Well-balanced schemes for Euler equations with gravity

Praveen. C
praveen@tifrbng.res.in

Center for Applicable Mathematics
Tata Institute of Fundamental Research
Bangalore 560065

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Euler equations with gravity

Flow properties

$$\rho = \text{density}, \qquad u = \text{velocity}$$

$$p = \text{pressure}, \qquad E = \text{total energy}$$

Gravitational potential ϕ; force per unit volume of fluid

$$-\rho \nabla \phi$$

System of conservation laws

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0 \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) &= -\rho \frac{\partial \phi}{\partial x} \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(E + p) u &= -\rho u \frac{\partial \phi}{\partial x}
\end{aligned}$$

Euler equations with gravity

Perfect gas assumption

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right], \qquad \gamma = \frac{c_p}{c_v} > 1$$

In compact notation

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = -\begin{bmatrix} 0 \\ \rho \\ \rho u \end{bmatrix} \frac{\partial \phi}{\partial x}$$

where

$$\boldsymbol{q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E + p)u \end{bmatrix}$$

Hydrostatic solutions

- Fluid at rest

$$u_e = 0$$

- Mass and energy equation satisfied
- Momentum equation

$$\frac{\mathrm{d}p_e}{\mathrm{d}x} = -\rho_e \frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{1}$$

- Need additional assumptions to solve this equation
- Assume ideal gas and some temperature profile $T_e(x)$

$$p_e(x) = \rho_e(x) R T_e(x), \qquad R = \text{gas constant}$$

integrate (1) to obtain

$$p_e(x) = p_0 \exp\left(- \int_{x_0}^{x} \frac{\phi'(s)}{R T_e(s)} \mathrm{d}s \right)$$

Hydrostatic solutions

- If the hydrostatic state is *isothermal*, i.e., $T_e(x) = T_e = \text{const}$, then

$$p_e(x) \exp\left(\frac{\phi(x)}{RT_e}\right) = \text{const} \tag{2}$$

  Density

$$\rho_e(x) = \frac{p_e(x)}{RT_e}$$

- If the hydrostatic solution is *polytropic* then we have following relations

$$p_e \rho_e^{-\nu} = \text{const}, \qquad p_e T_e^{-\frac{\nu}{\nu-1}} = \text{const}, \qquad \rho_e T_e^{-\frac{1}{\nu-1}} = \text{const} \tag{3}$$

  where $\nu > 1$ is some constant. From (1) and (3), we obtain

$$\frac{\nu R T_e(x)}{\nu - 1} + \phi(x) = \text{const} \tag{4}$$

  E.g., pressure is

$$p_e(x) = C_1 \left[C_2 - \phi(x)\right]^{\frac{\nu-1}{\nu}}$$

Existing schemes

- Isothermal case: Xing and Shu [2], well-balanced WENO scheme
- If $\nu = \gamma$ we are in isentropic case

$$h(x) + \phi(x) = \text{const}$$

  has been considered by Kappeli and Mishra [1].
- Desveaux et al: Relaxation schemes, general hydrostatic states

Well-balanced scheme

- Scheme is well-balanced if it exactly preserves hydrostatic solution.
- General evolutionary PDE

$$\frac{\partial \boldsymbol{q}}{\partial t} = R(\boldsymbol{q})$$

- Stationary solution $\boldsymbol{q}_e$

$$R(\boldsymbol{q}_e) = 0$$

- We are interested in computing small perturbations

$$\boldsymbol{q}(x,0) = \boldsymbol{q}_e(x) + \varepsilon \tilde{\boldsymbol{q}}(x,0), \qquad \varepsilon \ll 1$$

- Perturbations are governed by linear equation

$$\frac{\partial \tilde{\boldsymbol{q}}}{\partial t} = R'(\boldsymbol{q}_e)\tilde{\boldsymbol{q}}$$

Well-balanced scheme

- Some numerical scheme

$$\frac{\partial \boldsymbol{q}_h}{\partial t} = R_h(\boldsymbol{q}_h)$$

- $\boldsymbol{q}_{h,e} =$ interpolation of $\boldsymbol{q}_e$ onto the mesh
- Scheme is well balanced if

$$R_h(\boldsymbol{q}_{h,e}) = 0 \quad \Longrightarrow \quad \frac{\partial \boldsymbol{q}_h}{\partial t} = 0$$

- Suppose scheme is not well-balanced $R_h(\boldsymbol{q}_{h,e}) \neq 0$. Solution

$$\boldsymbol{q}_h(x,t) = \boldsymbol{q}_{h,e}(x) + \varepsilon \tilde{\boldsymbol{q}}_h(x,t)$$

Well-balanced scheme

- Linearize the scheme around $\boldsymbol{q}_{h,e}$

$$\frac{\partial}{\partial t}(\boldsymbol{q}_{h,e} + \varepsilon \tilde{\boldsymbol{q}}_h) = R_h(\boldsymbol{q}_{h,e} + \varepsilon \tilde{\boldsymbol{q}}_h) = R_h(\boldsymbol{q}_{h,e}) + \varepsilon R_h'(\boldsymbol{q}_{h,e})\tilde{\boldsymbol{q}}_h$$

or

$$\frac{\partial \tilde{\boldsymbol{q}}_h}{\partial t} = \frac{1}{\varepsilon} R_h(\boldsymbol{q}_{h,e}) + R_h'(\boldsymbol{q}_{h,e})\tilde{\boldsymbol{q}}_h$$

- Scheme is consistent of order r: $R_h(\boldsymbol{q}_{h,e}) = Ch^r \|\boldsymbol{q}_{h,e}\|$

$$\frac{\partial \tilde{\boldsymbol{q}}_h}{\partial t} = \frac{1}{\varepsilon} Ch^r \|\boldsymbol{q}_{h,e}\| + R_h'(\boldsymbol{q}_{h,e})\tilde{\boldsymbol{q}}_h$$

- $\varepsilon \ll 1$ then first term may dominate the second term; need $h \ll 1$
- Canonical approach

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(p + \rho u^2) = -\rho \frac{\partial \phi}{\partial x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho u)_i + \frac{1}{\Delta x}[\hat{\boldsymbol{f}}_{i+\frac{1}{2}} - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}] = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$$

Scope of present work

- Second order finite volume scheme
- Ideal gas model: well-balanced for both isothermal and polytropic solutions
- Most numerical fluxes can be used

Source term [2]

Define

$$\psi(x) = -\int_{x_0}^{x} \frac{\phi'(s)}{RT(s)} \mathrm{d}s, \qquad x_0 \text{ is arbitrary}$$

