Jameson-Schmidt-Turkel (JST) scheme

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Dissipation operators

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} - \eta\frac{\partial^4 u}{\partial x^4}, \qquad \mu > 0, \quad \eta > 0$$

Consider a single Fourier mode with wave number k

$$u(x,t) = u_k(t)\mathrm{e}^{\mathrm{i}kx}$$

The amplitude evolves according to

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} + \mathrm{i}aku_k = -\mu k^2 u_k - \eta k^4 u_k$$

whose exact solution is

$$u_k(t) = u_k(0)\mathrm{e}^{\mathrm{i}k(x-at)}\mathrm{e}^{-(\mu k^2+\eta k^4)t}$$

Dissipation operators

Second and fourth order terms lead to loss of energy/amplitude. They add dissipation. The fourth derivative term damps high frequency components more strongly. To solve the hyperbolic PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

we replace with

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} - \eta\frac{\partial^4 u}{\partial x^4}, \qquad \mu > 0, \quad \eta > 0$$

where, the artificial coefficients μ, η vanish under mesh refinement

$$\mu \to 0 \qquad \eta \to 0 \qquad \text{as} \quad h \to 0$$

Now construct a central difference approximation for this modified equation. This is the method of **artificial dissipation**.

Upwind scheme

The upwind scheme for

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + a^+\frac{u_j - u_{j-1}}{h} + a^-\frac{u_{j+1} - u_j}{h} = 0$$

or in finite volume form

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{h} = 0, \qquad f_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a|(u_{j+1} - u_j)$$

satisfies the modified equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = h|a|\frac{\partial^2 u}{\partial x^2} + \mathcal{O}\left(h^2\right)$$

The artificial viscosity coefficient $\mu = h|a|$. Alternately, if we start with

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = h|a|\frac{\partial^2 u}{\partial x^2}$$

Upwind scheme

and discretize with central differences

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + a\frac{u_{j+1} - u_{j+1}}{2h} = h|a|\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}$$

then we obtain the upwind scheme.

JST Scheme

1-D Euler equations and FVM

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}F(U) = 0 \quad \Longrightarrow \quad \frac{\mathrm{d}U_j}{\mathrm{d}t} + \frac{F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}}{h} = 0$$

Numerical flux

$$F_{j+\frac{1}{2}} = \frac{1}{2}(F_j + F_{j+1}) - d_{j+\frac{1}{2}}$$

Dissipation: second order + fourth order dissipation

$$d_{j+\frac{1}{2}} = d^{(2)}_{j+\frac{1}{2}} + d^{(4)}_{j+\frac{1}{2}}$$

Finite volume scheme

$$\frac{\mathrm{d}U_j}{\mathrm{d}t} + \frac{F_{j+1} - F_{j-1}}{2h} = \frac{d^{(2)}_{j+\frac{1}{2}} - d^{(2)}_{j-\frac{1}{2}}}{h} + \frac{d^{(4)}_{j+\frac{1}{2}} - d^{(4)}_{j-\frac{1}{2}}}{h}$$

JST Scheme

Maximum wave speed at $x_{j+\frac{1}{2}}$, e.g.,

$$\lambda_{j+\frac{1}{2}} = \max(|u_j| + a_j, |u_{j+1}| + a_{j+1})$$

Second order dissipation

$$d^{(2)}_{j+\frac{1}{2}} = \varepsilon^{(2)}_{j+\frac{1}{2}} \lambda_{j+\frac{1}{2}} (U_{j+1} - U_j)$$

$$\frac{d^{(2)}_{j+\frac{1}{2}} - d^{(2)}_{j-\frac{1}{2}}}{h} \approx \varepsilon^{(2)} \lambda h \frac{\partial^2 U}{\partial x^2}$$

Fourth order dissipation

$$d^{(4)}_{j+\frac{1}{2}} = -\varepsilon^{(4)}_{j+\frac{1}{2}} \lambda_{j+\frac{1}{2}} (U_{j+2} - 3U_{j+1} + 3U_j - U_{j-1})$$

$$\frac{d^{(4)}_{j+\frac{1}{2}} - d^{(4)}_{j-\frac{1}{2}}}{h} \approx -\varepsilon^{(4)} \lambda h^3 \frac{\partial^4 U}{\partial x^4}$$

