Non-linear hyperbolic conservation laws

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Non-linear hyperbolic PDE

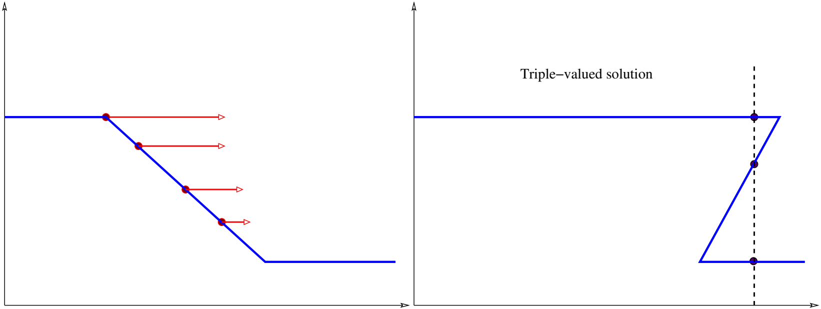
- Linear convection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- Non-linear convection (Burger) equation

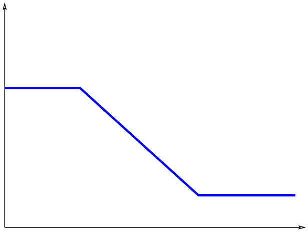
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

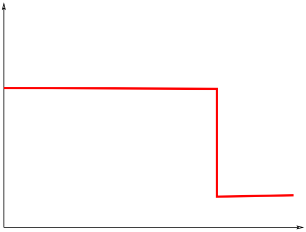
Convection speed depends on solution u



Non-linear hyperbolic PDE

- Even if initial condition is very smooth, solution can become discontinuous at some time. This is called a shock





  After shock is formed, it propagates at a speed given by Rankine-Hugoniot (RH) condition.

- Not differentiable $\Longrightarrow$ does not satisfy PDE
  Notion of weak solution[1]
- Discontinuous solutions occur in many physical models: Compressible flow of gases

[1] S. Kesavan: Topics in Functional Analysis and Applications

Scalar conservation laws

General form

$$u_t + f(u)_x = 0$$

Integrate over any spatial domain $[a, b]$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x &= \int_a^b u_t \mathrm{d}x = -\int_a^b f_x \mathrm{d}x \\ &= f(u(a,t)) - f(u(b,t)) \\ &= \text{[inflow at a] - [outflow at b]} \end{aligned}$$

u is neither created nor destroyed; the total amount of u contained inside any given interval $[a, b]$ can change only due to flux of u across the two end points.

- u is called a conserved quantity
- $f(u)$ is its flux
- Conservation principle is more fundamental; does not require derivatives.
- Under smoothness assumption, we get the PDE

Burgers equation

Famous non-linear hyperbolic PDE

$$u_t + uu_x = 0$$

or in conservation form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

The flux function

$$f(u) = \frac{u^2}{2}$$

is smooth and convex.

Remark: The two forms of the equation are mathematically equivalent only for smooth solutions. For discontinuous solutions, the conservation form must be used.

Rankine-Hugoniot condition

- A discontinuity in the solution is admissible provided it satisfies the RH condition

$$f(u_r) - f(u_l) = s \cdot (u_r - u_l)$$

  where s is the velocity with which the discontinuity moves.
- If the discontinuity is stationary $(s = 0)$ then $f(u_l) = f(u_r)$
- It is basically a consequence of conservation property.
- Burgers equation: $f(u) = \frac{u^2}{2}$

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{\frac{u_r^2}{2} - \frac{u_l^2}{2}}{u_r - u_l} = \frac{1}{2}(u_l + u_r)$$

- Weak solution: piecewise smooth solution which satisfies RH condition at shocks

Method of characteristics

$$u_t + f_x = 0 \quad \Longrightarrow \quad u_t + a(u)u_x = 0, \qquad a(u) = f'(u)$$

Characteristic curve

$$x = x(t) \qquad \text{such that} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = a(u(x,t))$$

Along the characteristic curve

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0$$

Solution is constant along each characteristic curve. Since the slope of the curve depends on u only, the characteristics are straight lines.

