Numerical implementation of feedback stabilization of Boussinesq model

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Outline

Joint work with Mythily Ramaswamy and Jean-Pierre Raymond

- Boussinesq model
  - Stationary solution
  - Instability
- Finite dim feedback control
- FEM for linearized Boussinesq model: State space model
- Approach
  - Discrete Leray projector
  - Unstable sub-space
  - Riccati eqn and optimal feedback control
- Briefly about FEniCS
- Some results
- Extended system

Navier-Stokes-Boussinesq model

- Non-isothermal flow

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) + \nabla p = \nabla\cdot\tau(\boldsymbol{v}) - \rho g \boldsymbol{e}_y$$

- Density variations assumed to be small; important only in the gravity term

$$\rho_0\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) + \nabla p = \nabla\cdot\tau(\boldsymbol{v}) - \rho g \boldsymbol{e}_y$$

- Density depends on temperature

$$\rho(T) = \rho_0[1 - \beta(T - T_0)], \qquad \beta = \text{coefficient of thermal expansion}$$

- Momentum eqn

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla(p/\rho_0 + gy) = \frac{1}{\rho_0}\nabla\cdot\tau(\boldsymbol{v}) + g(T - T_0)\boldsymbol{e}_y$$

Navier-Stokes-Boussinesq model

- Energy equation

$$\rho_0 C_p \left(\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T \right) = \kappa \Delta T + f, \quad \kappa = \text{coef. of thermal diffusion}$$

- Non-dimensionalize

$$x \to \frac{x}{L}, \qquad \boldsymbol{v} \to \frac{\boldsymbol{v}}{U}, \qquad p \to \frac{p}{\rho_0 U^2}, \qquad \theta = \frac{T - T_0}{T_1 - T_0}$$

$$\mathrm{Re} = \frac{\rho_0 U L}{\mu}, \qquad \mathrm{Gr} = \frac{\beta L^3 \rho_0^2 g (T_1 - T_0)}{\mu^2}, \qquad \mathrm{Pr} = \frac{\mu C_p}{\kappa}$$

Navier-Stokes-Boussinesq model

- Non-dimensional form

$$
\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p &= \frac{1}{\mathrm{Re}} \nabla \cdot \tau(\boldsymbol{v}) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \theta \boldsymbol{e}_y \\
\frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \nabla \theta &= \frac{1}{\mathrm{RePr}} \Delta \theta + f \\
\nabla \cdot \boldsymbol{v} &= 0
\end{aligned}
$$

  where

$$
\tau(\boldsymbol{v}) = 2\epsilon(\boldsymbol{v}), \qquad \epsilon(\boldsymbol{v}) = \frac{1}{2}[\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\top]
$$

- The heat source function is given by[1]

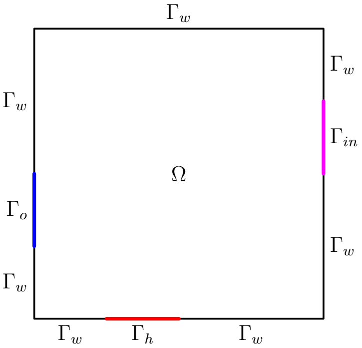
$$
f = 7 \sin(2\pi x) \cos(2\pi y)
$$

  We use the parameters

$$
\mathrm{Re} = 100, \qquad \mathrm{Pr} = 0.7, \qquad \mathrm{Gr} = \frac{\mathrm{Re}^2}{0.9} \approx 11111.1
$$

[1] Weiwei Hu, PhD Thesis, 2012

Stationary problem



Boundary conditions

$$
\begin{aligned}
\boldsymbol{v} &= 0 && \text{on} \quad \Gamma_{in} \cup \Gamma_h \cup \Gamma_w \\
-p\boldsymbol{n} + \frac{1}{\mathrm{Re}}\tau(\boldsymbol{v})\boldsymbol{n} &= 0 && \text{on} \quad \Gamma_o \\
\theta &= 0 && \text{on} \quad \Gamma_{in} \cup \Gamma_w \\
\frac{1}{\mathrm{RePr}}\frac{\partial \theta}{\partial n} &= 0 && \text{on} \quad \Gamma_h \cup \Gamma_o
\end{aligned}
$$

Solution space

$$
\begin{aligned}
\boldsymbol{V}_0 &= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}_{|\Gamma_{in} \cup \Gamma_h \cup \Gamma_w} = 0\} \\
W_0 &= \{\theta \in H^1(\Omega) : \theta_{|\Gamma_{in} \cup \Gamma_w} = 0\} \\
Q &= L^2(\Omega)
\end{aligned}
$$

Stationary problem

Find $(\boldsymbol{v}, \theta, p) \in \boldsymbol{V}_0 \times W_0 \times Q$ such that $\forall\ (\boldsymbol{\varphi}, \chi, \psi) \in \boldsymbol{V}_0 \times W_0 \times Q$

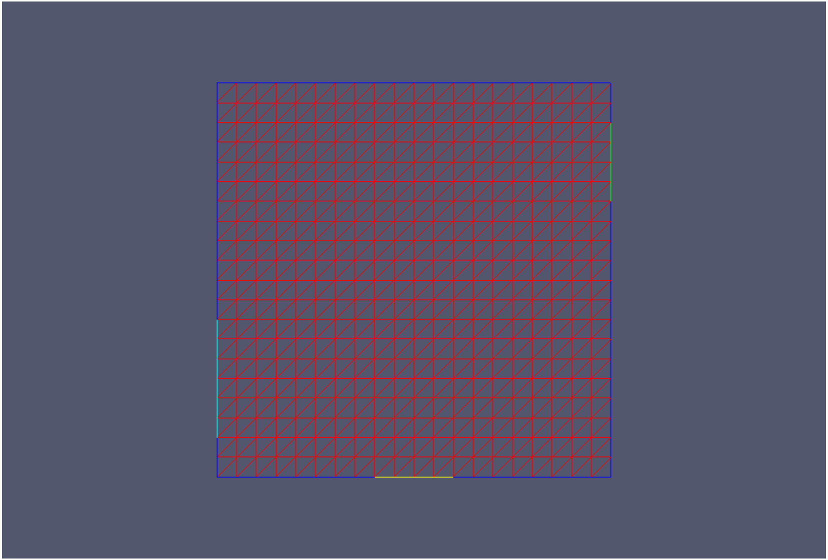
$$
\begin{aligned}
(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) + \frac{2}{\mathrm{Re}}\,(\epsilon(\boldsymbol{v}), \epsilon(\boldsymbol{\varphi})) - \frac{\mathrm{Gr}}{\mathrm{Re}^2}\,(\theta \boldsymbol{e}_y, \boldsymbol{\varphi}) &= 0 \\
(\boldsymbol{v} \cdot \nabla \theta, \chi) + \frac{1}{\mathrm{RePr}}\,(\nabla \theta, \nabla \chi) - (f, \chi) &= 0 \\
(\nabla \cdot \boldsymbol{v}, \psi) &= 0
\end{aligned}
$$

- Solving steady Navier-Stokes at $\mathrm{Re} = 100$ is numerically unstable
- Solve the steady Navier-Stokes on a sequence of Reynolds numbers:

$$50 \to 60 \to 70 \to 80 \to 85 \to 90 \to 95 \to 100$$

using solution from previous Re as initial guess in the Newton method

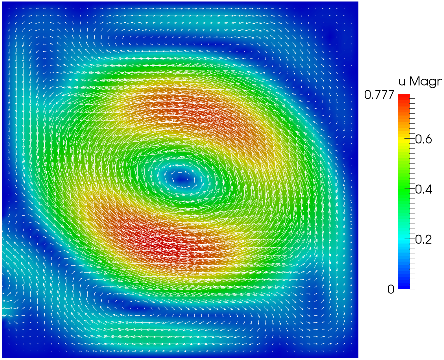
Sample FEM mesh: 20×20



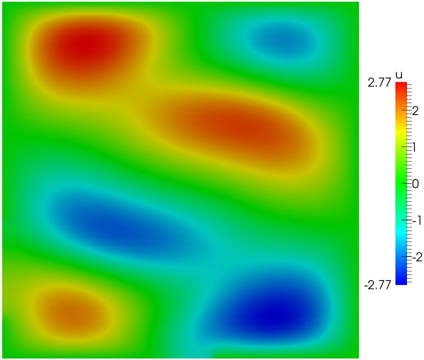
Stationary solution: $\boldsymbol{P}_2 - P_2 - P_1$, 50×50 mesh

```
$ python steady.py
```

