Flux vector splitting scheme

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Flux Vector Splitting schemes

- Split flux F: $F(U) = F^+(U) + F^-(U)$

$$
\begin{aligned}
F^+ &= \text{due to waves moving to right, positive speed} \\
F^- &= \text{due to waves moving to left, negative speed}
\end{aligned}
$$

$$\Downarrow$$

$\dfrac{\partial F^+}{\partial U}$ has positive eigenvalues, $\dfrac{\partial F^-}{\partial U}$ has negative eigenvalues

- Numerical flux function

$$F_{i+1/2} = F(U_i, U_{i+1}) = F^+(U_i) + F^-(U_{i+1})$$

- It is consistent: if $U_i = U_{i+1} = U$

$$F_{i+\frac{1}{2}} = F(U, U) = F^+(U) + F^-(U) = F(U)$$

Steger-Warming scheme

- Euler flux satisfy the homogeneity property

$$F(U) = A(U)U, \quad A(U) = \frac{\partial F}{\partial U}$$

- Split jacobians using eigenvalue splitting

$$A(U) = A^+(U) + A^-(U), \quad A^\pm(U) = R(U)\Lambda^\pm(U)R^{-1}(U)$$

- Flux splitting based on eigenvalues

$$F(U) = F^+(U) + F^-(U), \quad F^\pm(U) = A^\pm(U)U$$

- Steger-Warming flux

$$F_{i+1/2} = F^+(U_i) + F^-(U_{i+1}) = A^+(U_i)U_i + A^-(U_{i+1})U_{i+1}$$

- Vijayasundaram flux

$$F_{i+1/2} = A^+(U_{i+\frac{1}{2}})U_i + A^-(U_{i+\frac{1}{2}})U_{i+1}, \qquad U_{i+\frac{1}{2}} = \frac{1}{2}(U_i + U_{i+1})$$

Steger-Warming Scheme

- Conserved vector U in terms of the eigenvectors of A

$$U = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3, \qquad \alpha_1 = \alpha_3 = \frac{\rho}{2\gamma}, \qquad \alpha_2 = \rho\frac{\gamma - 1}{\gamma}$$

or

$$U = R\alpha, \qquad \alpha = [\alpha_1,\ \alpha_2,\ \alpha_3]^\top$$

- Split fluxes given by

$$F^\pm = A^\pm U = R\Lambda^\pm R^{-1}(R\alpha) = \alpha_1\lambda_1^\pm r_1 + \alpha_2\lambda_2^\pm r_2 + \alpha_3\lambda_3^\pm r_3$$

- Upwind property: if $\gamma \in [1, 5/3]$

$$\frac{\partial F^+}{\partial U} \geq 0, \quad \frac{\partial F^-}{\partial U} \leq 0$$

- Adds excessive numerical dissipation: poor resolution of contact waves
- $F^\pm$ not differentiable at sonic and stagnation points; Laney recommends smoothing the eigenvalues

$$\tilde{\lambda}_i^\pm = \frac{1}{2}\left(\lambda_i \pm \sqrt{\lambda_i^2 + \delta^2}\right), \qquad i = 1, 2, 3$$

van Leer splitting

- Mach number: $M = \frac{u}{a}$
- If $M > 1$: all eigenvalues are positive

$$u - a > 0, \quad u > 0, \quad u + a > 0$$

  and if $M < -1$, all eigenvalues are negative
- Euler flux: polynomials in M

$$F = \begin{bmatrix} \rho a M \\ \frac{\rho a^2}{\gamma}(\gamma M^2 + 1) \\ \rho a^3 M \left(\frac{1}{2} M^2 + \frac{1}{\gamma - 1} \right) \end{bmatrix}$$

- Each component flux of the form

$$F = G(\rho, a) H(M)$$

- Split polynomial $H(M)$ such that we have upwinding and smoothness
  1. $H(M) = H^+(M) + H^-(M)$
  2. $H^+(M) = 0$ for $M \le -1$ and $H^+(M) = H(M)$ for $M \ge 1$.
  3. $\frac{\mathrm{d}}{\mathrm{d}M} H^+(-1) = 0$, $\frac{\mathrm{d}}{\mathrm{d}M} H^+(1) = \frac{\mathrm{d}}{\mathrm{d}M} H(1)$.

