# Linear hyperbolic conservation laws

Praveen. C
praveen@math.tifrbng.res.in

Tata Institute of Fundamental Research
Center for Applicable Mathematics
Bangalore 560065
http://math.tifrbng.res.in/~praveen

March 4, 2013

# Partial Differential Equations

- One space and one time: $u(x,t)$
  - Hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

  - Parabolic equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

  - Convection-diffusion equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}$$

# Simplest hyperbolic PDE

- Linear, scalar, convection (advection) equation for $u(x,t)$

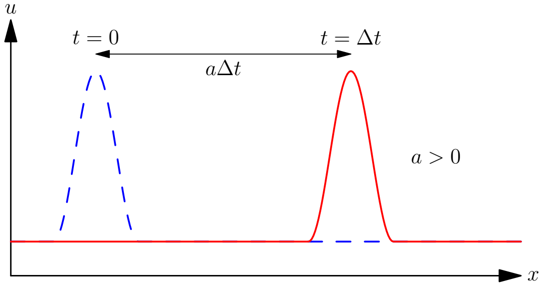
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}$$

with initial condition

$$u(x,0) = u_0(x)$$

- Exact solution

$$u(x,t) = u_0(x - at)$$




We can even put a discontinuous initial condition which is just transported at speed $a$ by the PDE.

# Hyperbolic PDE

## Wave

A phenomenon in which some recognizable feature propagates with a recognizable speed

## Hyperbolic PDE

A PDE which has wave-like solutions

- Waves propagate in specific directions:
- Linear, convection equation
  - $a > 0 \implies$ wave moves to the right
  - $a < 0 \implies$ wave moves to the left
  - $a$ is the speed at which waves propagate
  - Finite speed of propagation
  - Preserves shape of initial condition
  - Preserves minimum and maximum value

# Hyperbolic PDE

- Scalar, convection equation

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right) u = 0$$

  contains one wave
- Second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

  - can be factored

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right) u = 0$$

  - contains two waves, with speed $+a$ and $-a$
  - In fact, general solution is

$$u(x,t) = f(x - at) + g(x + at)$$

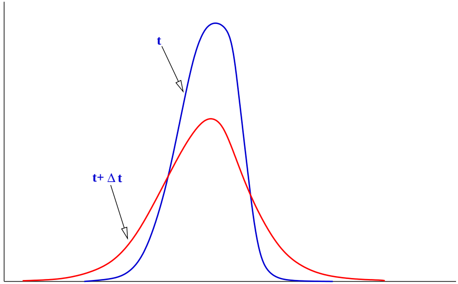
# Parabolic PDE

- Example: Heat equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}$$

with initial condition

$$u(x,0) = u_0(x)$$



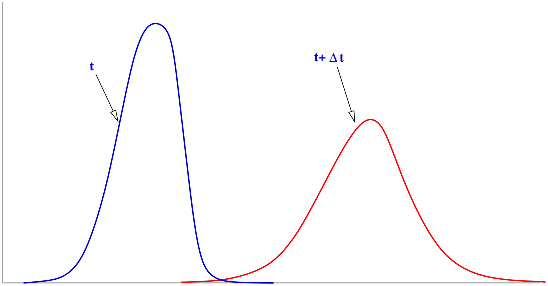

- No waves; initial condition is damped or dissipated

# Convection-diffusion PDE

- Convection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

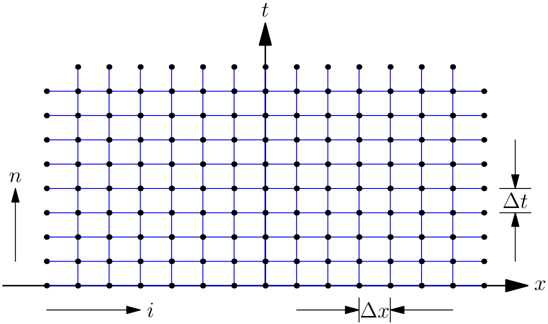
contains convection and diffusion




- Damped wave-like solutions

# FDM for $u_t + au_x = 0$

- Given $u(x, 0) = u_0(x)$, find solution for $t > 0$: Initial Value Problem
- Space-time grid