Velocity vectors



Temperature



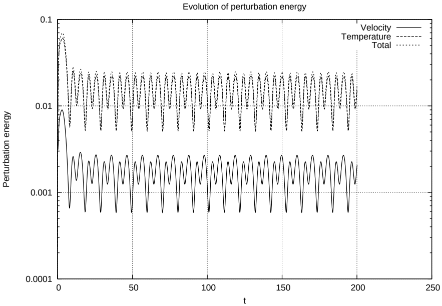
Stability of stationary state

Add a small perturbation to stationary state: Initial condition

$$\begin{bmatrix} \boldsymbol{v} \\ \theta \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_s \\ \theta_s \end{bmatrix} + 0.1 \begin{bmatrix} \boldsymbol{v}_e \\ \theta_e \end{bmatrix}$$

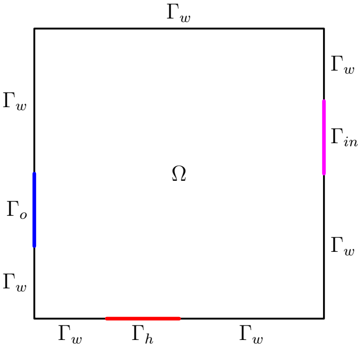
Solve non-linear Boussinesq model; measure the energy in the perturbations

$$\frac{1}{2}\int_\Omega |\boldsymbol{v} - \boldsymbol{v}_s|^2 \mathrm{d}x, \qquad \frac{1}{2}\int_\Omega (\theta - \theta_s)^2 \mathrm{d}x$$



The system settles down into a periodic state.

Boundary controls



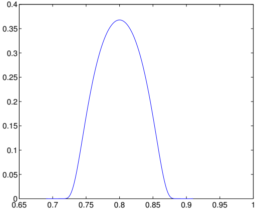
$$
\begin{aligned}
\boldsymbol{v} &= 0 && \text{on} \quad \Gamma_h \cup \Gamma_w \\
\boldsymbol{v} &= (v_c, 0) && \text{on} \quad \Gamma_{in} \\
-p\boldsymbol{n} + \frac{1}{\mathrm{Re}}\tau(\boldsymbol{v})\boldsymbol{n} &= 0 && \text{on} \quad \Gamma_o \\
\theta &= \theta_c && \text{on} \quad \Gamma_{in} \\
\theta &= 0 && \text{on} \quad \Gamma_w \\
\frac{1}{\mathrm{RePr}}\frac{\partial\theta}{\partial n} &= 0 && \text{on} \quad \Gamma_o \\
\frac{1}{\mathrm{RePr}}\frac{\partial\theta}{\partial n} &= f_c && \text{on} \quad \Gamma_h
\end{aligned}
$$

$v_c = u_1\alpha(y)$, $\theta_c = u_2\beta(y)$, $f_c = u_3\gamma(x)$, (u_1, u_2, u_3): control variables

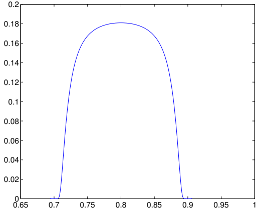
$$
\alpha(y) = \exp\left(-\frac{0.0001}{[(0.7-y)(0.9-y)]^2}\right), \quad \beta(y) = 0.2\exp\left(-\frac{0.00001}{[(0.7-y)(0.9-y)]^2}\right)
$$

$$
\gamma(x) = 0.4\exp\left(-\frac{0.00001}{[(0.4-x)(0.6-x)]^2}\right)
$$

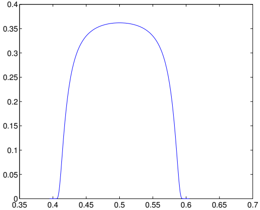
Boundary controls



(a) $b_{\mathbf{v}}(y) = e^{-\frac{0.0001}{[(0.7-y)(0.9-y)]^2}}$



(b) $b_{\theta_{\mathrm{I}}}(y) = 0.2e^{-\frac{0.00001}{[(0.7-y)(0.9-y)]^2}}$

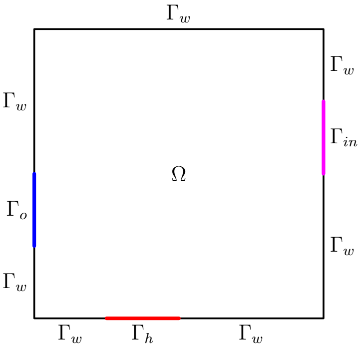


(c) $b_{\theta_{\mathrm{H}}}(x) = 0.4e^{-\frac{0.00001}{[(0.4-x)(0.6-x)]^2}}$

(From Hu)

Model of Hu

stress tensor: $\sigma(\boldsymbol{v}, p) = -pI + \frac{1}{\mathrm{Re}}\tau(\boldsymbol{v})$



$$
\begin{aligned}
\boldsymbol{v} &= 0 && \text{on} \quad \Gamma_h \cup \Gamma_w \\
\alpha_1 \sigma(\boldsymbol{v}, p)\boldsymbol{n} + \beta_1 \boldsymbol{v} &= \beta_1 (v_c, 0) && \text{on} \quad \Gamma_{in} \\
-p\boldsymbol{n} + \frac{1}{\mathrm{Re}}\tau(\boldsymbol{v})\boldsymbol{n} &= 0 && \text{on} \quad \Gamma_o \\
\frac{\alpha_2}{\mathrm{RePr}} \frac{\partial \theta}{\partial n} + \beta_2 \theta &= \beta_2 \theta_c && \text{on} \quad \Gamma_{in} \\
\theta &= 0 && \text{on} \quad \Gamma_w \\
\frac{1}{\mathrm{RePr}} \frac{\partial \theta}{\partial n} &= 0 && \text{on} \quad \Gamma_o \\
\frac{1}{\mathrm{RePr}} \frac{\partial \theta}{\partial n} &= f_c && \text{on} \quad \Gamma_h
\end{aligned}
$$

Divergence constraint via penalty approach: $(\nabla \cdot \boldsymbol{v}, \psi) = \varepsilon(p, \psi)$, $\forall \psi \in Q$

Fin. dim. feedback control approach

- Stationary state is unstable
  - small perturbations take the state away
- Aim: apply control to drive state towards stationary state
- Model perturbation (z) by linearization around stationary state

$$E\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bu, \qquad z = \begin{bmatrix} v \\ \theta \\ p \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

- Determine control by feedback law: $u = -Kz$ to achieve

$$\|z(t)\| \to 0, \qquad \text{as} \qquad t \to \infty$$

  - Minimal norm control: $A - BK$ is stable (real$(\lambda) < 0$)
  - LQR

$$\text{minimize } J = \frac{1}{2}\int_0^\infty \|Cz(t)\|^2 \,\mathrm{d}t + \frac{1}{2}\int_0^\infty \|Ru(t)\|^2 \,\mathrm{d}t$$

Control strategies

$$E\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bu, \qquad E = E^\top > 0, \qquad u = -Kz$$