Then

$$\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} \int_{x_0}^{x} \frac{\phi'(s)}{RT(s)} \mathrm{d}s = -\frac{\phi'(x)}{RT(x)}$$

and

$$\frac{\partial}{\partial x} \exp(\psi(x)) = \exp(\psi(x)) \frac{\partial \psi}{\partial x} = -\exp(\psi(x)) \frac{\phi'(x)}{RT(x)}$$

so that

$$-\rho(x) \frac{\partial \phi}{\partial x} = p(x) \exp(-\psi(x)) \frac{\partial}{\partial x} \exp(\psi(x))$$

Euler equations

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} = \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix} \exp(-\psi(x)) \frac{\partial}{\partial x} \exp(\psi(x))$$

1-D finite volume scheme

- Divide domain into N finite volumes each of size Δx
- i'th cell $= (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$
- semi-discrete finite volume scheme for the i'th cell

$$\frac{\mathrm{d}\boldsymbol{q}_i}{\mathrm{d}t} + \frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2}} - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}}{\Delta x} = \mathrm{e}^{-\psi_i} \left(\frac{\mathrm{e}^{\psi_{i+\frac{1}{2}}} - \mathrm{e}^{\psi_{i-\frac{1}{2}}}}{\Delta x} \right) \begin{bmatrix} 0 \\ p_i \\ p_i u_i \end{bmatrix} \tag{5}$$

- ψ_i, $\psi_{i+\frac{1}{2}}$ etc. are consistent approximations to the function $\psi(x)$
- consistent numerical flux $\hat{\boldsymbol{f}}_{i+\frac{1}{2}} = \hat{\boldsymbol{f}}(\boldsymbol{q}^L_{i+\frac{1}{2}}, \boldsymbol{q}^R_{i+\frac{1}{2}})$

1-D finite volume scheme

Def: Property C

The numerical flux $\hat{\boldsymbol{f}}$ is said to satisfy Property C if for any two states

$$\boldsymbol{q}^L = [\rho^L, 0, p/(\gamma-1)] \quad \text{and} \quad \boldsymbol{q}^R = [\rho^R, 0, p/(\gamma-1)]$$

we have

$$\hat{\boldsymbol{f}}(\boldsymbol{q}^L, \boldsymbol{q}^R) = [0, p, 0]^\top$$

- states $\boldsymbol{q}^L$, $\boldsymbol{q}^R$ in the above definition correspond to a stationary contact discontinuity.
- Property C $\Longrightarrow$ numerical flux exactly support a stationary contact discontinuity.
- Examples of such numerical flux: Roe, HLLC

1-D finite volume scheme

- First order scheme

$$\boldsymbol{q}^L_{i+\frac{1}{2}} = \boldsymbol{q}_i, \qquad \boldsymbol{q}^R_{i+\frac{1}{2}} = \boldsymbol{q}_{i+1}$$

- Higher order scheme: To obtain the states $\boldsymbol{q}^L_{i+\frac{1}{2}}, \boldsymbol{q}^R_{i+\frac{1}{2}}$, reconstruct the following set of variables

$$\boldsymbol{w} = \left[\rho \mathrm{e}^{-\psi}, \ u, \ p\mathrm{e}^{-\psi}\right]^\top$$

- Once $\boldsymbol{w}^L_{i+\frac{1}{2}}$ etc. are computed, the primitive variables are obtained as

$$\rho^L_{i+\frac{1}{2}} = \mathrm{e}^{\psi_{i+\frac{1}{2}}}(w_1)^L_{i+\frac{1}{2}}, \quad u^L_{i+\frac{1}{2}} = (w_2)^L_{i+\frac{1}{2}}, \quad p^L_{i+\frac{1}{2}} = \mathrm{e}^{\psi_{i+\frac{1}{2}}}(w_3)^L_{i+\frac{1}{2}}, \quad \text{etc}$$

Well-balanced property

Theorem

The finite volume scheme (5) together with a numerical flux which satisfies property C and reconstruction of $\boldsymbol{w}$ variables is well-balanced in the sense that the initial condition given by

$$u_i = 0, \qquad p_i \exp(-\psi_i) = const, \qquad \forall\ i \tag{6}$$

is preserved by the numerical scheme.

__**Proof**__: Start computation with an initial condition that satisfies (6). Since we reconstruct the variables $\boldsymbol{w}$, at any interface $i + \frac{1}{2}$ we have

$$(w_2)^L_{i+\frac{1}{2}} = (w_2)^R_{i+\frac{1}{2}} = 0, \qquad (w_3)^L_{i+\frac{1}{2}} = (w_3)^R_{i+\frac{1}{2}}$$

Hence

$$u^L_{i+\frac{1}{2}} = u^R_{i+\frac{1}{2}} = 0, \qquad p^L_{i+\frac{1}{2}} = p^R_{i+\frac{1}{2}} = p_i \exp(\psi_{i+\frac{1}{2}} - \psi_i) =: p_{i+\frac{1}{2}}$$

Well-balanced property

and at $i - \frac{1}{2}$

$$u^L_{i-\frac{1}{2}} = u^R_{i-\frac{1}{2}} = 0, \qquad p^L_{i-\frac{1}{2}} = p^R_{i-\frac{1}{2}} = p_i \exp(\psi_{i-\frac{1}{2}} - \psi_i) =: p_{i-\frac{1}{2}}$$

Since the numerical flux satisfies property C, we have

$$\hat{\boldsymbol{f}}_{i-\frac{1}{2}} = [0, p_{i-\frac{1}{2}}, 0]^\top, \qquad \hat{\boldsymbol{f}}_{i+\frac{1}{2}} = [0, p_{i+\frac{1}{2}}, 0]^\top$$

Mass and energy equations are already well balanced, i.e.,

$$\frac{\mathrm{d}\boldsymbol{q}_i^{(1)}}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\boldsymbol{q}_i^{(3)}}{\mathrm{d}t} = 0$$

Momentum equation: on the left we have

$$\frac{\hat{\boldsymbol{f}}^{(2)}_{i+\frac{1}{2}} - \hat{\boldsymbol{f}}^{(2)}_{i-\frac{1}{2}}}{\Delta x} = \frac{p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}}{\Delta x}$$

Well-balanced property

while on the right

$$p_i \mathrm{e}^{-\psi_i} \frac{\mathrm{e}^{\psi_{i+\frac{1}{2}}} - \mathrm{e}^{\psi_{i-\frac{1}{2}}}}{\Delta x} = \frac{p_i \mathrm{e}^{\psi_{i+\frac{1}{2}} - \psi_i} - p_i \mathrm{e}^{\psi_{i-\frac{1}{2}} - \psi_i}}{\Delta x} = \frac{p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}}{\Delta x}$$

and hence

$$\frac{\mathrm{d}\boldsymbol{q}_i^{(2)}}{\mathrm{d}t} = 0$$

This proves that the initial condition is preserved under any time integration scheme. □

Approximation of source term

- How to approximate ψ_i, $\psi_{i+\frac{1}{2}}$, etc. ? Need some quadrature
- well-balanced property independent of quadrature rule to compute ψ.
- To preserve isothermal/polytropic solutions exactly, the quadrature rule has to be exact for these cases.
- To compute the source term in the i'th cell, we define the function $\psi(x)$ as follows

$$\psi(x) = -\int_{x_i}^{x} \frac{\phi'(s)}{RT(s)} \mathrm{d}s$$

where we chose the reference position as x_i.