- Numerical solution $u_i^n$

$$u_i^n \approx u(x_i, t^n)$$

Numerical solution computed only at grid points

# FDM for $u_t + au_x = 0$

- Forward difference in time

$$\frac{\partial}{\partial t}u(x_i, t^n) \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

- Three choices for $\frac{\partial}{\partial x}$
  1. Backward difference

$$\frac{\partial}{\partial x}u(x_i, t^n) \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

  2. Forward difference

$$\frac{\partial}{\partial x}u(x_i, t^n) \approx \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

  3. Central difference

$$\frac{\partial}{\partial x}u(x_i, t^n) \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

# FDM for $u_t + au_x = 0$

- Forward-time and backward-space finite difference scheme

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

approximated as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

- Re-arranging

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

- Given initial condition $u_i^0$ for all $i$, we march forward in time

# FDM for $u_t + au_x = 0$

- Three numerical schemes
  1. Backward difference: First order accurate
  $$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n) = (1-\sigma)u_i^n + \sigma u_{i-1}^n$$
  2. Forward difference: First order accurate
  $$u_i^{n+1} = u_i^n - \sigma(u_{i+1}^n - u_i^n) = (1+\sigma)u_i^n - \sigma u_{i+1}^n$$
  3. Central difference: Second order accurate in space, first order in time
  $$u_i^{n+1} = u_i^n - \frac{1}{2}\sigma(u_{i+1}^n - u_{i-1}^n)$$
- Courant-Friedrich-Levy number or CFL number
$$\sigma = \frac{a\Delta t}{\Delta x}$$

# Lax-Wendroff scheme

Taylor's formula

$$u(x_j, t^{n+1}) = u(x_j, t^n) + \Delta t u_t(x_j, t^n) + \frac{1}{2}\Delta t^2 u_{tt}(x_j, t^n) + \mathcal{O}\left(\Delta t^3\right)$$

Use the PDE

$$u_t = -a u_x \qquad u_{tt} = a^2 u_{xx}$$

to get

$$u(x_j, t^{n+1}) = u(x_j, t^n) - a\Delta t u_x(x_j, t^n) + \frac{1}{2}a^2\Delta t^2 u_{xx}(x_j, t^n) + \mathcal{O}\left(\Delta t^3\right)$$

Approximate $u_x$ and $u_{xx}$ by central differences

$$u_j^{n+1} = u_j^n - a\Delta t \frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{1}{2}a^2\Delta t^2 \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + \mathcal{O}\left(\Delta t^3\right)$$

LW scheme

$$u_j^{n+1} = u_j^n - \frac{1}{2}\sigma(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}\sigma^2(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

This scheme is second order accurate in space and time.

# Leapfrog and Lax-Friedrich scheme

- Forward time, central space (FTCS): unstable scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

- Lax-Friedrich (LxF) scheme

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

$$u_j^{n+1} = \frac{1}{2}(1+\sigma)u_{j-1}^n + \frac{1}{2}(1-\sigma)u_{j+1}^n$$

- Leapfrog scheme

$$\frac{u_j^{n+1} - u_i^{n-1}}{2\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

$$u_i^{n+1} = u_i^{n-1} + \sigma(u_{i+1}^n - u_{i-1}^n)$$

# Fourier Stability Analysis

Take $k$'th Fourier mode, i.e., mode with wavenumber $k$

$$u_j^n = \hat{u}_k^n \mathrm{e}^{\mathrm{i}kx_j}$$

How does the scheme change this mode ? FTBS scheme:

$$u_j^{n+1} = (1-\sigma)u_j^n + \sigma u_{j-1}^n$$

$$\hat{u}_k^{n+1}\mathrm{e}^{\mathrm{i}kx_j} = (1-\sigma)\hat{u}_k^n\mathrm{e}^{\mathrm{i}kx_j} + \sigma\hat{u}_k^n\mathrm{e}^{\mathrm{i}k(x_j-h)}$$

Amplitude changes as

$$\hat{u}_k^{n+1} = \hat{u}_k^n[1-\sigma+\sigma\mathrm{e}^{-\mathrm{i}\xi}], \qquad \xi = kh$$

For stability, the amplitude must not increase with time

$$\left|\frac{\hat{u}_k^{n+1}}{\hat{u}_k^n}\right| = \left|1-\sigma+\sigma\mathrm{e}^{-\mathrm{i}\xi}\right| \le 1 \qquad \forall\ \xi$$

If $a > 0$ (i.e., $\sigma > 0$) then above condition is satisied iff $\boxed{0 \le \sigma \le 1}$.

