Euler with Gravity: Combining Well-Balancing with low Mach

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The Compressible Euler Equations with Gravity

The compressible Euler equations coupled to a gravitational source term are:

$$\begin{aligned} \rho_t + \nabla(\rho\mathbf{v}) &= 0 \\ (\rho\mathbf{v})_t + \nabla(p + \rho\mathbf{v}\otimes\mathbf{v}) &= -\rho\nabla\phi \\ E_t + \nabla(E\mathbf{v} + p\mathbf{v}) &= 0 \end{aligned}$$

▷ These equations are used to model inviscid compressible fluid dynamics.

▷ To close the problem, an *equation of state* (EOS) is added. An EOS is a relation of the form

$$p = p(\rho, T).$$

Hydrostatic Equilibria

To find a static solution of the problem, we set $\mathbf{v}, \mathbf{v}_t = 0$. Applying these conditions to the Euler equations above, we find $\rho_t = E_t = 0$ and the *hydrostatic equation*

$$\nabla p = -\rho\nabla\phi.$$

▷ Applying an EOS leads to an ordinary differential equation.

▷ A solution of this ODE is called *hydrostatic equilibrium*.

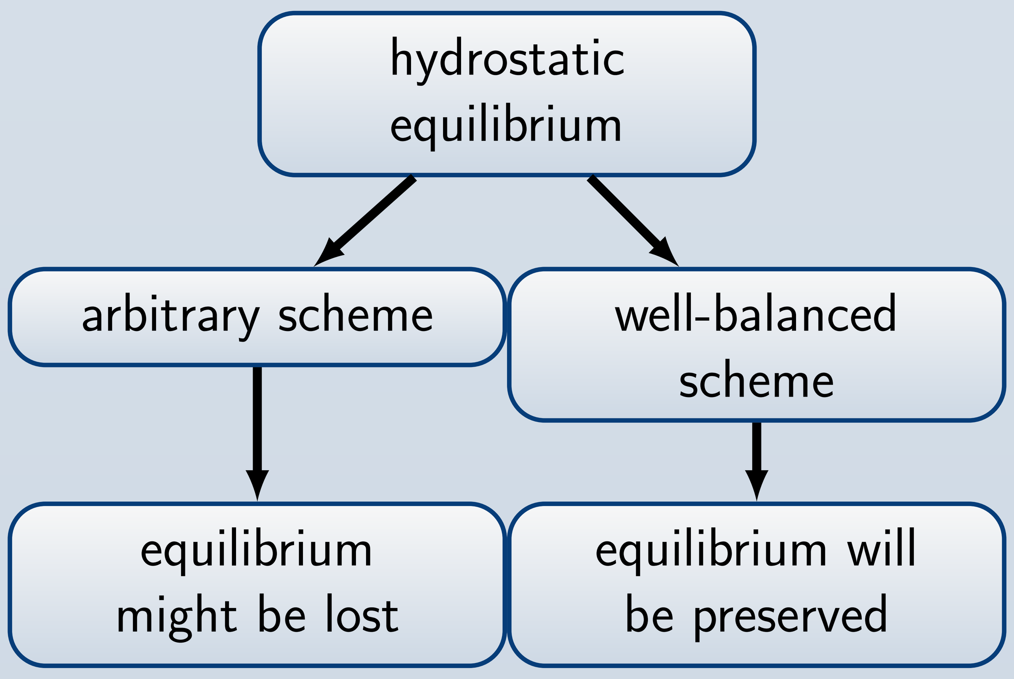
Application of a Finite Volume Scheme

▷ For one spatial dimension, a finite volume scheme would be a semi-discrete scheme of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}_i + \frac{\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}}{\Delta x} = \mathbf{S}_i.$$

▷ Any time integrator can be used to evolve these ODEs numerically.

▷ In general, a finite volume scheme is not able to preserve a hydrostatic equilibrium.



Our Well-Balanced Scheme

We use a second-order finite volume scheme developed by Praveen Chandrashekar (based on [1]). The scheme consists of three steps:

▷ *Define* functions α and β such that

$$\tilde{\rho} = \rho_0\,\alpha, \quad \tilde{p} = p_0\,\beta$$

for a hydrostatic equilibrium given by $\tilde{\rho}, \tilde{p}$.