van Leer: Mass flux

$$H(M) = M = M^+(M) + M^-(M)$$

- To satisfy all conditions, $M^\pm$ must be quadratic polynomial in M

$$M^+ = \begin{cases} 0 & M \le -1 \\ \left(\frac{M+1}{2}\right)^2 & -1 < M < 1 \\ M & M \ge 1 \end{cases}$$

$$M^- = \begin{cases} M & M \le -1 \\ -\left(\frac{M-1}{2}\right)^2 & -1 < M < 1 \\ 0 & M \ge 1 \end{cases}$$

van Leer: momentum flux

Split $\gamma M^2 + 1$ using cubic polynomials in M

$$(\gamma M^2 + 1) = (\gamma M^2 + 1)^+ + (\gamma M^2 + 1)^-$$

$$(\gamma M^2 + 1)^+ = \begin{cases} 0 & M \leq -1 \\ \left(\frac{M+1}{2}\right)^2 [(\gamma - 1)M + 2] & -1 < M < +1 \\ \gamma M^2 + 1 & M \geq 1 \end{cases}$$

$$(\gamma M^2 + 1)^- = \begin{cases} \gamma M^2 + 1 & M \leq -1 \\ -\left(\frac{M-1}{2}\right)^2 [(\gamma - 1)M - 2] & -1 < M < +1 \\ 0 & M \geq 1 \end{cases}$$

van Leer: Energy flux

Split energy flux using quartic polynomials in M

$$F_3^+ = \begin{cases} 0 & M \leq -1 \\ \frac{[(\gamma-1)u+2a]^2 F_1^+}{2(\gamma+1)(\gamma-1)} & -1 < M < 1 \\ F_3 & M > 1 \end{cases}$$

$$F_3^- = \begin{cases} F_3 & M \leq -1 \\ \frac{[(\gamma-1)u-2a]^2 F_1^-}{2(\gamma+1)(\gamma-1)} & -1 < M < 1 \\ 0 & M > 1 \end{cases}$$

van Leer flux

- Final flux formulae

$$F^{\pm} = \pm\frac{1}{4}\rho a(M \pm 1)^2 \begin{bmatrix} 1 \\ \frac{(\gamma-1)u \pm 2a}{\gamma} \\ \frac{[(\gamma-1)u \pm 2a]^2}{2(\gamma+1)(\gamma-1)} \end{bmatrix}$$

- Has upwind property of split flux jacobians

$$\frac{\partial F^+}{\partial U} \geq 0, \quad \frac{\partial F^-}{\partial U} \leq 0$$

- Adds excessive dissipation for contact discontinuity

Liou and Steffen (1993)

Separate flux into convective and pressure parts

$$F = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho H u \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \\ \rho H u \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} = M \begin{bmatrix} \rho a \\ \rho u a \\ \rho H a \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} = M F_c + F_p$$

Flux splitting

$$F^{\pm} = M^{\pm} F_c + F_p^{\pm}, \qquad F_p^{\pm} = \begin{bmatrix} 0 \\ p^{\pm} \\ 0 \end{bmatrix}$$

$M^{\pm}$ is same as in van Leer scheme. Pressure $p = p^+ + p^-$

$$p^+ = p \begin{cases} 0 & M \le -1 \\ \frac{1}{2}(1+M) & -1 < M < 1 \\ 1 & M \ge 1 \end{cases}, \qquad p^- = p \begin{cases} 1 & M \le -1 \\ \frac{1}{2}(1-M) & -1 < M < 1 \\ 0 & M \ge 1 \end{cases}$$

Remark: See the papers on AUSM family of schemes.