- (A, B) stabilizable (Hautus): For each eigenvalue λ, real$(\lambda) > 0$

$$A^\top v = \lambda E^\top v, \qquad B^\top v \neq 0$$

- Minimal norm control: $K = R^{-1}B^\top \Pi E$

$$\text{(GABE)}: \qquad A^\top \Pi E + E\Pi A - E\Pi B R^{-1} B^\top \Pi E = 0$$

- (A, C) is detectable if $\exists L$ such that $A - LC$ is stable
  (A, C) is detectable iff $(A^\top, C^\top)$ is stabilizable.
- LQR control: $K = R^{-1}B^\top \Pi E$

$$\text{(GARE)}: \qquad A^\top \Pi E + E\Pi A - E\Pi B R^{-1} B^\top \Pi E + Q = 0, \qquad Q = C^\top C$$

- Eigenvalues reflected about the imaginary axis

Linearized problem

Let $(\boldsymbol{v}_s, \theta_s, p_s)$ = stationary solution
Perturbations $(\boldsymbol{v}, \theta, p)$ around this state is governed by

$$\begin{aligned}
\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}_s \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_s + \nabla p &= \frac{1}{\mathrm{Re}} \nabla \cdot \tau(\boldsymbol{v}) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} \theta \boldsymbol{e}_y \\
\frac{\partial \theta}{\partial t} + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{v} \cdot \nabla \theta_s &= \frac{1}{\mathrm{RePr}} \Delta \theta \\
\nabla \cdot \boldsymbol{v} &= 0
\end{aligned}$$

Find $(\boldsymbol{v}(t), \theta(t), p(t)) \in \boldsymbol{V} \times W \times Q$ such that $\forall (\boldsymbol{\varphi}, \chi, \psi) \in \boldsymbol{V}_0 \times W_0 \times Q$

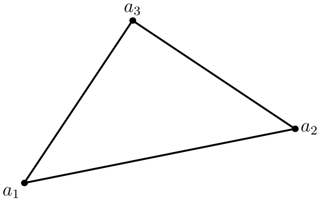
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{v}, \boldsymbol{\varphi}) + (\boldsymbol{v}_s \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) & \\
+ \frac{2}{\mathrm{Re}} (\epsilon(\boldsymbol{v}), \epsilon(\boldsymbol{\varphi})) - \frac{\mathrm{Gr}}{\mathrm{Re}^2} (\theta \boldsymbol{e}_y, \boldsymbol{\varphi}) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t} (\theta, \chi) + (\boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{v} \cdot \nabla \theta_s, \chi) + \frac{1}{\mathrm{RePr}} (\nabla \theta, \nabla \chi) - (f_c, \chi)_{\Gamma_h} &= 0 \\
-(\nabla \cdot \boldsymbol{v}, \psi) &= 0
\end{aligned}$$

Linearized problem

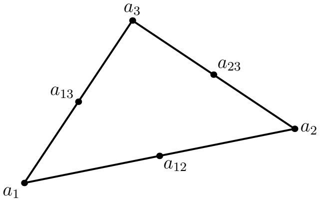
or

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}, \boldsymbol{\varphi}) &= \mathcal{A}_{vv}(\boldsymbol{v}, \boldsymbol{\varphi}) + \mathcal{A}_{v\theta}(\theta, \boldsymbol{\varphi}) + \mathcal{A}_{vp}(p, \boldsymbol{\varphi}) \\
\frac{\mathrm{d}}{\mathrm{d}t}(\theta, \chi) &= \mathcal{A}_{\theta v}(\boldsymbol{v}, \chi) + \mathcal{A}_{\theta\theta}(\theta, \chi) + \mathcal{A}_{\theta h}(f_c, \chi) \\
0 &= \mathcal{A}_{pv}(\boldsymbol{v}, \psi)
\end{aligned}$$

Finite element method



Pressure



Velocity and temperature

Define

$$\begin{aligned} I_{in}^{\boldsymbol{v}} &= \text{velocity dof on } \Gamma_{in} \text{ (} x \text{ component only)} \\ I_f^{\boldsymbol{v}} &= \text{unknown (free) velocity dof} \\ I_{in}^{\theta} &= \text{temperature dof on } \Gamma_{in} \\ I_f^{\theta} &= \text{unknown (free) temperature dof} \\ I^p &= \text{pressure dof} \end{aligned}$$

$$\boldsymbol{v}_h = \sum_{j \in I_f^{\boldsymbol{v}}} v_j \boldsymbol{\varphi}_j + \sum_{j \in I_{in}^{\boldsymbol{v}}} v_{c,j} \boldsymbol{\varphi}_j, \quad \theta_h = \sum_{j \in I_f^{\theta}} \theta_j \chi_j + \sum_{j \in I_{in}^{\theta}} \theta_{c,j} \chi_j, \quad p_h = \sum_{j \in I^p} p_j \psi_j$$

Finite element method

Define the vector of unknown dof

$$v = (v_i), \quad i \in I_f^{\boldsymbol{v}}, \qquad \theta = (\theta_i), \quad i \in I_f^{\theta}, \qquad p = (p_i), \quad i \in I^p$$

By construction, $\boldsymbol{v}_h$ and θ_h satisfy the homogeneous Dirichlet bc, i.e.,

$$\begin{aligned} \boldsymbol{v}_h &= 0, && \text{on} \quad \Gamma_h \cup \Gamma_w \\ \boldsymbol{v}_h \cdot \boldsymbol{e}_y &= 0, && \text{on} \quad \Gamma_{in} \\ \theta_h &= 0, && \text{on} \quad \Gamma_w \end{aligned}$$

FE Galerkin method

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_h, \boldsymbol{\varphi}_i) &= \mathcal{A}_{vv}(\boldsymbol{v}_h, \boldsymbol{\varphi}_i) + \mathcal{A}_{v\theta}(\theta_h, \boldsymbol{\varphi}_i) + \mathcal{A}_{vp}(p_h, \boldsymbol{\varphi}_i), \quad i \in I_f^{\boldsymbol{v}} \\ \frac{\mathrm{d}}{\mathrm{d}t}(\theta_h, \chi_i) &= \mathcal{A}_{\theta v}(\boldsymbol{v}_h, \chi_i) + \mathcal{A}_{\theta\theta}(\theta_h, \chi_i) + \mathcal{A}_{\theta h}(f_c, \chi_i), \quad i \in I_f^{\theta} \\ 0 &= \mathcal{A}_{pv}(\boldsymbol{v}_h, \psi_i), \quad i \in I^p \end{aligned}$$

Finite element method

Consider the velocity equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_h, \boldsymbol{\varphi}_i) = \sum_{j \in I_f^{\boldsymbol{v}}} \frac{\mathrm{d}v_j}{\mathrm{d}t}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) + \sum_{j \in I_{in}^{\boldsymbol{v}}} \frac{\mathrm{d}v_{c,j}}{\mathrm{d}t}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i), \qquad i \in I_f^{\boldsymbol{v}}$$

We will ignore the term involving $\frac{\mathrm{d}v_{c,j}}{\mathrm{d}t}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_h, \{\boldsymbol{\varphi}_i\}) = M_v \frac{\mathrm{d}v}{\mathrm{d}t}, \qquad M_v = \{(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i)\} \qquad i, j \in I_f^{\boldsymbol{v}}$$

Next

$$\begin{aligned} \mathcal{A}_{vv}(\boldsymbol{v}_h, \boldsymbol{\varphi}_i) &= \sum_{j \in I_f^{\boldsymbol{v}}} v_j \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) + \sum_{j \in I_{in}^{\boldsymbol{v}}} v_{c,j} \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) \\ &= \sum_{j \in I_f^{\boldsymbol{v}}} v_j \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) + u_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \alpha_j \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i), \quad i \in I_f^{\boldsymbol{v}} \end{aligned}$$

In matrix form

$$\mathcal{A}_{vv}(\boldsymbol{v}_h, \{\boldsymbol{\varphi}_i\}) = A_{vv} v + B_{vv} u_1$$