Riemann problem

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

with initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$$

This has a self-similar solution

$$u(x,t) = w_R\left(\frac{x}{t}; u_l, u_r\right)$$

Riemann problem for Burgers equation: Shock

Initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Solution

$$u(x,t) = \begin{cases} 1 & x < st \\ 0 & x > st \end{cases} \qquad s = \frac{1}{2}(u_l + u_r) = \frac{1}{2}$$

Riemann problem for Burgers equation: Rarefaction

Initial condition

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Solution

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x < t \\ 1 & x > t \end{cases}$$

This solution is continuous but not differentiable everywhere.

Riemann problem for Burgers equation: Entropy-violating solution

Initial condition

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Solution

$$u(x,t) = \begin{cases} 0 & x < st \\ 1 & x > st \end{cases} \qquad s = \frac{1}{2}(u_l + u_r) = \frac{1}{2}$$

There are an infinite number of entropy violating solutions for this problem.

Entropy condition

- The future is determined by the present, uniquely.
- Characteristic curves, when drawn forward in time, can intersect into the shock curve.
- But characteristics cannot emanate from the shock and go into the future.
- A discontinuity between u_l and u_r is physically admissible if

$$f'(u_l) \geq s \geq f'(u_r)$$

  where s is the shock speed given by the RH condition.
- For a convex flux like Burger's equation, entropy condition can be written as

$$u_l > u_r$$

- Among all possible weak solutions, there is only one solution which satisfies the entropy solution. (For systems, this is not known.)

Scalar conservation law

Given a smooth flux function $f : \mathbb{R} \to \mathbb{R}$ and a bounded initial condition u_0

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) &= 0, \quad x \in \mathbb{R}, \quad t > 0 \qquad (1) \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R} \qquad (2) \end{aligned}$$

We also define

$$a(u) = f'(u)$$

which is the slope of the characteristics.

Finite volume method

Divide space domain into finite volumes

$$\Omega = \bigcup_j (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$$

Integrate conservation law over finite volume $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and time slab (t^n, t^{n+1})

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \mathrm{d}x\mathrm{d}t = 0$$

Cell average value

$$u_j(t) = \frac{1}{h_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t)\mathrm{d}x$$

gives conservation law (exact)

$$(u_j^{n+1} - u_j^n)h_j + \int_{t^n}^{t^{n+1}} [f(x_{j+\frac{1}{2}}, t) - f(x_{j-\frac{1}{2}}, t)]\mathrm{d}t = 0$$

Finite volume method

Approximate time integral of flux

$$\int_{t^n}^{t^{n+1}} f(x_{j+\frac{1}{2}}, t)\mathrm{d}t \approx f(x_{j+\frac{1}{2}}, t^n)\Delta t$$

leads to finite volume method

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n}{h_j} = 0$$

Cell average values are the unknowns in the finite volume method.

$$v_j^n \approx u_j(t^n) = \frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\mathrm{d}x$$

We do not deal with point values. However for smooth solutions

$$u_j(t) - u(x_j, t) = \mathcal{O}\left(h_j^2\right) \qquad \text{(Show this)}$$

Finite volume method

The finite volume solution is made of piecewise constant values

$$v(x,t) = v_j^n, \quad x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad t \in [t^n, t^{n+1})$$

At $x = x_{j+\frac{1}{2}}$ there are two values of v; it is not obvious how to approximate the flux $f(x_{j+\frac{1}{2}}, t^n)$. The simplest choice is to average

$$f(x_{j+\frac{1}{2}}, t^n) \approx \frac{1}{2}[f(v_j^n) + f(v_{j+1}^n)] \qquad \text{or} \qquad f(x_{j+\frac{1}{2}}, t^n) \approx f\left(\frac{v_j^n + v_{j+1}^n}{2}\right)$$