Approximation of source term

- To approximate the integrals we define the piecewise constant temperature as follows

$$T(x) = \hat{T}_{i+\frac{1}{2}}, \qquad x_i < x < x_{i+1} \tag{7}$$

where $\hat{T}_{i+\frac{1}{2}}$ is the logarithmic average given by

$$\hat{T}_{i+\frac{1}{2}} = \frac{T_{i+1} - T_i}{\log T_{i+1} - \log T_i}$$

- The integrals are evaluated using the approximation of the temperature given in (7) leading to the following expressions for ψ.

$$\begin{aligned}
\psi_i &= 0 \\
\psi_{i-\frac{1}{2}} &= -\frac{1}{R\hat{T}_{i-\frac{1}{2}}} \int_{x_i}^{x_{i-\frac{1}{2}}} \phi'(s)\mathrm{d}s = \frac{\phi_i - \phi_{i-\frac{1}{2}}}{R\hat{T}_{i-\frac{1}{2}}} \\
\psi_{i+\frac{1}{2}} &= -\frac{1}{R\hat{T}_{i+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} \phi'(s)\mathrm{d}s = \frac{\phi_i - \phi_{i+\frac{1}{2}}}{R\hat{T}_{i+\frac{1}{2}}}
\end{aligned}$$

Approximation of source term

- Gravitational potential required at faces $\phi_{i+\frac{1}{2}}$
- ϕ is governed by Poisson equation and hence is a smooth function. We can interpolate

$$\phi_{i+\frac{1}{2}} = \frac{1}{2}(\phi_i + \phi_{i+1})$$

Sufficient to obtain second order accuracy. Then

$$\psi_{i-\frac{1}{2}} = \frac{\phi_i - \phi_{i-1}}{2R\hat{T}_{i-\frac{1}{2}}}, \qquad \psi_i = 0, \qquad \psi_{i+\frac{1}{2}} = \frac{\phi_i - \phi_{i+1}}{2R\hat{T}_{i+\frac{1}{2}}} \tag{8}$$

Approximation of source term

Theorem

The source term discretization given by (8) is second order accurate.

Proof: The source term in (5) has the factor

$$e^{-\psi_i} \frac{e^{\psi_{i+\frac{1}{2}}} - e^{\psi_{i-\frac{1}{2}}}}{\Delta x} = \frac{\exp\left(\frac{\phi_i - \phi_{i+1}}{2R\hat{T}_{i+\frac{1}{2}}}\right) - \exp\left(\frac{\phi_i - \phi_{i-1}}{2R\hat{T}_{i-\frac{1}{2}}}\right)}{\Delta x} \qquad \text{using } (8)$$

Using a Taylor expansion around x_i we get

$$\frac{1}{\hat{T}_{i-\frac{1}{2}}} = \frac{1}{T_i}[1 + O(\Delta x^2)], \qquad \frac{1}{\hat{T}_{i+\frac{1}{2}}} = \frac{1}{T_i}[1 + O(\Delta x^2)]$$

Approximation of source term

and

$$
\begin{aligned}
& \mathrm{e}^{\frac{\phi_i-\phi_{i+1}}{2R\hat{T}_{i+\frac{1}{2}}}} - \mathrm{e}^{\frac{\phi_i-\phi_{i-1}}{2R\hat{T}_{i-\frac{1}{2}}}} \\
= \; & \mathrm{e}^{\frac{1}{2RT_i}(-\phi_i'\Delta x - \phi_i''\Delta x^2 + O(\Delta x^3))} - \mathrm{e}^{\frac{1}{2RT_i}(+\phi_i'\Delta x - \phi_i''\Delta x^2 + O(\Delta x^3))} \\
= \; & \left[1 + \frac{1}{2RT_i}(-\phi_i'\Delta x - \phi_i''\Delta x^2) + \frac{1}{2(2RT_i)^2}(\phi_i'\Delta x)^2 + O(\Delta x^3)\right] \\
& - \left[1 + \frac{1}{2RT_i}(\phi_i'\Delta x - \phi_i''\Delta x^2) + \frac{1}{2(2RT_i)^2}(\phi_i'\Delta x)^2 + O(\Delta x^3)\right] \\
= \; & -\frac{1}{RT_i}\phi'(x_i)\Delta x + O(\Delta x^3)
\end{aligned}
$$

Hence the source term discretization is second order accurate. □

Theorem

Any hydrostatic solution which is isothermal or polytropic is exactly preserved by the finite volume scheme (5).

Proof: Take initial condition to be a hydrostatic solution. We have to verify that the initial condition satisfies equation (6).

Isothermal case: $\hat{T}_{i+\frac{1}{2}} = T_e = \mathrm{const}$, and using (2) we obtain

$$\frac{p_{i+1}\mathrm{e}^{-\psi_{i+1}}}{p_i\mathrm{e}^{-\psi_i}} = \frac{p_{i+1}}{p_i}\mathrm{e}^{\psi_i-\psi_{i+1}} = \frac{p_{i+1}}{p_i}\exp\left(\frac{\phi_{i+1}-\phi_i}{RT_e}\right) = \frac{p_{i+1}\exp(\phi_{i+1}/RT_e)}{p_i\exp(\phi_i/RT_e)}$$

Polytropic case:

$$\frac{p_{i+1}\mathrm{e}^{-\psi_{i+1}}}{p_i\mathrm{e}^{-\psi_i}} = \frac{p_{i+1}}{p_i}\mathrm{e}^{\psi_i-\psi_{i+1}} = \frac{p_{i+1}}{p_i}\exp\left(\frac{\phi_{i+1}-\phi_i}{R\hat{T}_{i+\frac{1}{2}}}\right)$$

But from (3), (4) we have

$$\frac{\phi_{i+1}-\phi_i}{R\hat{T}_{i+\frac{1}{2}}} = -\frac{\frac{\nu R}{\nu-1}(T_{i+1}-T_i)}{R\frac{T_{i+1}-T_i}{\log(T_{i+1})-\log(T_i)}} = \log\left(\frac{T_i}{T_{i+1}}\right)^{\frac{\nu}{\nu-1}}$$

and hence

$$\frac{p_{i+1}\mathrm{e}^{-\psi_{i+1}}}{p_i\mathrm{e}^{-\psi_i}} = \frac{p_{i+1}T_{i+1}^{-\nu/(\nu-1)}}{p_iT_i^{-\nu/(\nu-1)}} = 1$$