Finite element method

Similarly

$$\mathcal{A}_{v\theta}(\theta_h, \{\boldsymbol{\varphi}_i\}) = A_{v\theta}\theta + B_{v\theta}u_2$$

$$\mathcal{A}_{vp}(p_h, \{\boldsymbol{\varphi}_i\}) = A_{vp}p$$

Velocity equation

$$M_v \frac{\mathrm{d}v}{\mathrm{d}t} = A_{vv}v + A_{v\theta}\theta + A_{vp}p + B_v u, \qquad B_v = [B_{vv},\ B_{v\theta},\ 0]$$

In this way, the linear system can be written as

$$\begin{bmatrix} M_v & 0 & 0 \\ 0 & M_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v \\ \theta \\ p \end{bmatrix} = \begin{bmatrix} A_{vv} & A_{v\theta} & A_{vp} \\ A_{\theta v} & A_{\theta\theta} & 0 \\ A_{vp}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \\ p \end{bmatrix} + \begin{bmatrix} B_v \\ B_\theta \\ B_p \end{bmatrix} u$$

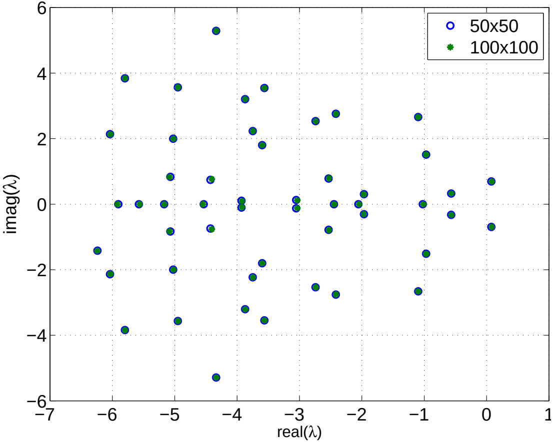
We now have a state space representation of our system

$$E\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bu$$

Eigenvalues from FEM

$$Az_e = \lambda E z_e, \qquad z_e = \begin{bmatrix} v_e \\ \theta_e \\ p_e \end{bmatrix}$$

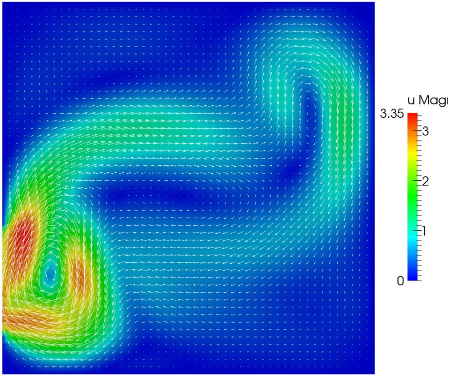
```
$ python linear.py
```



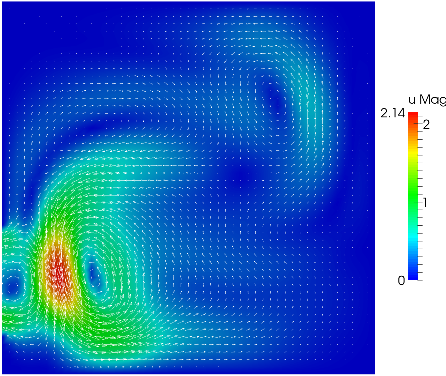
Eigenvalues from FEM

No.	50×50	100×100
1	0.0758 - 0.6948i	0.0803 - 0.6945i
2	0.0758 + 0.6948i	0.0803 + 0.6945i
3	-0.5672 - 0.3259i	-0.5664 - 0.3283i
4	-0.5672 + 0.3259i	-0.5664 + 0.3283i
5	-0.9699 - 1.5131i	-0.9716 - 1.5145i
6	-0.9699 + 1.5131i	-0.9716 + 1.5145i
7	-1.0229 + 0.0000i	-1.0162 + 0.0000i
8	-1.0971 - 2.6590i	-1.0971 - 2.6592i
9	-1.0971 + 2.6590i	-1.0971 + 2.6592i
10	-1.9657 - 0.3050i	-1.9708 - 0.3032i
11	-1.9657 + 0.3050i	-1.9708 + 0.3032i

Velocity eigenvectors v_e

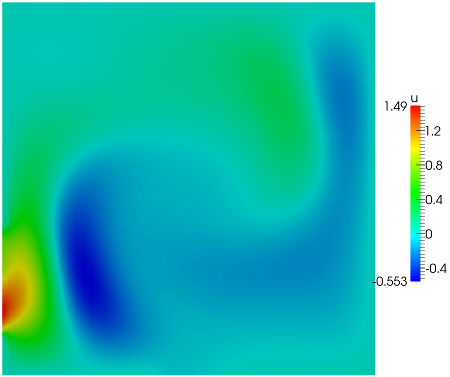


Real part

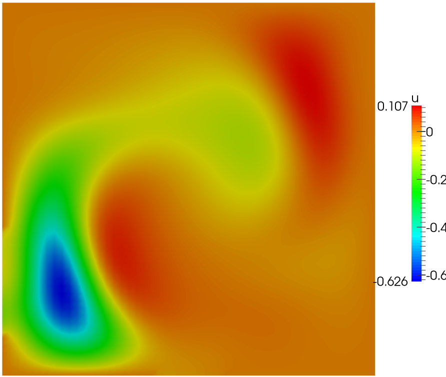


imaginary part

Temperature eigenvectors θ_e

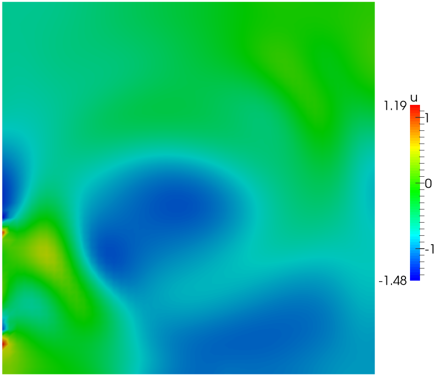


Real part

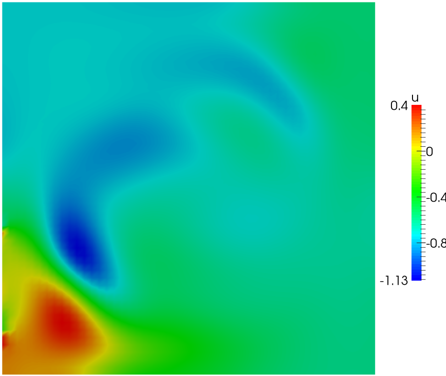


imaginary part

Pressure eigenvectors p_e



Real part



imaginary part

State-space model

Let us rewrite the linearized system

$$E\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Bu$$

into the differential and algebraic parts[2]

$$\begin{aligned} E_{11}\frac{\mathrm{d}y}{\mathrm{d}t} =& A_{11}y + A_{12}p + B_1 u \\ 0 =& A_{12}^\top y - B_2 u \end{aligned} \tag{1}$$

where

$$y = \begin{bmatrix} v \\ \theta \end{bmatrix}, \quad E_{11} = \begin{bmatrix} M_v & 0 \\ 0 & M_\theta \end{bmatrix}, \quad A_{11} = \begin{bmatrix} A_{vv} & A_{v\theta} \\ A_{\theta v} & A_{\theta\theta} \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} A_{vp} \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{vv} & 0 & 0 \\ 0 & B_{\theta\theta} & B_{\theta h} \end{bmatrix}, \quad B_2 = -\begin{bmatrix} B_{pv} & 0 & 0 \end{bmatrix}$$

[2] L. Thevenet, PhD Thesis, 2009

Discrete Leray projector

Denote by P the oblique projector defined as

$$\begin{aligned} P &= I_{n_y} - A_{12}(A_{12}^\top E_{11}^{-1} A_{12})^{-1} A_{12}^\top E_{11}^{-1} \\ &= \begin{bmatrix} I_{n_v} - A_{vp}(A_{vp}^\top M_v^{-1} A_{vp})^{-1} A_{vp}^\top M_v^{-1} & 0 \\ 0 & I_{n_\theta} \end{bmatrix} \end{aligned}$$