▷ *Reconstruct* the vector

$$\mathbf{W} := \left[\rho/\alpha, \mathbf{v}^T, p/\beta\right]^T$$

using any consistent reconstruction

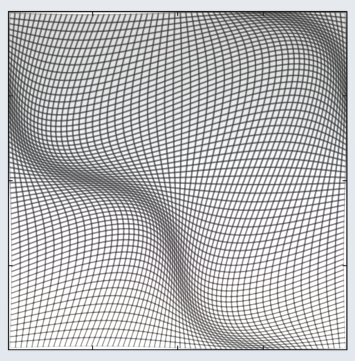
▷ *Discretize the source term* correctly, for one spatial dimension e.g.

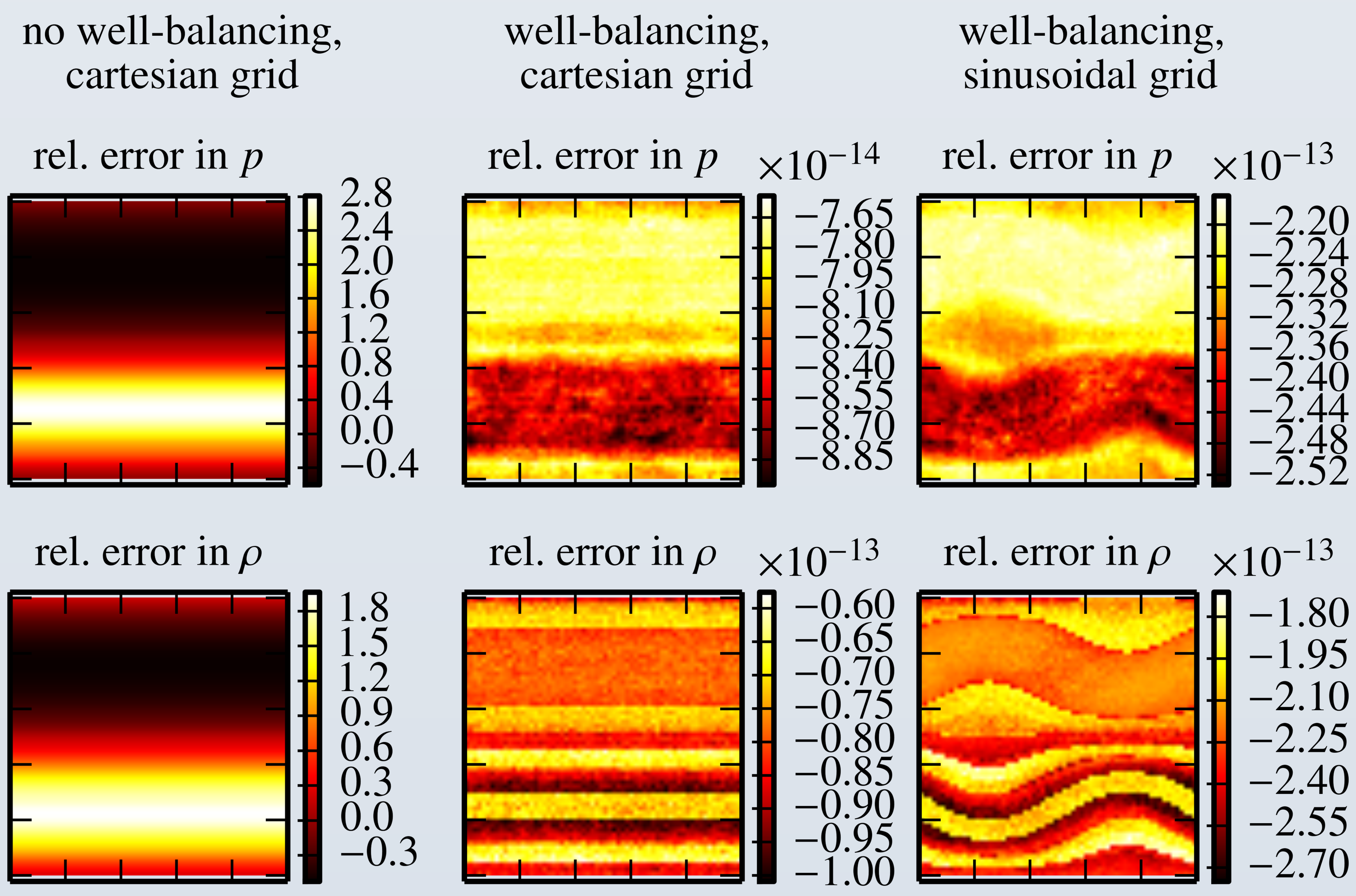
$$\mathbf{S}_i = \left[0, \frac{p_0}{\rho_0}\frac{\beta_{i+\frac{1}{2}} - \beta_{i-\frac{1}{2}}}{\Delta x}\frac{\rho_i}{\alpha_i}, 0\right]^T$$