Hence in both cases, the initial condition is preserved by the finite volume scheme. □

Summary of the scheme

Using the approximations given by (8), the semi-discrete finite volume scheme is given by

$$\frac{\mathrm{d}\boldsymbol{q}_i}{\mathrm{d}t} + \frac{\hat{\boldsymbol{f}}_{i+\frac{1}{2}} - \hat{\boldsymbol{f}}_{i-\frac{1}{2}}}{\Delta x} = \frac{\mathrm{e}^{\hat{\beta}_{i+\frac{1}{2}}(\phi_i - \phi_{i+1})} - \mathrm{e}^{\hat{\beta}_{i-\frac{1}{2}}(\phi_i - \phi_{i-1})}}{\Delta x} \begin{bmatrix} 0 \\ p_i \\ p_i u_i \end{bmatrix}$$

where we have introduced the quantity

$$\hat{\beta}_{i+\frac{1}{2}} = \frac{1}{2R\hat{T}_{i+\frac{1}{2}}}$$

As an example of reconstruction, we discuss the minmod-type scheme for the interface $i + \frac{1}{2}$ which is given by

$$\boldsymbol{w}^L_{i+\frac{1}{2}} = \boldsymbol{w}_i + \frac{1}{2} m(\theta(\boldsymbol{w}_i - \boldsymbol{w}_{i-1}), (\boldsymbol{w}_{i+1} - \boldsymbol{w}_{i-1})/2, \theta(\boldsymbol{w}_{i+1} - \boldsymbol{w}_i))$$

Summary of the scheme

$$\boldsymbol{w}^R_{i+\frac{1}{2}} = \boldsymbol{w}_{i+1} - \frac{1}{2} m(\theta(\boldsymbol{w}_{i+1} - \boldsymbol{w}_i), (\boldsymbol{w}_{i+2} - \boldsymbol{w}_{i+1})/2, \theta(\boldsymbol{w}_{i+2} - \boldsymbol{w}_{i+1}))$$

where $\theta \in [1, 2]$ and $m(\cdot, \cdot, \cdot)$ is the minmod limiter function given by

$$m(a, b, c) = \begin{cases} s \min(|a|, |b|, |c|) & \text{if } s = \text{sign}(a) = \text{sign}(b) = \text{sign}(c) \\ 0 & \text{otherwise} \end{cases}$$

The variables $\boldsymbol{w}$ are defined using the potential relative to $x_{i+\frac{1}{2}}$

$$\psi(x) = -\int_{x_{i+\frac{1}{2}}}^{x} \frac{\phi'(s)}{RT(s)} \mathrm{d}s$$

Summary of the scheme

Then

$$\begin{aligned}
\psi_{i-1} &= \frac{\phi_i - \phi_{i-1}}{R\hat{T}_{i-\frac{1}{2}}} + \frac{\phi_{i+\frac{1}{2}} - \phi_i}{R\hat{T}_{i+\frac{1}{2}}} = 2\hat{\beta}_{i-\frac{1}{2}}(\phi_i - \phi_{i-1}) + \hat{\beta}_{i+\frac{1}{2}}(\phi_{i+1} - \phi_i) \\
\psi_i &= \frac{\phi_{i+\frac{1}{2}} - \phi_i}{R\hat{T}_{i+\frac{1}{2}}} = \hat{\beta}_{i+\frac{1}{2}}(\phi_{i+1} - \phi_i) \\
\psi_{i+1} &= -\frac{\phi_{i+1} - \phi_{i+\frac{1}{2}}}{R\hat{T}_{i+\frac{1}{2}}} = -\hat{\beta}_{i+\frac{1}{2}}(\phi_{i+1} - \phi_i) \\
\psi_{i+2} &= -\frac{\phi_{i+1} - \phi_{i+\frac{1}{2}}}{R\hat{T}_{i+\frac{1}{2}}} - \frac{\phi_{i+2} - \phi_{i+1}}{R\hat{T}_{i+\frac{3}{2}}} = -\hat{\beta}_{i+\frac{1}{2}}(\phi_{i+1} - \phi_i) - 2\hat{\beta}_{i+\frac{3}{2}}(\phi_{i+2}
\end{aligned}$$

In terms of the above ψ_i's, the variables $\boldsymbol{w}$ are defined as follows

$$\boldsymbol{w}_j = \begin{bmatrix} \rho_j \mathrm{e}^{-\psi_j} \\ u_j \\ p_j \mathrm{e}^{-\psi_j} \end{bmatrix}, \qquad j = i-1, i, i+1, i+2$$

Summary of the scheme

Since $\psi_{i+\frac{1}{2}} = 0$ we obtain the reconstructed values as

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}^L_{i+\frac{1}{2}} = \boldsymbol{w}^L_{i+\frac{1}{2}}, \qquad \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}^R_{i+\frac{1}{2}} = \boldsymbol{w}^R_{i+\frac{1}{2}}$$

For the first and last cells, we extrapolate the potential from inside the domain to the faces located on the domain boundary

$$\phi_{\frac{1}{2}} = \frac{3}{2}\phi_1 - \frac{1}{2}\phi_2, \qquad \phi_{N+\frac{1}{2}} = \frac{3}{2}\phi_N - \frac{1}{2}\phi_{N-1}$$

Isothermal examples: well-balanced test

Density and pressure are given by

$$\rho_e(x) = p_e(x) = \exp(-\phi(x))$$

$$N = 100, 1000, \qquad \text{final time} = 2$$

	Potential 1	Potential 2	Potential 3
$\phi(x)$	x	$\frac{1}{2}x^2$	$\sin(2\pi x)$

Table: Potential functions used for well-balanced tests

Isothermal examples: well-balanced test

Potential	Cells	Density	Velocity	Pressure
x	100	8.21676e-15	4.98682e-16	9.19209e-15
	1000	8.00369e-14	1.51719e-14	9.15152e-14
$\frac{1}{2}x^2$	100	1.01874e-14	2.49332e-16	1.06837e-14
	1000	1.05202e-13	4.10434e-16	1.11861e-13
$\sin(2\pi x)$	100	1.12466e-14	5.79978e-16	1.74966e-14
	1000	1.16191e-13	2.93729e-15	1.76361e-13