We look for solution to (1) of the form

$$y = y_1 + y_2, \qquad y_1 = P^\top y, \qquad y_2 = (I - P)^\top y$$

From their definitions we have

$$A_{12}^\top y_1 = 0, \qquad P^\top y_2 = 0$$

Discrete Leray projector

Equation satisfied by $y_1 = P^\top y$

The couple (y_1, y_2) satisfies

$$PE_{11}\frac{\mathrm{d}y_1}{\mathrm{d}t} = PA_{11}y_1 + PB_{12}u, \qquad y_1(0) = P^\top y_0 \tag{2}$$

$$y_2 = E_{11}^{-1}A_{12}(A_{12}^\top E_{11}^{-1}A_{12})^{-1}B_2u \tag{3}$$

where

$$B_{12} = B_1 + A_{11}E_{11}^{-1}A_{12}(A_{12}^\top E_{11}^{-1}A_{12})^{-1}B_2$$

Moreover, the pressure p is given by the expression

$$p = (A_{12}^\top E_{11}^{-1}A_{12})^{-1}\left[B_2\frac{\mathrm{d}u}{\mathrm{d}t} - A_{12}^\top E_{11}^{-1}(A_{11}y + B_1u)\right] \tag{4}$$

Discrete Leray projector

Using

$$PE_{11} = E_{11}P^\top, \qquad P^\top y_1 = y_1$$

The term $y_1 = P^\top y$, where y is solution to (1) obeys the equation

$$E_{11}\frac{\mathrm{d}y_1}{\mathrm{d}t} = PA_{11}y_1 + PB_{12}u, \qquad P^\top y_1(0) = P^\top y_0(0) \tag{5}$$

Riccati equation

We will find the control $u(t)$ which solves the following LQR problem

$$\inf\left\{I(y_1,u)=\frac{1}{2}\int_0^\infty y_1^\top E_{11}y_1\mathrm{d}t+\frac{1}{2}\int_0^\infty u^\top R u\mathrm{d}t \quad \middle| \quad (y_1,u)\text{ solves }(5)\right\}$$

If the system (5) is stabilizable and detectable then the LQR problem has a unique optimal control law given by

$$u=-Ky_1,\qquad K=R^{-1}B_{12}^\top P^\top \Pi$$

where Π is the unique positive semi-definite matrix solution of the GARE

$$A_{11}^\top P^\top E_{11}^{-1}\Pi+\Pi E_{11}^{-1}PA_{11}-\Pi E_{11}^{-1}PB_{12}R^{-1}B_{12}^\top P^\top E_{11}^{-1}\Pi+E_{11}=0 \tag{6}$$

1. Computing P is difficult since it is a full matrix
2. Solving for Π is difficult since we have a large Riccati equation

Project equation (5) onto the unstable subspace associated to (PA_{11},PE_{11}).

Projection onto the unstable subspace

- Need eigenvalues/vectors of (PA_{11}, PE_{11})
- P is expensive to compute

(A, E) and (PA_{11}, PE_{11})

λ is a finite eigenvalue associated to the eigenvector $z = [y, p]^\top$ for the matrix pencil (A, E) if and only if λ is a finite eigenvalue associated to the eigenvector y for the matrix pencil (PA_{11}, PE_{11}).

- We will assume that there are n_u complex unstable eigenvalues.

$$\Lambda_u = \text{diag}(\lambda_1, \dots, \lambda_{n_u})$$

- Let us denote by

$$\tilde{V}_y \in \mathbb{C}^{n_y \times n_u}, \qquad \tilde{Z}_y \in \mathbb{C}^{n_y \times n_u}$$

  the matrix of complex eigenvectors of (PA_{11}, PE_{11}) and of $(PA_{11}^\top, PE_{11}^\top)$ respectively, i.e., we have

$$PA_{11}\tilde{V}_y = PE_{11}\tilde{V}_y\Lambda_u, \qquad PA_{11}^\top \tilde{Z}_y = PE_{11}^\top \tilde{Z}_y\Lambda_u \tag{7}$$

Projection onto the unstable subspace

- Similarly we denote by

$$\tilde{V} \in \mathbb{C}^{(n_y+n_p)\times n_u}, \qquad \tilde{Z} \in \mathbb{C}^{(n_y+n_p)\times n_u}$$

  the matrices that satisfy

$$A\tilde{V} = E\tilde{V}\Lambda_u, \qquad A^\top \tilde{Z} = E^\top \tilde{Z}\Lambda_u$$

- Thanks to previous Lemma we have

$$\tilde{V} = \begin{bmatrix} \tilde{V}_y \\ \tilde{V}_p \end{bmatrix}, \qquad \tilde{Z} = \begin{bmatrix} \tilde{Z}_y \\ \tilde{Z}_p \end{bmatrix}$$

  where $\tilde{V}_y$, $\tilde{Z}_y$ satisfy equation (7).
- Thus using (A, E) we can determine the unstable eigenspace of (PA_{11}, PE_{11}) without computing P.

Projection onto the unstable subspace

In order to obtain a real subspace, we consider the matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \qquad U = \mathrm{diag}(S, \ldots, S) \in \mathbb{C}^{n_u \times n_u}$$

U is a unitary matrix consisting of $\frac{1}{2} n_u$ diagonal blocks. Define

$$V = \tilde{V} U^* \in \mathbb{R}^{(n_y + n_p) \times n_u}, \qquad Z = \tilde{Z} U^\top \in \mathbb{R}^{(n_y + n_p) \times n_u}$$

and

$$V_y = \tilde{V}_y U^* \in \mathbb{R}^{n_y \times n_u}, \qquad Z_y = \tilde{Z}_y U^\top \in \mathbb{R}^{n_y \times n_u}$$

Similarly to the complex case, these matrices obey the relations

$$V = \begin{bmatrix} V_y \\ V_p \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_y \\ Z_p \end{bmatrix}$$

Projector Q on the unstable subspace

$$V_y = [e_1, \ldots, e_{n_u}] \in \mathbb{R}^{n_y \times n_u}, \qquad Z_y = [\varepsilon_1, \ldots, \varepsilon_{n_u}] \in \mathbb{R}^{n_y \times n_u}$$

$$Qy = \sum_{k=1}^{n_u} (y, \varepsilon_k) e_k$$

The projector Q is given by the relation

$$Q = V_y Z_y^\top E_{11}$$

Equation satisfied by Qy_1

Let us consider y_1 solution to (5). Then Qy_1 satisfies the equation

$$Z_y^\top E_{11} \frac{\mathrm{d}y}{\mathrm{d}t} = Z_y^\top A_{11} y + Z_y^\top B_{12} u \tag{8}$$

$$Z_y^\top E_{11} \frac{\mathrm{d}y_1}{\mathrm{d}t} = Z_y^\top P A_{11} y_1 + Z_y^\top P B_{12} u$$

Using

$$P^\top Z_y = Z_y, \qquad Z_y^\top E_{11} = Z_y^\top E_{11} Q, \qquad Z_y^\top P A_{11} = Z_y^\top A_{11} Q$$

we obtain

$$Z_y^\top E_{11} Q \frac{\mathrm{d}y_1}{\mathrm{d}t} = Z_y^\top A_{11} Q y_1 + Z_y^\top B_{12} u$$

$\square$

Riccati equation

Find u minimizing the cost functional

$$I(Qy_1, u) = \frac{1}{2}\int_0^\infty (Qy_1)^\top E_{11}(Qy_1)\mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top R u \mathrm{d}t$$