JST Scheme

Switching functions

$$\nu_j = \frac{|p_{j-1} - 2p_j + p_{j+1}|}{|p_{j-1} + 2p_j + p_{j+1}|}, \qquad \varepsilon^{(2)}_{j+\frac{1}{2}} = \kappa^{(2)} \max(\nu_{j-1}, \nu_j, \nu_{j+1}, \nu_{j+2})$$

$$\varepsilon^{(4)}_{j+\frac{1}{2}} = \max\left[0, \kappa^{(4)} - \varepsilon^{(2)}_{j+\frac{1}{2}}\right], \quad \frac{1}{4} \le \kappa^{(2)} \le \frac{1}{2}, \qquad \frac{1}{64} \le \kappa^{(4)} \le \frac{1}{32}$$

In smooth regions of flow, $\nu_j = \mathcal{O}\left(h^2\right)$

$$\varepsilon^{(2)}_{j+\frac{1}{2}} = \mathcal{O}\left(h^2\right), \quad d^{(2)}_{j+\frac{1}{2}} = \mathcal{O}\left(h^3\right), \quad \varepsilon^{(4)} = \mathcal{O}\left(1\right), \quad d^{(4)}_{j+\frac{1}{2}} = \mathcal{O}\left(h^3\right)$$

Fourth order dissipation damps high frequencies which helps to reach steady state quickly.

Near a shock, $\nu_j = \mathcal{O}\left(1\right)$

$$\varepsilon^{(2)}_{j+\frac{1}{2}} = \mathcal{O}\left(1\right), \quad d^{(2)}_{j+\frac{1}{2}} = \mathcal{O}\left(h\right), \quad \varepsilon^{(4)}_{j+\frac{1}{2}} = 0, \quad d^{(4)}_{j+\frac{1}{2}} = 0$$

JST Scheme: Matrix dissipation

Upwind scheme like Roe scheme able to compute shocks without oscillations. Dissipation is scaled by the eigenvectors. Roe flux

$$F_{j+\frac{1}{2}} = \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}|A|_{j+\frac{1}{2}}(U_{j+1} - U_j), \qquad |A| = R|\Lambda|R^{-1}$$

Replace $\lambda_{j+\frac{1}{2}}$ with a Roe-type matrix in dissipation terms

$$d^{(2)}_{j+\frac{1}{2}} = \varepsilon^{(2)}_{j+\frac{1}{2}}|A|_{j+\frac{1}{2}}(U_{j+1} - U_j)$$

$$d^{(4)}_{j+\frac{1}{2}} = -\varepsilon^{(4)}_{j+\frac{1}{2}}|A|_{j+\frac{1}{2}}(U_{j+2} - 3U_{j+1} + 3U_j - U_{j-1})$$

As in Roe scheme dissipation can vanish at sonic points and stagnation points. Eigenvalues in Λ matrix are not allowed to vanish. We can prevent them from becoming too small compared to maximum wave speed $\lambda_m = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$. Acoustic eigenvalues are modified as

$$|\tilde{\lambda}_i| = \max(|\lambda_i|, \alpha\lambda_m), \qquad i = 1, 3$$

JST Scheme: Matrix dissipation

and convective eigenvalue as

$$|\tilde{\lambda}_2| = \max(|\lambda_2|, \beta\lambda_m)$$

Typical values

$$\alpha \approx 0.25, \qquad \beta \approx 0.025$$

TVD property

The JST scheme with scalar dissipation may still produce oscillations at strong shocks. For scalar problems, if the numerical flux is written as

$$F_{j+\frac{1}{2}} = \frac{1}{2}(F_j + F_{j+1}) - \frac{1}{2}Q_{j+\frac{1}{2}}(U_{j+1} - U_j)$$

then it is TVD provided

$$Q_{j+\frac{1}{2}} \geq |A|_{j+\frac{1}{2}}, \qquad A = F'(U)$$

A modified switch proposed by Swanson and Turkel (1992) makes the scheme TVD for scalar problems

$$\nu_j = \frac{|p_{j-1} - 2p_j + p_{j+1}|}{|p_j - p_{j-1}| + |p_{j+1} - p_j| + \epsilon}, \qquad \kappa^{(2)} = \frac{1}{2}$$

TVD property

This leads to too much dissipation of shocks and in practice a weaker form of the switching function is used

$$\nu_j = \frac{|p_{j-1} - 2p_j + p_{j+1}|}{(1-\omega)P_j + \omega Q_j}, \qquad 0 \leq \omega \leq 1$$

$$P_j = |p_j - p_{j-1}| + |p_{j+1} - p_j|, \qquad Q_j = |p_{j-1} + 2p_j + p_{j+1}|$$

$\omega = 0$ gives the TVD switch and $\omega = 1$ gives the JST switch. Typically, a value of $\omega = \frac{1}{2}$ can be used for problems with strong shocks.