Table: Error in density, velocity and pressure for isothermal example

Isentropic examples: well-balanced test

Isentropic hydrostatic solution

$$T_e(x) = 1 - \frac{\gamma - 1}{\gamma}\phi(x), \qquad \rho_e = T_e^{\frac{1}{\gamma-1}}, \qquad p_e = \rho_e^\gamma$$

$$N = 100, 1000, \qquad \text{final time} = 2$$

Potential	Cells	Density	Velocity	Pressure
x	100	6.86395e-15	2.65535e-16	7.88869e-15
	1000	7.03820e-14	7.79350e-16	8.03623e-14
$\frac{1}{2}x^2$	100	1.06604e-14	2.27512e-16	1.04128e-14
	1000	1.10726e-13	1.15415e-15	1.09185e-13
$\sin(2\pi x)$	100	1.27570e-14	5.18212e-16	1.65185e-14
	1000	1.29020e-13	1.12837e-15	1.66566e-13

Table: Error in density, velocity and pressure for isentropic example

Polytropic examples: well-balanced test

Polytropic hydrostatic solutions

$$T_e(x) = 1 - \frac{\nu - 1}{\nu}\phi(x), \qquad \rho_e = T_e^{\frac{1}{\nu-1}}, \qquad p_e = \rho_e^{\nu}$$

$$\nu = 1.2, \qquad N = 100, 1000, \qquad \text{final time} = 2$$

Potential	Cells	Density	Velocity	Pressure
x	100	6.86395e-15	2.65535e-16	7.88869e-15
	1000	7.03820e-14	7.79350e-16	8.03623e-14
$\frac{1}{2}x^2$	100	1.06604e-14	2.27512e-16	1.04128e-14
	1000	1.10726e-13	1.15415e-15	1.09185e-13
$\sin(2\pi x)$	100	1.27570e-14	5.18212e-16	1.65185e-14
	1000	1.29020e-13	1.12837e-15	1.66566e-13

Table: Error in density, velocity and pressure for polytropic example

Non-isothermal example

- Stationary solution

$$\phi(x) = \frac{1}{2}x^2, \qquad \rho_e(x) = \exp(-x), \qquad p_e(x) = (1+x)\exp(-x)$$

- corresponds to a non-uniform temperature profile

$$T_e(x) = 1 + x$$

- Neither isothermal nor polytropic; present scheme will not be able to preserve the exact hydrostatic solution
- Instead, we construct an approximation to the above hydrostatic solution by numerically integrating the hydrostatic equations (1)

$$p_1 = p_e(x_1), \qquad \rho_1 = \frac{p_1}{RT_e(x_1)}$$

$$p_i = p_{i-1}\exp(-2\hat{\beta}_{i-\frac{1}{2}}(\phi_i - \phi_{i-1})), \qquad \rho_i = \frac{p_i}{RT_e(x_i)}, \qquad i = 2, 3, \ldots,$$

Non-isothermal example

- The above solution satisfies equation (6) and hence is preserved by the numerical scheme.
- Solution converges at second order; velocity is zero upto machine precision indicating that we have a stationary solution

Cells	ρ error	ρ rate	Velocity	p error	p rate
50	5.41510e-06	-	3.90665e-16	8.51248e-06	
100	1.37964e-06	1.97	1.06754e-15	2.16486e-06	1.97
200	3.48173e-07	1.98	4.82755e-16	5.45846e-07	1.98
400	8.74530e-08	1.99	1.94554e-15	1.37043e-07	1.99
800	2.19146e-08	1.99	2.62298e-15	3.43336e-08	1.99
1600	5.48521e-09	1.99	6.56911e-15	8.59273e-09	1.99

Table: Convergence of error for hydrostatic solution of section (34).

Evolution of small perturbations

The initial condition is taken to be the following

$$\phi = \frac{1}{2}x^2, \quad u = 0, \quad \rho(x) = \exp(-\phi(x))$$

Add small perturbation to equilibrium pressure

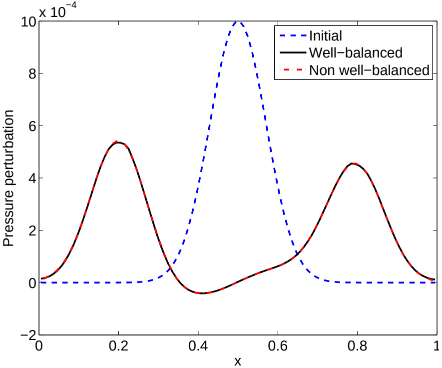
$$p(x) = \exp(-\phi(x)) + \varepsilon \exp(-100(x - 1/2)^2), \qquad 0 < \varepsilon \ll 1$$

Non-well-balanced scheme

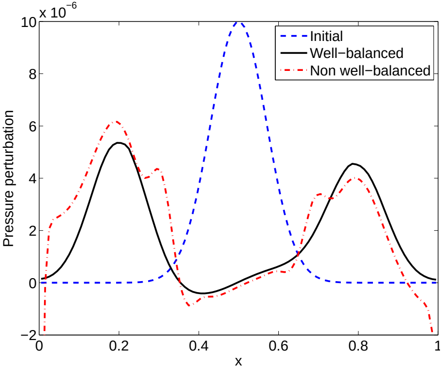
$$\frac{\partial \phi}{\partial x}(x_i) \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}, \qquad \text{reconstruct } \rho, u, p$$

Using exact derivative of potential does not improve results. In practice, ϕ is only available at grid points.