Using the expression of Q

$$Qy_1 = V_y Z_y^\top E_{11} y_1 \in \mathbb{R}^{n_y} \qquad \text{Define} \qquad z = Z_y^\top E_{11} y_1 \in \mathbb{R}^{n_u}$$

which solves

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Z_y^\top A_{11} V_y z + Z_y^\top B_{12} u \tag{9}$$

and the control problem is equivalent to the new one ($Qy_1 = V_y z$)

$$J(z, u) = \frac{1}{2}\int_0^\infty z^\top V_y^\top E_{11} V_y z \mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top R u \mathrm{d}t$$

where (z, u) is solution to (9)

Riccati equation

If the system (9) is stabilizable and detectable then the LQR problem has a unique optimal control law given by

$$u = -Kz, \qquad K = R^{-1} B_{12}^\top Z_y \pi$$

where π is the unique positive semi-definite matrix solution of the ARE

$$V_y^\top A_{11}^\top Z_y \pi + \pi Z_y^\top A_{11} V_y - \pi Z_y^\top B_{12} R^{-1} B_{12}^\top Z_y \pi + V_y^\top E_{11} V_y = 0 \tag{10}$$

- Leray projector P is not involved
- $\pi \in \mathbb{R}^{n_u \times n_u}$ is of small size

Let us write this in the standard notation

$$A_u^\top \pi + \pi A_u - \pi B_u R^{-1} B_u^\top \pi + Q_u = 0$$

where

$$A_u = Z_y^\top A_{11} V_y, \quad B_u = Z_y^\top B_{12}, \quad Q_u = V_y^\top E_{11} V_y$$

Riccati equation

In Matlab we can solve for $\pi = \texttt{Pu}$ using the `care` function

```
Pu = care(Au, Bu, Qu, R);
```

We have

$$z = Z_y^\top E_{11} y_1 = Z_y^\top E_{11} y, \qquad Z_y^\top B_{12} = Z^\top B$$

Hence feedback law becomes

$$u = -\tilde{K} y, \qquad \tilde{K} = R^{-1}(B^\top Z)\pi(Z_y^\top E_{11}) \tag{11}$$

Summary of computing feedback law

- Form the matrices A_{11}, A_{12}, E_{11}, B_1, B_2, A, B, E
- Compute $B_{12} = B_1 + A_{11}E_{11}^{-1}A_{12}(A_{12}^\top E_{11}^{-1}A_{12})^{-1}B_2$

$$\begin{bmatrix} E_{11} & A_{12} \\ A_{12}^\top & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad B_{12} = B_1 + A_{11}X$$

- Compute unstable eigenvalues/vectors of (A, E), $(A^\top, E^\top)$.
  Form real matrices V_y, Z_y, Z.
- Solve small Riccati equation, size $n_u \times n_u$

$$A_u^\top \pi + \pi A_u - \pi B_u R^{-1} B_u^\top \pi + Q_u = 0$$

  where

$$A_u = Z_y^\top A_{11} V_y, \quad B_u = Z_y^\top B_{12}, \quad Q_u = V_y^\top E_{11} V_y$$

- Feedback operator

$$\tilde{K} = R^{-1}(B^\top Z)\pi(Z_y^\top E_{11})$$

Enlarging the stabilized subspace

Order eigenvalues based on real part

$$\lambda_1 \geq \lambda_2 \geq \ldots, \lambda_{n_u} > 0 > \lambda_{n_u+1} \geq \ldots \geq \lambda_n \geq \ldots$$

Add a shift $s \in \mathbb{R}$ to the eigenvalues

$$\lambda_1 + s \geq \lambda_2 + s \geq \ldots, \lambda_{n_u} + s > \lambda_{n_u+1} + s \geq \ldots \geq \lambda_n + s > 0 > \lambda_{n+1} \geq \ldots$$

Enlarge stabilized subspace

$$V_y = [e_1, \ldots, e_{n_u}, e_{n_u+1}, \ldots, e_n], \qquad Z_y = [\varepsilon_1, \ldots, \varepsilon_{n_u}, \varepsilon_{n_u+1}, \ldots \varepsilon_n]$$

$$A_u = Z_y^\top A_{11} V_y + sI \in \mathbb{R}^{n \times n}$$

Solve Riccati equation of size $n \times n$ to get $\tilde{K}$

Solving the Boussinesq equations

After spatial discretization

$$E\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = N(\tilde{y}; u), \qquad \tilde{y} = \begin{bmatrix} y \\ p \end{bmatrix}, \qquad y = \begin{bmatrix} v \\ \theta \end{bmatrix}, \qquad u = -\tilde{K}(y - y_s)$$

First time step: BDF1 (Backward euler)

$$E\frac{\tilde{y}^1 - \tilde{y}^0}{\Delta t} = N(\tilde{y}^1; u^1)$$

Second time step onwards: BDF2

$$E\frac{\frac{3}{2}\tilde{y}^n - 2\tilde{y}^{n-1} + \frac{1}{2}\tilde{y}^{n-2}}{\Delta t} = N(\tilde{y}^n; u^n), \qquad n = 2, 3, \ldots$$

Control

$$y^* = y^{n-1} \quad \text{or} \quad y^* = 2y^{n-1} - y^{n-2}, \qquad u^n = -\tilde{K}(y^* - y_s)$$

Non-linear systems solved using Newton method and UMFPACK (LU solver).

Intro to FEniCS

$$
\begin{aligned}
-\Delta u &= 1, \quad \text{in} \quad \Omega = [0,1]\times[0,1] \\
u &= 0, \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= [0,1]\times[0,1] \\
V_h &= \{v_h \in C(\Omega) : v_h|_K \in \mathbb{P}_1\} \\
u_h &\in V_h \text{ (Trial function)} \\
v_h &\in V_h \text{ (Test function)} \\
a &= \int_\Omega \nabla u_h \cdot \nabla v_h \mathrm{d}x \\
L &= \int_\Omega v_h \mathrm{d}x
\end{aligned}
$$

```
from dolfin import *
mesh = UnitSquareMesh(50,50)
V = FunctionSpace(mesh,'CG',1)
u = TrialFunction(V)
v = TestFunction(V)
a = inner(grad(u),grad(v))*dx
L = v*dx
bc= DirichletBC(V,0,'on_boundary')
u = Function(V)
solve(a==L,u,bc)
plot(u,interactive=True)
```

Find $u_h \in V_h \cap H_0^1(\Omega)$ s.t.

$a(u_h, v_h) = L(v_h), \forall v_h \in V_h \cap H_0^1(\Omega)$

File `demo.py`

Run the code

```
$ python demo.py
```

Boussinesq model

Finite element spaces

```
self.V = VectorFunctionSpace(mesh, "CG", self.udeg)  # velocity
self.W = FunctionSpace(mesh, "CG", self.tdeg)        # temp
self.Q = FunctionSpace(mesh, "CG", self.pdeg)        # pressure
self.X = MixedFunctionSpace([self.V, self.W, self.Q])
```

Trial and test functions

```
u,T,p = TrialFunctions(self.X)
v,S,q = TestFunctions(self.X)
```

Assemble the full mass matrix (includes all dof)

```
Ma = assemble(inner(u,v)*dx + T*S*dx)
```

Convert to sparse storage format

```
rows, cols, values = Ma.data()
import scipy.sparse as sps
Ma = sps.csc_matrix((values, cols, rows))
```

Boussinesq model

Indices of degrees of freedom

```
# Returns dof indices which are free
# freeinds = free indices of velocity, temperature, pressure
# pinds    = free indices of pressure
def get_indices(self):
   # Collect all dirichlet boundary dof indices
   bcinds = []
   for b in self.bc:
      bcdict = b.get_boundary_values()
      bcinds.extend(bcdict.keys())

   # total number of dofs
   N = self.X.dim()

   # indices of free nodes
   freeinds = np.setdiff1d(range(N),bcinds,assume_unique=True).↩
       astype(np.int32)

   # pressure indices
   pinds = self.X.sub(2).dofmap().dofs()

   return freeinds, pinds
```

