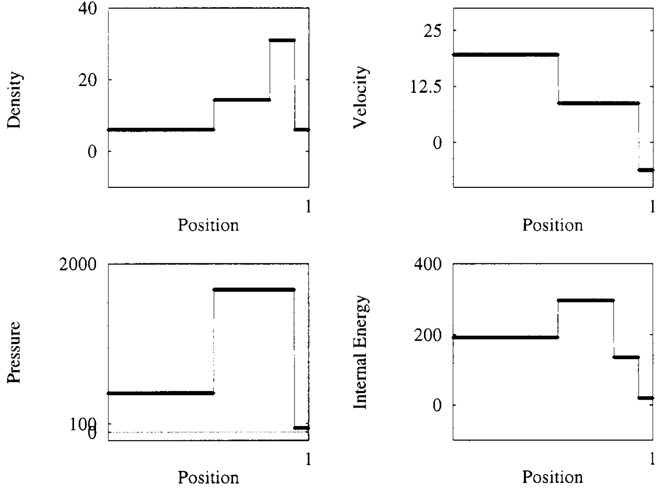


Fig. 4.11. Test 5: Exact solution for density, velocity, pressure and specific internal energy at time t = 0.035 units