# Fourier Stability Analysis

| $a$ | FTBS | FTFS | FTCS | LW | LF |
|---|---|---|---|---|---|
| $> 0$ | Stable<br>$0 \leq \sigma \leq 1$ | Unstable | Unstable | Stable<br>$0 \leq \sigma \leq 1$ | Stable<br>$0 \leq \sigma \leq 1$ |
| $< 0$ | Unstable | Stable<br>$-1 \leq \sigma \leq 0$ | Unstable | Stable<br>$-1 \leq \sigma \leq 0$ | Stable<br>$-1 \leq \sigma \leq 0$ |
| | Unstable | Unstable | Unstable | Stable<br>$\|\sigma\| \leq 1$ | Stable<br>$\|\sigma\| \leq 1$ |

**Upwind scheme**: Switch between backward difference and forward difference depending on whether $a > 0$ or $a < 0$.

### Hyperbolic problems

Finite difference scheme must be chosen based on the sign/direction of waves present in the problem

# Fourier Stability Analysis

**Remark**: Stable central schemes can be constructed. For example, the implicit Euler scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = 0$$

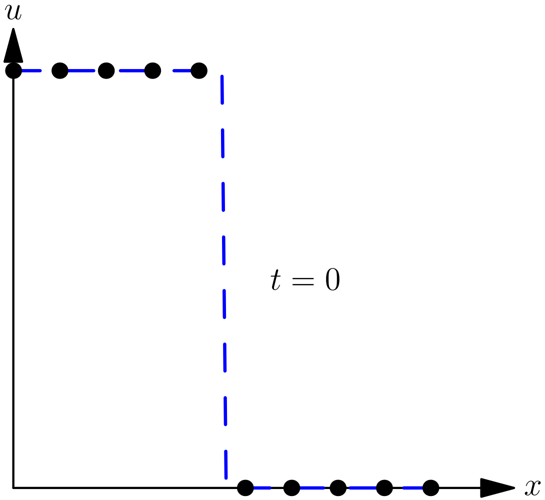
is unconditionally Fourier stable. The semi-discrete scheme

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + a\frac{u_{j+1} - u_{j-1}}{2h} = 0$$

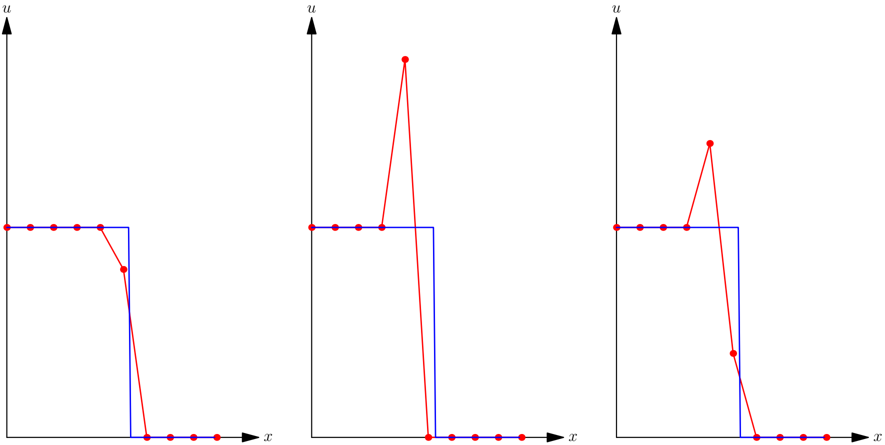
can be integrated with RK4 scheme in which case it is conditionally stable (under a CFL condition).