Boussinesq model

Mass matrix (same as E in the slides)

```
M = Ma[freeinds,:][:,freeinds]
```

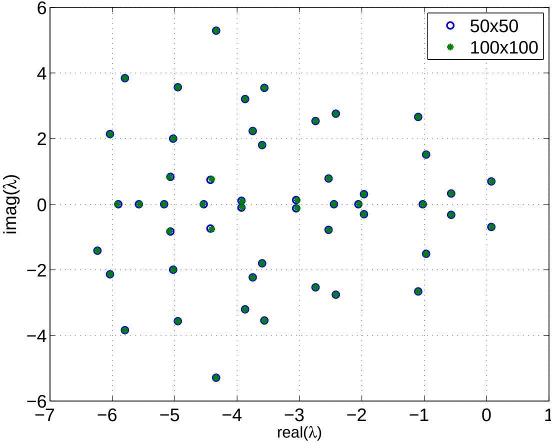
Save in Matlab readable format

```
import scipy.io as sio
sio.savemat('linear.mat', mdict={'M':M, 'A':A, 'B':B},
            oned_as='column')
```

Compute eigenvalues/vectors within python (using Scipy)

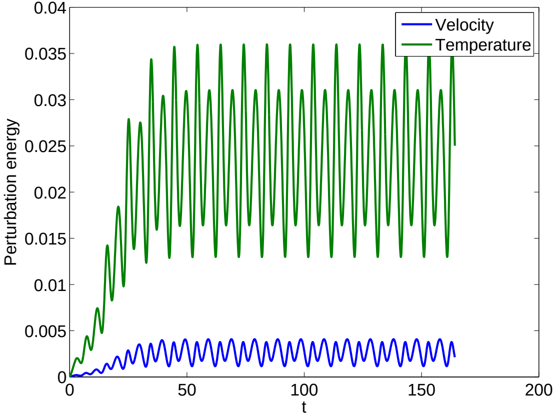
```
import scipy.sparse.linalg as la
vals, vecs = la.eigs(A, k=10, M=M, sigma=-1.0, which='LR')
```

Recall the eigenvalues



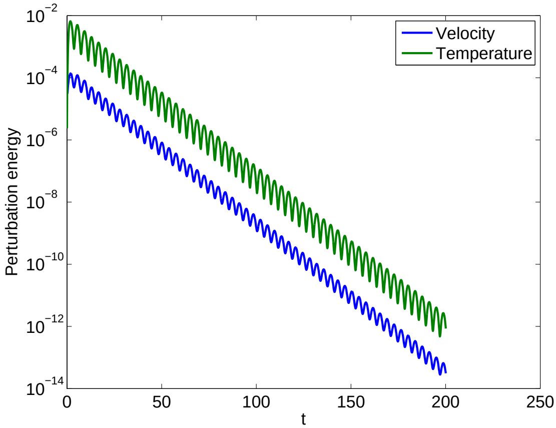
Case 1: $y_c = 0.7$

Stabilizing the two unstable eigenvalues.



Case 2: $y_c = 0.7$

Add shift $s = 1$ to the two unstable eigenvalues



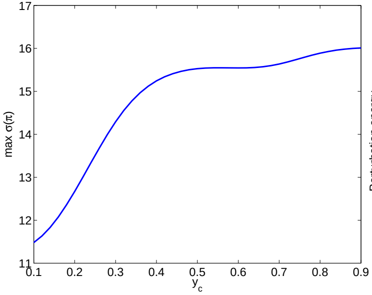
Best control location

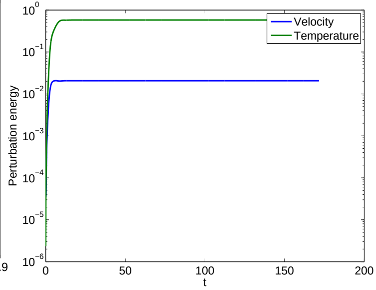
- Freedom to choose location of control zone y_c
- Norm of control

$$\|u\|^2 = z(0)^\top \pi(y_c) z(0)$$

- Minimize control norm for all possible initial conditions

$$y_{c,opt} = \text{argmin} \max \sigma(\pi(y_c))$$

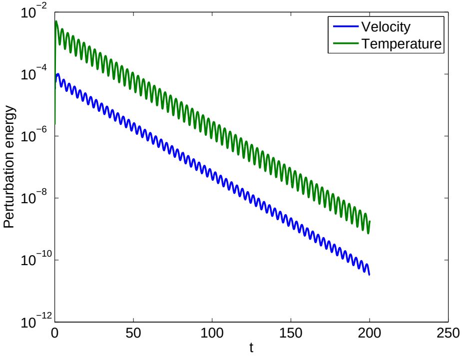




Case 3: (a) Dependance of $\max \sigma(\pi(y_c))$ wrt y_c. The minimum is attained at $y_c = 0.1$, (b) Evolution of perturbation energy

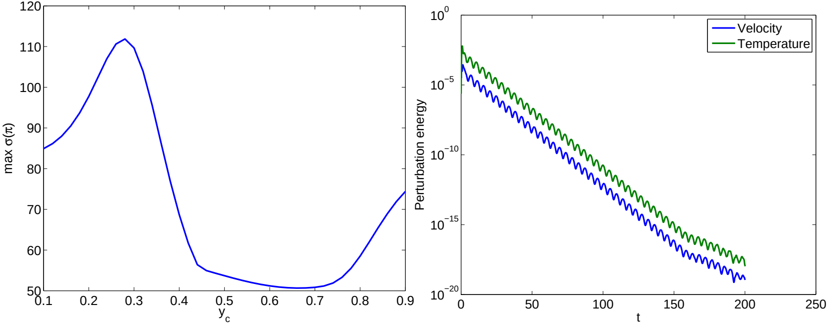
Case 4: $y_c = 0.7$

Add shift $s = 1$ to four leading eigenvalues.



Case 5

Add shift $s = 1$ to four leading eigenvalues.



Case 5: (a) Dependance of $\max \sigma(\pi(y_c))$ wrt y_c; minimum is attained at $y_c = 0.65$ (b) Evolution of perturbation energy with feedbackat $y_c = 0.65$

Getting the code

All the code is available here to browse (click below)
https://code.google.com/p/cfdlab/source/browse/trunk/fenics/2d/ns_control/

Use svn to get the code

```
svn co http://cfdlab.googlecode.com/svn/trunk/fenics/2d/↩
    ns_control
```

Extended system

Parameterize the boundary control as

$$v_c = w_1 \alpha(y), \qquad \theta_c = w_2 \beta(y), \qquad f_c = w_3 \gamma(x)$$

FE solution

$$\begin{aligned}
\boldsymbol{v}_h(x,t) &= \sum_{j \in I_f^{\boldsymbol{v}}} v_j(t) \boldsymbol{\varphi}_j(x) + w_1(t) \sum_{j \in I_{in}^{\boldsymbol{v}}} \alpha_j \boldsymbol{\varphi}_j(x) \\
\theta_h(x,t) &= \sum_{j \in I_f^{\theta}} \theta_j(t) \chi_j(x) + w_2(t) \sum_{j \in I_{in}^{\theta}} \beta_j \chi_j(x) \\
p_h(x,t) &= \sum_{j \in I^p} p_j(t) \psi_j(x)
\end{aligned}$$

We introduce models for w_1, w_2, w_3

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = r_i(w_i - u_i), \qquad i = 1, 2, 3$$

where (u_1, u_2, u_3) are our new control variables.