But this leads to a central difference type scheme, which does not respect the wave propagation property present in the hyperbolic problem. We will see how to construct good approximations to the flux in the form

$$f(x_{j+\frac{1}{2}}, t^n) \approx g(\ldots, v_j^n, v_{j+1}^n, \ldots) =: g_{j+\frac{1}{2}}^n$$

Finite volume method

where g is called the numerical flux function. In the simplest case

$$g_{j+\frac{1}{2}} = g(v_j, v_{j+1})$$

The finite volume method takes the form

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n}{h_j} = 0$$

More accurate flux integral (Trapezoidal rule)

$$\int_{t^n}^{t^{n+1}} f(x_{j+\frac{1}{2}}, t)\mathrm{d}t \approx \frac{1}{2}[g_{j+\frac{1}{2}}^n + g_{j+\frac{1}{2}}^{n+1}]\Delta t$$

or, mid-point integration rule

$$\int_{t^n}^{t^{n+1}} f(x_{j+\frac{1}{2}}, t)\mathrm{d}t \approx g(\ldots, v_{j-1}^{n+\frac{1}{2}}, v_j^{n+\frac{1}{2}}, v_{j+1}^{n+\frac{1}{2}}, \ldots)\Delta t$$

Finite volume method

where

$$v_j^{n+\frac{1}{2}} = \frac{1}{2}(v_j^n + v_j^{n+1})$$

These approximations lead to an implicit scheme which is second order accurate in time.

Remark: There are two approximations involved

- time integral through some quadrature
- numerical flux function

Method of lines approach

Integrate conservation law over one finite volume

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} \right) \mathrm{d}x = 0$$

Semi-discrete conservation law (exact)

$$h_j \frac{\mathrm{d}u_j}{\mathrm{d}t} + f(x_{j+\frac{1}{2}}, t) - f(x_{j-\frac{1}{2}}, t) = 0$$

Approximate flux with numerical flux function

$$h_j \frac{\mathrm{d}v_j}{\mathrm{d}t} + g_{j+\frac{1}{2}}(t) - g_{j-\frac{1}{2}}(t) = 0$$

$$g_{j+\frac{1}{2}}(t) = g(\ldots, v_j(t), v_{j+1}(t), \ldots)$$

System of ODE for the cell averages $(u_j(t))_j$; integrate in time using some ODE scheme like Runge-Kutta scheme. Explicit or implicit, high order accurate schemes can be constructed by this approach.

Writing previous schemes as FV schemes

$$u_t + au_x = 0$$

- Upwind scheme

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a^+ \frac{v_j^n - v_{j-1}^n}{\Delta x} + a^- \frac{v_{j+1}^n - v_j^n}{\Delta x} = 0$$

Numerical flux

$$\begin{aligned} g_{j+\frac{1}{2}} &= \frac{1}{2} a(v_j + v_{j+1}) - \frac{1}{2}|a|(v_{j+1} - v_j) \\ &= \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a|(v_{j+1} - v_j) \end{aligned}$$

Writing previous schemes as FV schemes

- Lax-Friedrichs

$$\frac{v_j^{n+1} - \frac{v_{j-1}^n + v_{j+1}^n}{2}}{\Delta t} + a\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = 0$$

Numerical flux

$$g_{j+\frac{1}{2}} = \frac{1}{2}a(v_j + v_{j+1}) - \frac{1}{2}\frac{\Delta x}{\Delta t}(v_{j+1} - v_j) = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2\lambda}(v_{j+1} - v_j)$$

- Lax-Wendroff

$$v_j^{n+1} = v_j^n - \frac{1}{2}a\lambda(v_{j+1}^n - v_{j-1}^n) + \frac{1}{2}a^2\lambda^2(v_{j-1}^n - 2v_j^n + v_{j+1}^n)$$

Numerical flux

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}\lambda a^2(v_{j+1} - v_j)$$