# FDM for $u_t + au_x = 0$

- Consider the case $a > 0$, $\sigma = 0.8$
- Initial condition with a step



# FDM for $u_t + au_x = 0$

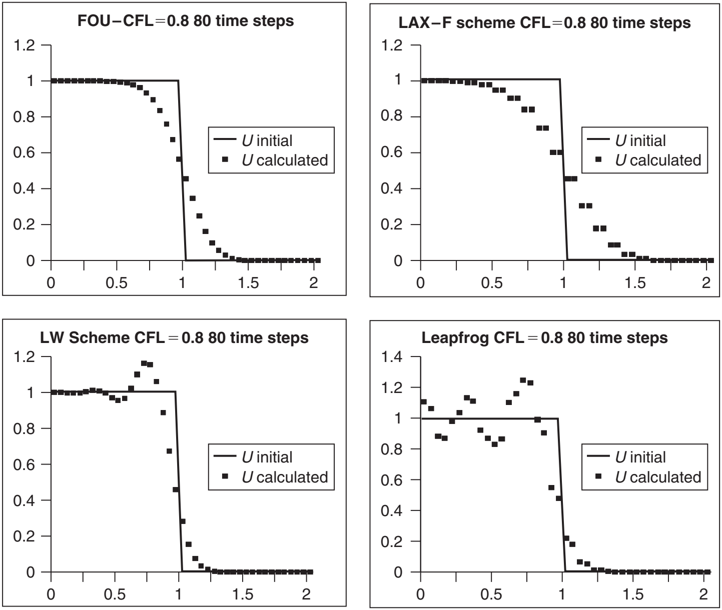



Backward
Monotone

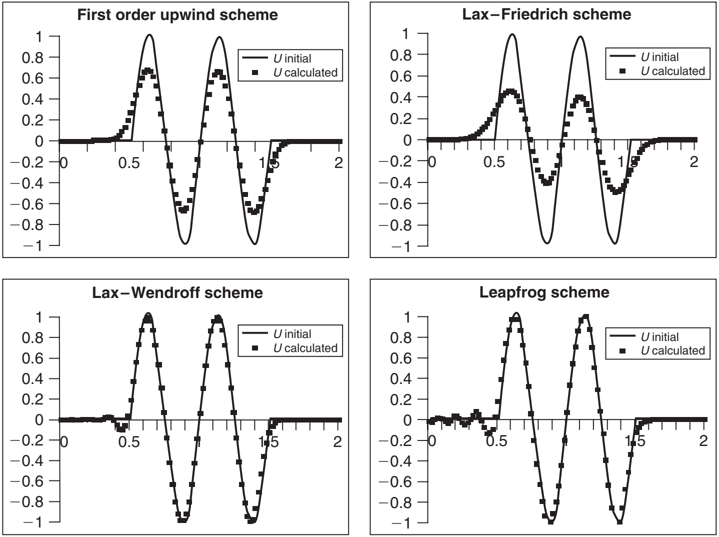
Forward
Non-monotone

Central
Non-monotone

# FDM for $u_t + au_x = 0$



# FDM for $u_t + au_x = 0$

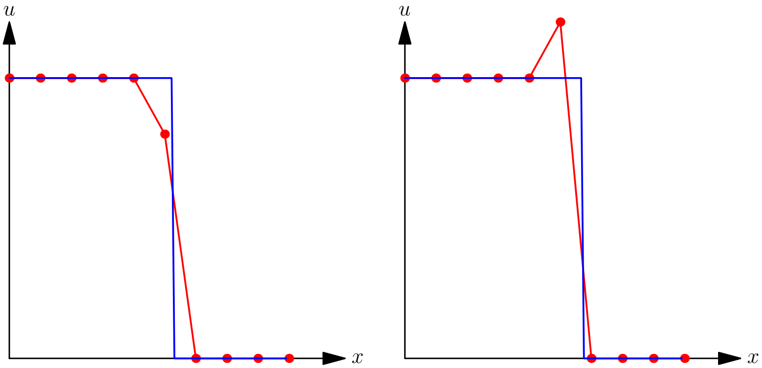


# FDM for $u_t + au_x = 0$

Monotone property

| $a$ | FTBS | FTFS | FTCS | LW | LxF |
|---|---|---|---|---|---|
| $a > 0$ | Yes | No | No | No | Yes |
| $a < 0$ | No | Yes | No | No | Yes |