Evolution of small perturbations

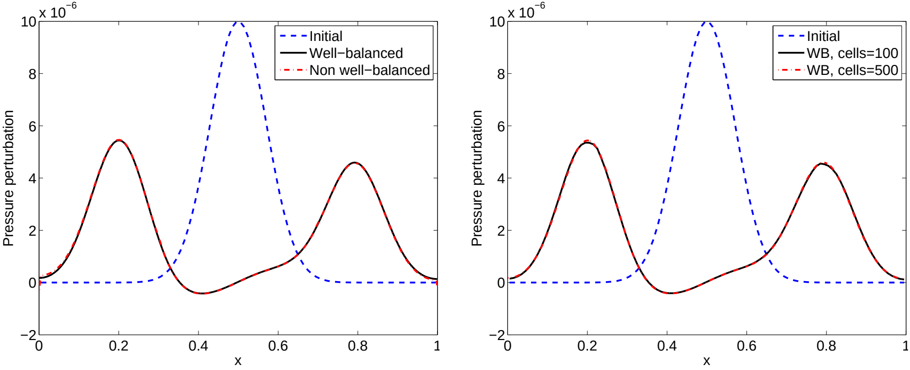


$\varepsilon = 10^{-3}$, $N = 100$ cells



$\varepsilon = 10^{-5}$, $N = 100$ cells

Evolution of small perturbations



$\varepsilon = 10^{-5}$, $N = 500$ cells

$\varepsilon = 10^{-5}$

Shock tube under gravitational field

Gravitational field

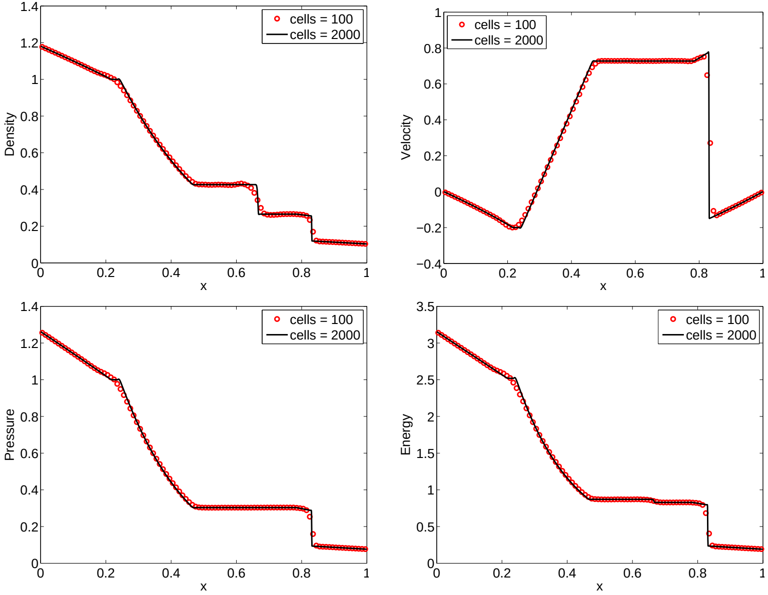
$$\phi(x) = x$$

The domain is $[0, 1]$ and the initial conditions are given by

$$(\rho, u, p) = \begin{cases} (1, 0, 1) & x < \frac{1}{2} \\ (0.125, 0, 0.1) & x > \frac{1}{2} \end{cases}$$

Solid wall boundary conditions. Final time $t = 0.2$, $N = 100, 2000$ cells

Shock tube under gravitational field



2-D Euler equations with gravity

2-D Euler equations in Cartesian coordinates

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{f}}{\partial x} + \frac{\partial \boldsymbol{g}}{\partial y} = \boldsymbol{s}$$

Here the conserved variables $\boldsymbol{q}$, fluxes $(\boldsymbol{f}, \boldsymbol{g})$ and source terms $\boldsymbol{s}$ are given by

$$\boldsymbol{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ (E+p)u \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ (E+p)v \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} 0 \\ -\rho \frac{\partial \phi}{\partial x} \\ -\rho \frac{\partial \phi}{\partial y} \\ -\rho(u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} \end{bmatrix}$$

In the above equations

$\rho =$ density, $(u, v) =$ Cartesian components of the velocity

$p =$ pressure, $E =$ total energy per unit volume

$\phi = \phi(x, y) =$ gravitational potential

Hydrostatic solution

The hydrostatic equilibrium is characterized by the following set of equations

$$u_e = v_e = 0, \qquad \frac{\partial p_e}{\partial x} = -\rho_e \frac{\partial \phi}{\partial x}, \qquad \frac{\partial p_e}{\partial y} = -\rho_e \frac{\partial \phi}{\partial y}$$

These equations can be integrated along $y = \mathrm{const}$ lines

$$p_e(x, y) = a(y) \exp\left(-\int_{x_0}^{x} \frac{\phi_x(s, y)}{RT(s, y)} \mathrm{d}s \right)$$

and $x = \mathrm{const}$ lines

$$p_e(x, y) = b(x) \exp\left(-\int_{y_0}^{y} \frac{\phi_y(x, s)}{RT(x, s)} \mathrm{d}s \right)$$

As in the 1-D case, we will exploit the structure of these solutions to construct the well-balanced scheme.

Source term

Define

$$\psi(x,y) = -\int_{x_0}^{x} \frac{\phi_x(s,y)}{RT(s,y)}\mathrm{d}s, \qquad \chi(x,y) = -\int_{y_0}^{y} \frac{\phi_y(x,s)}{RT(x,s)}\mathrm{d}s$$

Then the gravitational force can be written as

$$-\rho\phi_x = p\mathrm{e}^{-\psi}\frac{\partial}{\partial x}\mathrm{e}^{\psi}, \qquad -\rho\phi_y = p\mathrm{e}^{-\chi}\frac{\partial}{\partial y}\mathrm{e}^{\chi} \tag{9}$$

2-d finite volume scheme on Cartesian meshes

- Partition computational domain into rectangular cells

$$\Omega_{i,j} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$$

  with

$$x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \Delta x \quad \text{and} \quad y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = \Delta y$$

- semi-discrete finite volume scheme for the cell (i, j)

$$\Omega_{i,j} \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{q}_{i,j} + \hat{\boldsymbol{f}}_{i+\frac{1}{2},j} - \hat{\boldsymbol{f}}_{i-\frac{1}{2},j} + \hat{\boldsymbol{g}}_{i,j+\frac{1}{2}} - \hat{\boldsymbol{g}}_{i,j-\frac{1}{2}} = \hat{\boldsymbol{s}}_{i,j} \tag{10}$$

2-d finite volume scheme on Cartesian meshes

- The gravitational source term is discretized as

$$
\begin{aligned}
\hat{\boldsymbol{s}}_{i,j}^{(1)} &= 0 \\
\hat{\boldsymbol{s}}_{i,j}^{(2)} &= p_{i,j} \mathrm{e}^{-\psi_{i,j}} \left[\mathrm{e}^{\psi_{i+\frac{1}{2},j}} - \mathrm{e}^{\psi_{i-\frac{1}{2},j}} \right] \\
\hat{\boldsymbol{s}}_{i,j}^{(3)} &= p_{i,j} \mathrm{e}^{-\chi_{i,j}} \left[\mathrm{e}^{\chi_{i,j+\frac{1}{2}}} - \mathrm{e}^{\chi_{i,j-\frac{1}{2}}} \right] \\
\hat{\boldsymbol{s}}_{i,j}^{(4)} &= u_{i,j} \hat{\boldsymbol{s}}_{i,j}^{(2)} + v_{i,j} \hat{\boldsymbol{s}}_{i,j}^{(3)}
\end{aligned}
$$

- Following the steps in the 1-D case, we can write the source terms as

$$
\begin{aligned}
\hat{\boldsymbol{s}}_{i,j}^{(2)} &= p_{i,j} \left[\mathrm{e}^{\hat{\beta}_{i+\frac{1}{2},j}(\phi_{i+1,j} - \phi_{i,j})} - \mathrm{e}^{\hat{\beta}_{i-\frac{1}{2},j}(\phi_{i-1,j} - \phi_{i,j})} \right] \\
\hat{\boldsymbol{s}}_{i,j}^{(3)} &= p_{i,j} \left[\mathrm{e}^{\hat{\beta}_{i,j+\frac{1}{2}}(\phi_{i,j+1} - \phi_{i,j})} - \mathrm{e}^{\hat{\beta}_{i,j-\frac{1}{2}}(\phi_{i,j-1} - \phi_{i,j})} \right]
\end{aligned}
$$