Writing previous schemes as FV schemes

All of these numerical fluxes have same structure

$$\begin{aligned} g_{j+\frac{1}{2}} &= \frac{1}{2}a(v_j + v_{j+1}) - \frac{1}{2}q(v_{j+1} - v_j) \\ &= \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}q(v_{j+1} - v_j) \\ &= \text{central flux} + \text{dissipative flux} \end{aligned}$$

$$q = \text{numerical viscosity coefficient}$$

	LW	Upwind	LxF
q	λa^2	$\|a\|$	$\frac{\Delta x}{\Delta t}$

Non-linear conservation law

$$u_t + f_x = 0$$

- Lax-Friedrichs

$$\frac{v_j^{n+1} - \frac{v_{j-1}^n + v_{j+1}^n}{2}}{\Delta t} + \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x} = 0$$

Numerical flux

$$\begin{aligned} g_{j+\frac{1}{2}} &= \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}\frac{\Delta x}{\Delta t}(v_{j+1} - v_j) \\ &= \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2\lambda}(v_{j+1} - v_j) \end{aligned}$$

Non-linear conservation law

- Lax-Wendroff

$$v_j^{n+1} = v_j^n - \frac{1}{2}\lambda(f_{j+1}^n - f_{j-1}^n) + \frac{1}{2}\lambda^2[a_{j+\frac{1}{2}}(f_{j+1}^n - f_j^n) - a_{j-\frac{1}{2}}(f_j^n - f_{j-1}^n)]$$

Numerical flux

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}\lambda a_{j+\frac{1}{2}}(f_{j+1} - f_j), \qquad a_{j+\frac{1}{2}} = f'\left(\frac{v_j + v_{j+1}}{2}\right)$$

- Ritchmeyer two-step Lax-Wendroff method

$$\begin{aligned} v_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{\lambda}{2}[f(v_{j+1}^n) - f(v_j^n)] \\ v_j^{n+1} &= v_j^n - \lambda[f(v_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - f(v_{j-\frac{1}{2}}^{n+\frac{1}{2}})] \end{aligned}$$

Non-linear conservation law

- MacCormack method

$$
\begin{aligned}
v_j^* &= v_j^n - \lambda[f(v_{j+1}^n) - f(v_j^n)] \\
v_j^{n+1} &= \frac{1}{2}(v_j^n + v_j^*) - \frac{\lambda}{2}[f(v_j^*) - f(v_{j-1}^*)]
\end{aligned}
$$

- Upwind scheme for $u_t + uu_x = 0$: naive generalization

$$
\frac{v_j^{n+1} - v_j^n}{\Delta t} + (v_j^n)^+ \frac{v_j^n - v_{j-1}^n}{\Delta x} + (v_j^n)^- \frac{v_{j+1}^n - v_j^n}{\Delta x} = 0
$$

Cannot be written as a finite volume scheme !!!

Example (Non-conservative scheme for Burgers equation)
Consider initial condition

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Then naive upwind scheme

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + v_j^n \frac{v_j^n - v_{j-1}^n}{\Delta x} = 0$$

with initial condition

$$v_j^0 = \begin{cases} 1 & j \leq 0 \\ 0 & j > 0 \end{cases} \qquad \Longrightarrow \qquad v_j^n = \begin{cases} 1 & j \leq 0 \\ 0 & j > 0 \end{cases}$$

which is the wrong weak solution. The correct solution has a shock moving with speed $s = \frac{1}{2}$.

Example: (Non-conservative scheme for Burgers equation)
Consider initial condition

$$u(x,0) = \begin{cases} 1.2 & x < 0 \\ 0.4 & x > 0 \end{cases}$$

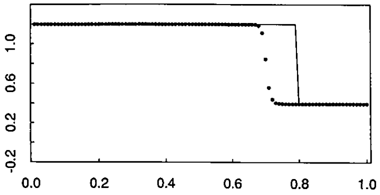


Figure 12.1. True and computed solutions to Burgers' equation using a nonconservative method.