# FDM for $u_t + au_x = 0$: Backward difference




$\nu = 0.8$ $\nu = 1.2$

- Scheme is monotone or maximum stable only if $\sigma \leq 1$
- This is called CFL condition
- Restriction on time step: **conditional stability**

$$\Delta t \leq \frac{\Delta x}{|a|} = \frac{\text{mesh size}}{\text{wave speed}}, \qquad a\Delta t \leq \Delta x$$

# Positive scheme

Semi-discrete scheme

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_j a_{ij}(u_j - u_i)$$

**Local extremum diminishing** (LED) if

$$a_{ij} \geq 0$$

Maxima do not increase and minima do not decrease. Suppose $u_i$ is a local maximum, i.e., $u_i \geq u_j$

$$u_j - u_i \leq 0 \quad \Longrightarrow \quad \frac{\mathrm{d}u_i}{\mathrm{d}t} \leq 0 \quad \Longrightarrow \quad u_i \text{ will not increase}$$

# Positive scheme

Fully discrete scheme: forward Euler scheme in time

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \sum_j a_{ij}(u_j^n - u_j^n)$$

$$\begin{aligned} u_i^{n+1} &= (1 - \Delta t \sum_j a_{ij}) u_i^n + \Delta t \sum_j a_{ij} u_j^n \\ &= \alpha_{ii} u_i^n + \sum_j \alpha_{ij} u_j^n \end{aligned}$$

If CFL condition is satisfied

$$\Delta t \le \frac{1}{\sum_j a_{ij}}, \qquad \alpha_{ii} \ge 0$$

# Positive scheme

then all coefficients are positive and sum to one

$$\alpha_{ii} + \sum_j \alpha_{ij} = 1$$

Hence

$$\min_j u_j^n \le u_i^{n+1} \le \max_j u_j^n$$

Solution remains bounded between minimum and maximum values.

This is known as *maximum stability* or *stability in maximum norm*. This is a more stronger condition than Fourier stability.

# Upwind scheme for $u_t + au_x = 0$

Write

$$a = a^+ + a^-, \qquad a^\pm = \frac{a \pm |a|}{2}, \qquad a^+ \geq 0, \qquad a^- \leq 0$$

CIR (Courant-Isaacson-Rees) splitting

$$u_t + a^+ u_x + a^- u_x = 0$$

Semi-discrete scheme: automatic switching b/w backward and forward difference

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} + a^+ \frac{u_j - u_{j-1}}{h} + a^- \frac{u_{j+1} - u_j}{h} = 0$$

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = a^+ \frac{u_{j-1} - u_j}{h} + (-a^-) \frac{u_{j+1} - u_j}{h}, \quad \text{Positive coefficients}$$

# Upwind scheme for $u_t + au_x = 0$

Fully discrete scheme (forward Euler time integration)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a^+ \frac{u_{j-1}^n - u_j^n}{h} + (-a^-) \frac{u_{j+1}^n - u_j^n}{h}$$

$$u_j^{n+1} = \sigma^+ u_{j-1}^n + (1 - |\sigma|) u_j^n + (-\sigma^-) u_{j+1}^n, \qquad \sigma^\pm = \frac{a^\pm \Delta t}{h}$$

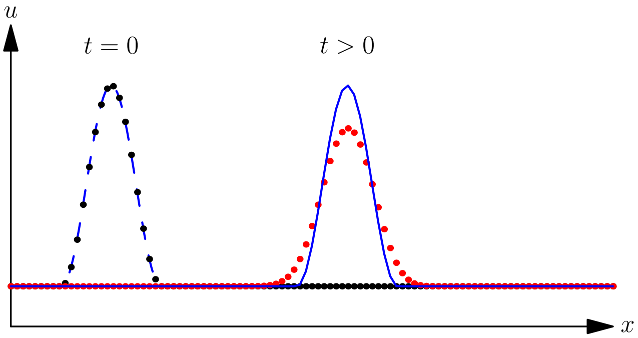
All coefficients are positive if CFL condition is satisfied

$$|\sigma| = \frac{|a|\Delta t}{h} \le 1$$

Upwind scheme is also Fourier stable under same CFL condition.

FTCS and Lax-Wendroff schemes are not positive. Lax-Friedrich scheme is positive under CFL condition $|\sigma| \le 1$.