2-d finite volume scheme on Cartesian meshes

- To obtain the values at the face $\boldsymbol{q}^L_{i+\frac{1}{2},j}$, $\boldsymbol{q}^R_{i+\frac{1}{2},j}$ we reconstruct the following set of variables

$$\boldsymbol{w} = [\rho \mathrm{e}^{-\psi}, u, v, p\mathrm{e}^{-\psi}]^\top$$

and to obtain $\boldsymbol{q}^L_{i,j+\frac{1}{2}}$, $\boldsymbol{q}^R_{i,j+\frac{1}{2}}$, we reconstruct the following set of variables

$$\boldsymbol{w} = [\rho \mathrm{e}^{-\chi}, u, v, p\mathrm{e}^{-\chi}]^\top$$

Theorem

The finite volume scheme (10) together with a numerical flux which satisfies property C and reconstruction of $\boldsymbol{w}$ variables is well-balanced in the sense that the initial condition given by

$$u_{i,j} = v_{i,j} = 0, \qquad p_{i,j} \exp(-\psi_{i,j}) = a_j, \qquad p_{i,j} \exp(-\chi_{i,j}) = b_i, \qquad \forall\, i, j \tag{11}$$

is preserved by the numerical scheme.

Theorem

Any hydrostatic solution which is isothermal or polytropic is exactly preserved by the finite volume scheme (10).

Isothermal hydrostatic solution

unit square, potential

$$\phi(x, y) = x + y$$

$$\rho_e(x, y) = \rho_0 \exp(-\rho_0 g(x+y)/p_0), \qquad p_e(x, y) = p_0 \exp(-\rho_0 g(x+y)/p_0)$$

$$\rho_0 = 1.21, \qquad p_0 = 1, \qquad g = 1, \qquad \text{final time} = 1$$

Grid	ρ	u	v	p
50×50	0.19050E-14	0.14660E-15	0.14439E-15	0.20428E-14
200×200	0.75677E-14	0.12908E-14	0.12853E-14	0.83936E-14

Table: Error in density, velocity and pressure for isothermal hydrostatic example

Isothermal hydrostatic solution

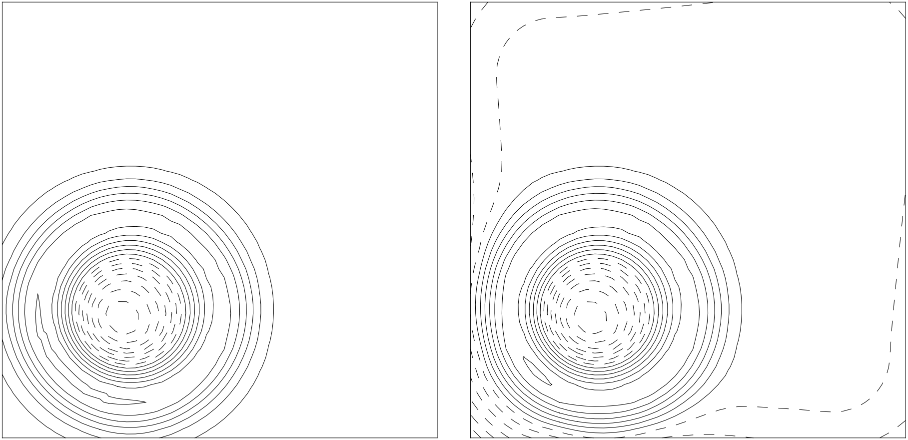
To study the accuracy of the scheme, we add an initial perturbation to the pressure and take the following initial condition

$$p(x,y,0) = p_0 \exp(-\rho_0 g(x+y)/p_0) + \eta \exp(-100\rho_0 g((x-0.3)^2+(y-0.3)^2)/p$$

mesh $= 50 \times 50$, transmissive bc, final time $= 0.15$

pressure perturbation with $\eta = 0.1$

Isothermal hydrostatic solution

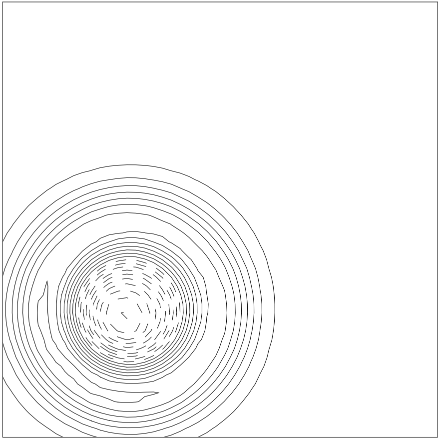


well-balanced non well-balanced

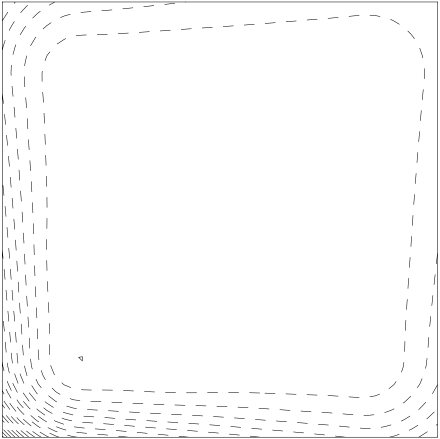
20 equally spaced contours between -0.03 and +0.03 are shown

Isothermal hydrostatic solution

pressure perturbation with $\eta = 0.001$



well-balanced
20 contours in [-0.00026, $+$0.00026]



non well-balanced
20 contours in [-0.02, $+$0.00026]

Polytropic hydrostatic solution

Unit square, potential $\phi(x, y) = x + y$

$$T_e = 1 - \frac{\nu - 1}{\nu}(x + y), \qquad p_e = T_e^{\frac{\nu}{\nu-1}}, \qquad \rho_e = T_e^{\frac{1}{\nu-1}}$$

$$\nu = 1.2, \qquad \text{final time} = 1$$

Grid	ρ	u	v	p
50×50	0.20449E-14	0.41148E-15	0.39802E-15	0.24637E-14
200×200	0.83747E-14	0.18037E-14	0.17986E-14	0.10107E-13