Shock location is wrong !!!

Conservation property is extremely important to correctly compute discontinuous solutions.

Murman-Roe scheme

Upwind flux for $u_t + au_x = 0$

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a|(v_{j+1} - v_j)$$

Murman-Roe flux for $u_t + f_x = 0$

$$g_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a_{j+\frac{1}{2}}|(v_{j+1} - v_j)$$

where

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1}-f_j}{v_{j+1}-v_j} & v_j \neq v_{j+1} \\ a(v_j) = f'(v_j) & v_j = v_{j+1} \end{cases}$$

Upwind property

$$g_{j+\frac{1}{2}} = \begin{cases} f_j & a_{j+\frac{1}{2}} \geq 0 \\ f_{j+1} & a_{j+\frac{1}{2}} < 0 \end{cases}$$

Godunov scheme (Riemann solver)

Finite volume solution is piecewise constant. At each cell face $x_{j+\frac{1}{2}}$ and each time t^n there is a Riemann problem

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0$$

with initial condition

$$w(x,0) = \begin{cases} v_j^n & x < x_{j+\frac{1}{2}} \\ v_{j+1}^n & x > x_{j+\frac{1}{2}} \end{cases}$$

Solution

$$w(x,t) = w_R\left(\frac{x - x_{j+\frac{1}{2}}}{t - t^n}, v_j^n, v_{j+1}^n\right)$$

Godunov flux

$$g_{j+\frac{1}{2}}^G = g^G(v_j, v_{j+1}) = f(w_R(0; v_j, v_{j+1}))$$

Requires knowledge of exact solution of Riemann problem w_R.

Approximate Riemann Solver

At $x = x_{j+\frac{1}{2}}$ linearise the conservation law locally

$$\frac{\partial w}{\partial t} + a_{j+\frac{1}{2}} \frac{\partial w}{\partial x} = 0, \qquad a_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{v_{j+1} - v_j}$$

Solution of Riemann problem

$$w(x,t) = \begin{cases} v_j^n & (x - x_{j+\frac{1}{2}}) < a_{j+\frac{1}{2}}(t - t^n) \\ v_{j+1}^n & (x - x_{j+\frac{1}{2}}) > a_{j+\frac{1}{2}}(t - t^n) \end{cases}$$

Numerical flux

$$g^R_{j+\frac{1}{2}} = f(w(x_{j+\frac{1}{2}}, t)) = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}|a_{j+\frac{1}{2}}|(v_{j+1} - v_j)$$

This is the Murman-Roe flux. It does not satisfy entropy condition.

Approximate Riemann Solver

Take initial condition for Burgers equation

$$v_k = \begin{cases} -1 & k \le j \\ +1 & k > j \end{cases} \qquad\qquad a_{j+\frac{1}{2}} = 0$$

Numerical solution contains stationary shock which violates entropy condition. The solution should develop a rarefaction. The reason for this failure is that the numerical dissipation vanishes at the shock.

$$a_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{v_{j+1} - v_j} = \frac{\frac{1}{2}(+1)^2 - \frac{1}{2}(-1)^2}{(+1) - (-1)} = 0$$

To fix this Harten proposed to increase the numerical dissipation whenever it is likely to become zero. The flux with **entropy fix**

$$g^R_{j+\frac{1}{2}} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2} D_{j+\frac{1}{2}}(v_{j+1} - v_j)$$

Approximate Riemann Solver

where the numerical viscosity D is not allowed to vanish

$$D_{j+\frac{1}{2}} = \begin{cases} |a_{j+\frac{1}{2}}| & |a_{j+\frac{1}{2}}| \geq \delta \\ (\delta^2 + a^2_{j+\frac{1}{2}})/(2\delta) & |a_{j+\frac{1}{2}}| < \delta \end{cases}, \qquad \delta = 0.01 - 0.05$$