# FDM for $u_t + au_x = 0$: Backward difference




- Numerical solution behaves like solution of convection-diffusion equation
- Numerical scheme has artificial dissipation or numerical dissipation
- Numerical dissipation $\Longrightarrow$ stable scheme
  But we must not have too much numerical dissipation

# Second order upwind scheme $u_t + au_x = 0$, $a > 0$

Since $a > 0$, we should use back ward difference scheme. We can construct second order accurate approximation to $u_x$ using upwind points $u_{i-2}, u_{i-2}, u_i$

$$\frac{u_{i-2} - 4u_{i-1} + 3u_i}{h} = \frac{\partial u}{\partial x}(x_i) + \mathcal{O}\left(h^2\right)$$

SOU scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i-2}^n - 4u_{i-1}^n + 3u_i^n}{h} = 0$$

or

$$u_i^{n+1} = -\sigma u_{i-2}^n + 4\sigma u_{i-1}^n + (1 - 3\sigma)u_i^n, \qquad \sigma = \frac{a\Delta t}{\Delta x} > 0$$

This scheme is not positive. It is Fourier stable under CFL condition $0 \le \sigma \le 1$.

# FDM for Parabolic equation

- Parabolic PDE

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$$

- No waves $\Longrightarrow$ no directional dependance
  Hence use central differencing for spatial derivatives

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

  or re-arranging

$$u_i^{n+1} = P u_{i-1}^n + (1 - 2P) u_i^n + P u_{i+1}^n$$

  with

$$P := \frac{\mu \Delta t}{\Delta x^2}$$

- Stability condition (same for Fourier and maximum stability)

$$P \leq \frac{1}{2} \qquad \Longrightarrow \qquad \Delta t \leq \frac{\Delta x^2}{2\mu}$$

# FDM for convection-diffusion equation

- Convection-diffusion equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \quad a > 0$$

- Combine appropriate scheme for hyperbolic and elliptic operators

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = \mu\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

Backward difference for convection term $a\frac{\partial u}{\partial x}$ (upwind scheme)
Central difference for diffusion term $\mu\frac{\partial^2 u}{\partial x^2}$

- Exercise: Find condition for this scheme to be LED

# Consistency and accuracy

- FTBS for $u_t + au_x = 0$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Plug in exact solution $u(x,t)$

$$\frac{u(x_i, t^n + \Delta t) - u(x_i, t^n)}{\Delta t} + a\frac{u(x_i, t^n) - u(x_i - \Delta x, t^n)}{\Delta x} = \tau_i^n$$

- $\tau_i^n$ = local truncation error
- Numerical scheme consistent with PDE if

$$\tau_i^n \to 0, \quad \text{as} \quad \Delta x \to 0, \quad \Delta t \to 0$$

# Consistency and accuracy

- Upwind scheme: truncation error

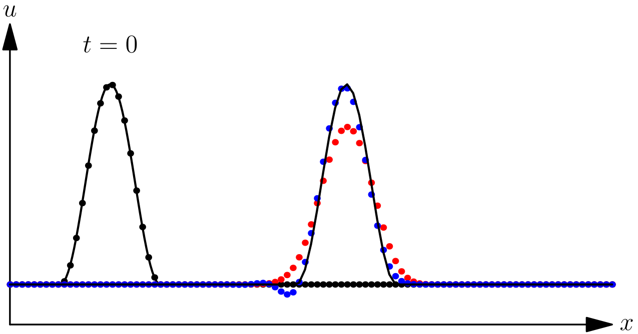
$$\tau_i^n = \frac{1}{2}|a|\Delta x(1-|\sigma|)\frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)$$

We say that this scheme is first order accurate

- For a second order accurate scheme

$$\tau_i^n = O(\Delta x^2)$$

Higher order accurate scheme $\Longrightarrow$ more accurate solution



# Convergence

Does the numerical solution converge to the exact solution as the grid is refined ?

$$\Delta x \to 0, \quad \Delta t \to 0 \quad \Longrightarrow \quad u_i^n \to u(x_i, t^n)$$

**Lax-Richtmyer Equivalence theorem**

A consistent finite difference scheme for a PDE for which the initial value problem is well-posed is convergent if and only if it is stable

$$\text{consistency} + \text{stability} = \text{convergence}$$