Table: Error in density, velocity and pressure

Polytropic hydrostatic solution

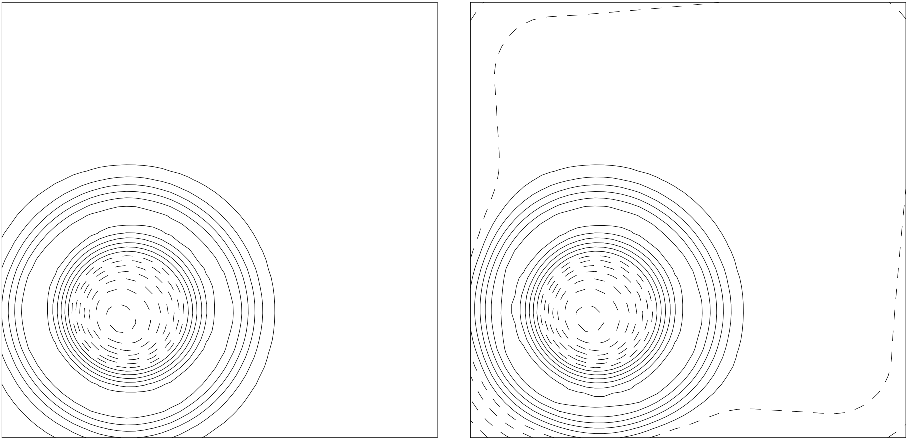
Perturbation of the initial pressure from the above polytropic solution

$$p(x, y, 0) = p_e(x, y) + \eta \exp(-100\rho_0 g((x - 0.3)^2 + (y - 0.3)^2)/p_0)$$

mesh $= 50 \times 50,$ transmissive bc, final time $= 0.15$

pressure perturbation with $\eta = 0.1$

Polytropic hydrostatic solution

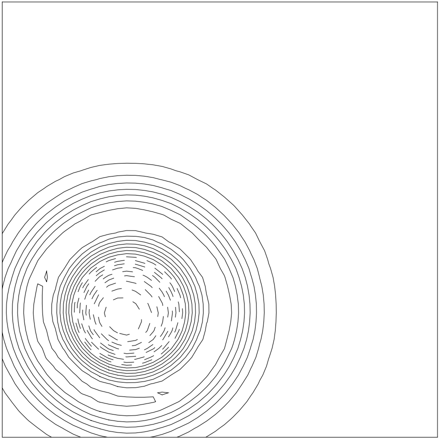


well-balanced non well-balanced

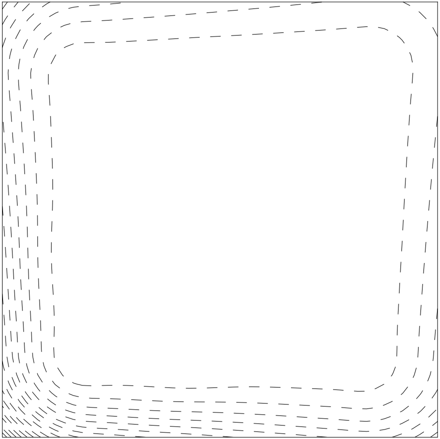
20 equally spaced contours between -0.03 and +0.03

Polytropic hydrostatic solution

pressure perturbation with $\eta = 0.001$



well-balanced
20 contours in [-0.00025,+0.00025]



non well-balanced
20 contours in [-0.015,+0.0003]

Rayleigh-Taylor instability

- isothermal radial solution with potential $\phi = r$: $\rho = p = \exp(-r)$
- Add perturbation: initial pressure and density are given by

$$p = \begin{cases} \mathrm{e}^{-r} & r \le r_0 \\ \mathrm{e}^{-\frac{r}{\alpha} + r_0 \frac{(1-\alpha)}{\alpha}} & r > r_0 \end{cases}, \qquad \rho = \begin{cases} \mathrm{e}^{-r} & r \le r_i \\ \frac{1}{\alpha}\mathrm{e}^{-\frac{r}{\alpha} + r_0 \frac{(1-\alpha)}{\alpha}} & r > r_i \end{cases}$$

$$r_i = r_0(1 + \eta \cos(k\theta)), \qquad \alpha = \exp(-r_0)/(\exp(-r_0) + \Delta_\rho)$$

- density jumps by an amount $\Delta_\rho > 0$ at interface $r = r_i$, pressure is continuous.

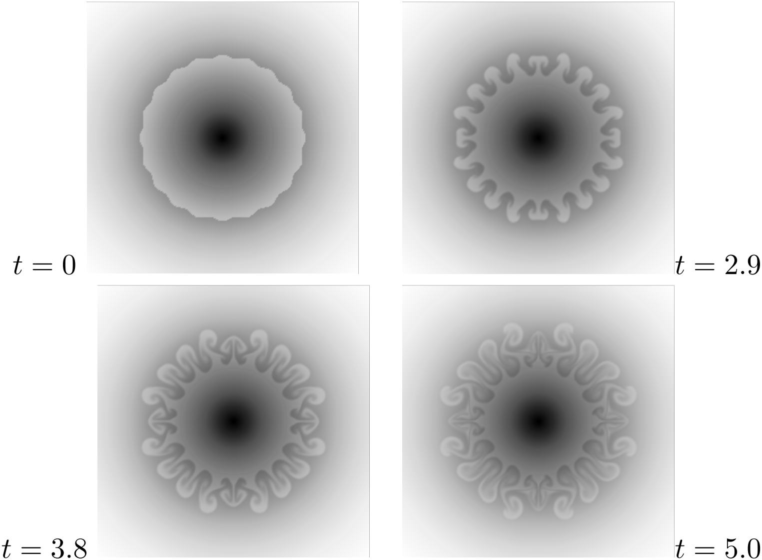
$$\Delta_\rho = 0.1, \quad \eta = 0.02, \quad k = 20, \quad \text{mesh} = 240 \times 240 \text{ cells}$$

$$\text{domain} = [-1, +1] \times [-1, +1].$$

Rayleigh-Taylor instability

- In the regions $r < r_0(1-\eta)$ and $r > r_0(1+\eta)$ the initial condition is in stable equilibrium
- but due to the discontinuous density, a Rayleigh-Taylor instability develops near interface defined by $r = r_i$.
- Due to well-balanced scheme, instability is concentrated only around the discontinuous interface

Rayleigh-Taylor instability



Extensions, ongoing work

- 2/3-D curvilinear meshes
- General equation of state, e.g., ideal gas with radiation pressure

  $$p = \rho RT + \frac{1}{3}aT^4$$

  No exact hydrostatic solutions known, preserve an approximate hydrostatic solution
- Weak formulation

  $$\text{find} \quad u \in V \quad \text{such that} \quad a(u, v) = \ell(v) \quad \forall v \in V$$

- Galerkin method

  $$\text{find} \quad u_h \in V_h \quad \text{such that} \quad a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h$$

  In practice

  $$\text{find} \quad u_h \in V_h \quad \text{such that} \quad a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_h$$

  Exact solution u is not a solution of above problem.
- Discontinuous Galerkin method: well-balanced for isothermal hydrostatic solution

Thank You

References

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[2] Yulong Xing and Chi-Wang Shu, *High order well-balanced WENO scheme for the gas dynamics equations under gravitational fields*, J. Sci. Comput., 54 (2013), pp. 645–662.