Zha-Bilgen flux vector splitting (1993)

$$F = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \\ Eu \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix} = u \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix}$$

Flux vector splitting

$$F^{\pm} = u^{\pm} U + \begin{bmatrix} 0 \\ p^{\pm} \\ (pu)^{\pm} \end{bmatrix}, \qquad u^{\pm} = \frac{1}{2}(u \pm |u|)$$

$p^{\pm}$ is same as in Liou-Steffen scheme.

$$(pu)^{+} = p \begin{cases} 0 & M \le -1 \\ \frac{1}{2}(u+a) & -1 < M < 1 \\ u & M \ge 1 \end{cases}, \qquad (pu)^{-} = p \begin{cases} u & M \le -1 \\ \frac{1}{2}(u-a) & -1 < M < 1 \\ 0 & M \ge 1 \end{cases}$$

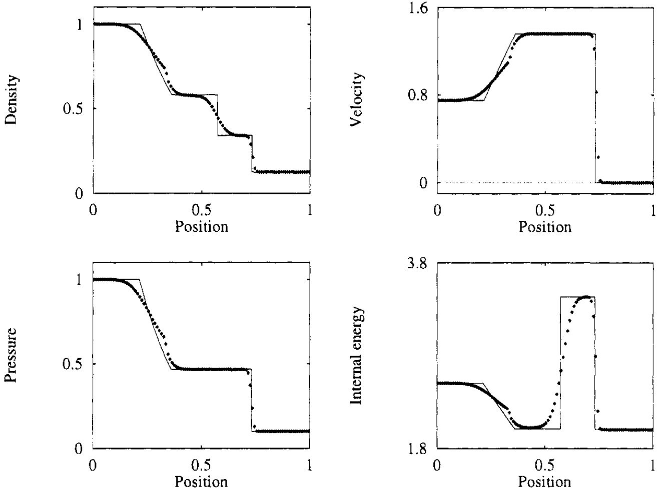

Fig. 8.3. Steger and Warming FVS scheme applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2 units

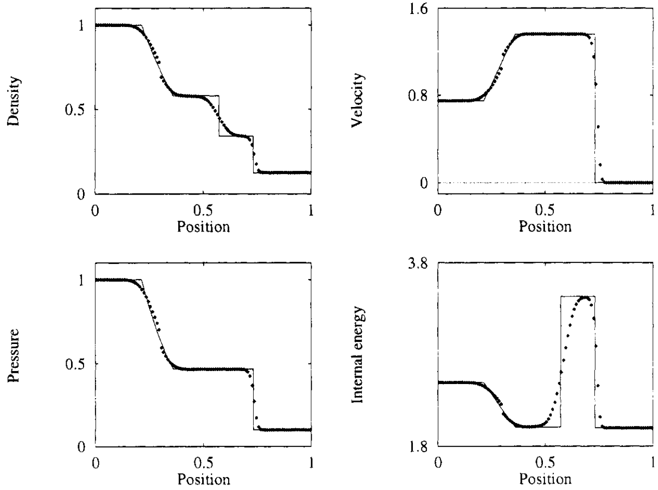

Fig. 8.4. Van Leer FVS scheme applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2 units

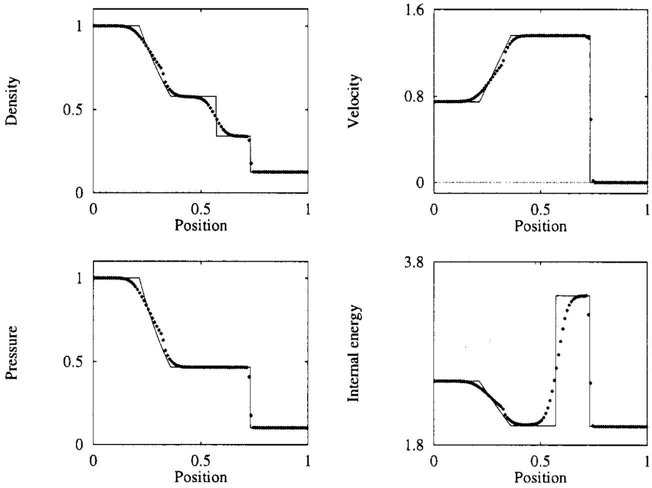

Fig. 8.5. Liou and Steffen scheme applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2 units