SLIP scheme

JST scheme uses pressure based switch for all flow components. It forces all variables to be treated equally though they may experience different changes through a discontinuity.

Jameson introduced SLIP (Symmetric limited posititive) scheme which makes use of limiters.

$$R(u,v) = 1 - \left| \frac{u-v}{|u|+|v|+\epsilon} \right|^q, \qquad q > 0$$

1. If u and v have opposite sign then $R(u,v) \approx 0$.
2. If u, v are close to one another, then $R(u,v) \approx 1$.

SLIP scheme

Define the limited average value

$$L(u,v) = R(u,v)\left(\frac{u+v}{2}\right)$$

At mesh face $x_{j+\frac{1}{2}}$, define left and right states

$$U_L = U_j + \frac{1}{2}L(\Delta U_{j+\frac{3}{2}}, \Delta U_{j-\frac{1}{2}}), \qquad U_R = U_{j+1} - \frac{1}{2}L(\Delta U_{j+\frac{3}{2}}, \Delta U_{j-\frac{1}{2}})$$

Dissipation flux is defined as

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}}(U_R - U_L), \qquad \alpha_{j+\frac{1}{2}} = \kappa^{(2)}\lambda_{j+\frac{1}{2}}$$

Now

$$U_R - U_L = \Delta U_{j+\frac{1}{2}} - L(\Delta U_{j+\frac{3}{2}}, \Delta U_{j-\frac{1}{2}})$$

SLIP scheme

Near a shock, we get second order dissipation

$$U_R - U_L = \Delta U_{j+\frac{1}{2}}, \qquad d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}}(U_{j+1} - U_j)$$

In smooth regions, we get fourth order dissipation

$$\begin{aligned} U_R - U_L &\approx \Delta U_{j+\frac{1}{2}} - \frac{1}{2}(\Delta U_{j+\frac{3}{2}} + \Delta U_{j-\frac{1}{2}}) \\ &= -\frac{1}{2}(U_{j+2} - 3U_{j+1} + 3U_j - U_{j-1}) \\ &\approx -\frac{1}{2}h^3\frac{\partial^3 U}{\partial x^3} \end{aligned}$$

CUSP scheme

Inspired by AUSM

$$F = uU_c + F_p = u \begin{bmatrix} \rho \\ \rho u \\ \rho H \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

Numerical flux

$$F_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} \frac{U_{c_j} + U_{c_{j+1}}}{2} - d_{c_{j+\frac{1}{2}}} + \frac{F_{p_j} + F_{p_{j+1}}}{2} - d_{p_{j+\frac{1}{2}}}$$

Upwinding is achieved by choosing

$$d_{c_{j+\frac{1}{2}}} = \frac{1}{2}|u_{j+\frac{1}{2}}|\Delta U_{c_{j+\frac{1}{2}}} = \frac{1}{2}|M_{j+\frac{1}{2}}|c_{j+\frac{1}{2}}\Delta U_{c_{j+\frac{1}{2}}}$$

$$d_{p_{j+\frac{1}{2}}} = \frac{1}{2}\mathrm{sign}(M) \begin{bmatrix} 0 \\ \Delta p_{j+\frac{1}{2}} \\ 0 \end{bmatrix}$$

CUSP scheme

Full upwinding is unstable in subsonic regions where waves propagate in both directions. The dissipation is taken to be

$$d_{c_{j+\frac{1}{2}}} = \frac{1}{2} f_1(M) c_{j+\frac{1}{2}} \Delta U_{c_{j+\frac{1}{2}}}, \qquad d_{p_{j+\frac{1}{2}}} = \frac{1}{2} f_2(M) \begin{bmatrix} 0 \\ \Delta p_{j+\frac{1}{2}} \\ 0 \end{bmatrix}$$

We need full upwinding in supersonic regions

$$f_1(M) = |M|, \qquad f_2(M) = \operatorname{sign}(M), \qquad |M| > 1$$

$$f_1(M) = \begin{cases} a_0 + a_2 M^2 + a_4 M^4 & |M| < 1 \\ |M| & |M| \geq 1 \end{cases}$$

$$f_1(\pm 1) = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}M} f_1(\pm 1) = \pm 1$$

$$a_2 = \frac{3}{2} - 2a_0, \qquad a_4 = a_0 - \frac{1}{2}$$

CUSP scheme

Only one free parameter a_0 which is usually taken to be $a_0 = \frac{1}{4}$.

$$f_2(M) = \begin{cases} \frac{1}{2}M(3 - M^2) & |M| < 1 \\ \mathrm{sign}(M) & |M| \geq 1 \end{cases}$$

Remark: There are other versions of CUSP scheme, see the references.

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