This scheme satisfies the well-balanced property.

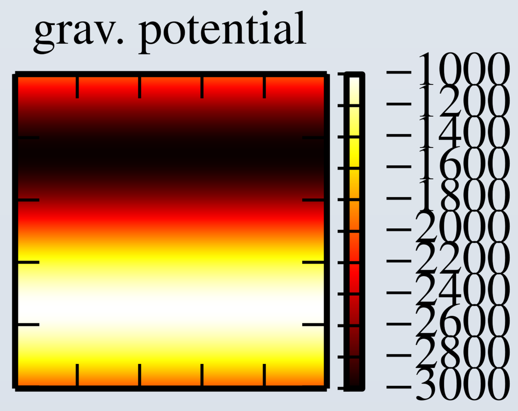
Numerical Tests

We use the *well-balanced scheme combined with a low Mach-solver* (Miczek-preconditioned Roe solver [2]). In one simulation we use a sinusoidal grid. *Right:* Structure of the sinusoidal grid (Source: [3]).





Left: Numerical simulation of a two-dimensional polytropic atmosphere over a long time.
Bottom: The gravitational Potential used for the simulations.



Advantages of the Scheme

▷ Applicable in *one, two or three spatial dimensions*.

▷ Adaptable for *any hydrostatic equilibrium* – even if it is not known analytically.

▷ Applicable on arbitrary curvilinear grids.

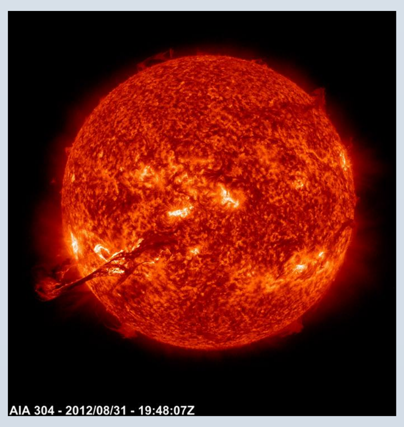
▷ Independent of the equation of state.

▷ Independent of the reconstruction scheme.

▷ Independent of the numerical flux function.
→ can be combined with a *low Mach* flux.

Astrophysical Application

▷ Stars are typically near a hydrostatic equilibrium.



▷ Dynamics such as convective mixing processes are orders of magnitude smaller than the background.

▷ This dynamics can only be resolved if the hydrostatic equilibrium is preserved near machine precision.

References

[1] P. Chandrashekar and C. Klingenberg, *A Second Order Well-Balanced Finite Volume Scheme for Euler Equations with Gravity*, SIAM Journal on Scientific Computing **37** (2015), no. 3, B382–B402.

[2] F. Miczek, F. K. Röpke, and P. V. F. Edelmann, *New numerical solver for flows at various Mach numbers*, Astronomy & Astrophysics **576** (2015).

[3] D. Zoar, Master Thesis, Universität Würzburg, 2016.

Acknowledgements

Simulations have been run using the Seven-League Hydro code on the bocksbeutel cluster of the Würzburg University.
The picture of the sun is taken from the NASA homepage www.nasa.gov.

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