Momentum eqn

$$(\partial_t \boldsymbol{v}_h, \boldsymbol{\varphi}_i) + \mathcal{A}_{vv}(\boldsymbol{v}_h, \boldsymbol{\varphi}_i) + \mathcal{A}_{vp}(p_h, \boldsymbol{\varphi}_i) + \mathcal{A}_{v\theta}(\theta_h, \boldsymbol{\varphi}_i) = 0, \qquad \forall\ i \in I_f^{\boldsymbol{v}}$$

$$\mathcal{A}_{vp}(\psi_i, \boldsymbol{v}_h) = 0, \qquad \forall\ i \in I^p$$

Substituting $\boldsymbol{v}_h, \theta_h, p_h$ we obtain

$$\begin{aligned}
\sum_{j \in I_f^{\boldsymbol{v}}} (\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i)\, \dot{v}_j = & - \sum_{j \in I_f^{\boldsymbol{v}}} \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) v_j - \sum_{j \in I_f^{\theta}} \mathcal{A}_{v\theta}(\chi_j, \boldsymbol{\varphi}_i)\theta_j - \sum_{j \in I^p} \mathcal{A}_{vp}(\psi_j, \boldsymbol{\varphi}_i) p_j \\
& - r_1 w_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \alpha_j\, (\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) - w_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \alpha_j \mathcal{A}_{vv}(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i) \\
& - w_2 \sum_{j \in I_{in}^{\theta}} \beta_j \mathcal{A}_{v\theta}(\chi_j, \boldsymbol{\varphi}_i) \\
& + r_1 u_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \alpha_j\, (\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i), \qquad \forall\ i \in I_f^{\boldsymbol{v}} \\
0 = & - \sum_{j \in I_f^{\boldsymbol{v}}} \mathcal{A}_{vp}(\psi_i, \boldsymbol{\varphi}_j) v_j - w_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \mathcal{A}_{vp}(\psi_i, \boldsymbol{\varphi}_j)\alpha_j, \qquad \forall\ i \in I^p
\end{aligned}$$

Momentum eqn

Written in matrix-vector form

$$
\begin{aligned}
M_v \frac{\mathrm{d}v}{\mathrm{d}t} =& A_{vv} v + A_{v\theta}\theta + A_{vw} w + A_{vp} p + B_v u \\
0 =& A_{vp}^\top v + A_{pw} w
\end{aligned}
$$

Heat eqn

$$(\partial_t \theta_h, \chi_i) + \mathcal{A}_{\theta v}(\boldsymbol{v}_h, \chi_i) + \mathcal{A}_{\theta\theta}(\theta_h, \chi_i) - (f_c, \chi_i)_{\Gamma_h} = 0, \qquad \forall\ i \in I_f^\theta$$

Substituting $\boldsymbol{v}_h, \theta_h, p_h$ we obtain

$$\begin{aligned}
\sum_{j \in I_f^\theta} (\chi_j, \chi_i)\, \dot{\theta}_j = &- \sum_{j \in I_f^{\boldsymbol{v}}} \mathcal{A}_{\theta v}(\boldsymbol{\varphi}_j, \chi_i) v_j - \sum_{j \in I_f^\theta} \mathcal{A}_{\theta\theta}(\chi_j, \chi_i)\theta_j \\
&- w_1 \sum_{j \in I_{in}^{\boldsymbol{v}}} \mathcal{A}_{\theta v}(\boldsymbol{\varphi}_j, \chi_i)\alpha_j - r_2 w_2 \sum_{j \in I_{in}^\theta} (\chi_j, \chi_i)\, \beta_j \\
&- w_2 \sum_{j \in I_{in}^\theta} \mathcal{A}_{\theta\theta}(\chi_j, \chi_i)\beta_j + r_2 u_2 \sum_{j \in I_{in}^\theta} (\chi_j, \chi_i)\, \beta_j \\
&+ w_3\, (\gamma, \chi_i)_{\Gamma_h}
\end{aligned}$$

This can be written in matrix-vector form as

$$M_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} = A_{\theta v} v + A_{\theta\theta}\theta + A_{\theta w} w + B_\theta u$$

State-space model

$$
\begin{bmatrix} M_v & 0 & 0 & 0 \\ 0 & M_\theta & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v \\ \theta \\ w \\ p \end{bmatrix} = \begin{bmatrix} A_{vv} & A_{v\theta} & A_{vw} & A_{vp} \\ A_{\theta v} & A_{\theta\theta} & A_{\theta w} & 0 \\ 0 & 0 & A_{ww} & 0 \\ A_{vp}^\top & 0 & A_{pw} & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \\ w \\ p \end{bmatrix} + \begin{bmatrix} B_v \\ B_\theta \\ -A_{ww} \\ 0 \end{bmatrix} u \tag{12}
$$

where

$$
A_{ww} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}
$$

Discrete Leray projection

Decompose the velocity as

$$v = v_1 + v_2$$

where

$$v_1 = P_v^\top v, \qquad v_2 = (I - P_v)^\top v$$

and

$$P_v = I_{n_v} - A_{vp}(A_{vp}^\top M_v^{-1} A_{vp})^{-1} A_{vp}^\top M_v^{-1}$$

Discrete Leray projection

$$
\begin{aligned}
P_v M_v \frac{\mathrm{d}v_1}{\mathrm{d}t} &= P_v A_{vv} v_1 + P_v A_{v\theta}\theta + P_v \tilde{A}_{vw} w + P_v B_v u \\
M_\theta \frac{\mathrm{d}\theta}{\mathrm{d}t} &= A_{\theta v} v_1 + A_{\theta\theta}\theta + \tilde{A}_{\theta w} w + B_\theta u \\
\frac{\mathrm{d}w}{\mathrm{d}t} &= A_{ww} w - A_{ww} u
\end{aligned} \tag{13}
$$

where

$$
\tilde{A}_{vw} = A_{vw} - A_{vv} M_v^{-1} A_{vp} (A_{vp}^\top M_v^{-1} A_{vp})^{-1} A_{pw}
$$

$$
\tilde{A}_{\theta w} = A_{\theta w} - A_{\theta v} M_v^{-1} A_{vp} (A_{vp}^\top M_v^{-1} A_{vp})^{-1} A_{pw}
$$

The pressure is given by

$$
\begin{aligned}
A_{vp}^\top M_v^{-1} A_{vp} p = &- A_{vp}^\top M_v^{-1} (A_{vv} v + A_{v\theta}\theta) - (A_{vp}^\top M_v^{-1} A_{vw} + A_{pw} A_{ww}) w \\
&- (A_{vp}^\top M_v^{-1} B_v + A_{pw}) u
\end{aligned}
$$

State-space model

Define

$$y = \begin{bmatrix} v \\ \theta \\ w \end{bmatrix}, \quad E_{11} = \begin{bmatrix} M_v & 0 & 0 \\ 0 & M_\theta & 0 \\ 0 & 0 & I_{n_w} \end{bmatrix}, \quad \tilde{A}_{11} = \begin{bmatrix} A_{vv} & A_{v\theta} & \tilde{A}_{vw} \\ A_{\theta v} & A_{\theta\theta} & \tilde{A}_{\theta w} \\ 0 & 0 & A_{ww} \end{bmatrix}$$

$$P = \begin{bmatrix} P_v & 0 & 0 \\ 0 & I_{n_\theta} & 0 \\ 0 & 0 & I_{n_w} \end{bmatrix}, \quad B = \begin{bmatrix} B_v \\ B_\theta \\ -A_{ww} \end{bmatrix}$$

Discrete Leray projection: $y = y_1 + y_2$, $y_1 = P^\top y$, $y_2 = (I - P)^\top y$

$$PE_{11} \frac{\mathrm{d}y_1}{\mathrm{d}t} = P\tilde{A}_{11} y_1 + PBu$$

Projection to unstable subspace, feedback law

- Projector onto unstable subspace: $Q = V_y Z_y^\top E_{11}$
- Qy_1 satisfies the equation

$$Z_y^\top E_{11} \frac{\mathrm{d}y}{\mathrm{d}t} = Z_y^\top \tilde{A}_{11} y + Z_y^\top B u$$

- $z = Z_y^\top E_{11} y_1$ satisfies the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Z_y^\top \tilde{A}_{11} V_y z + Z_y^\top B u$$

- π: solution of associated Bernoulli equation. Feedback law is given by

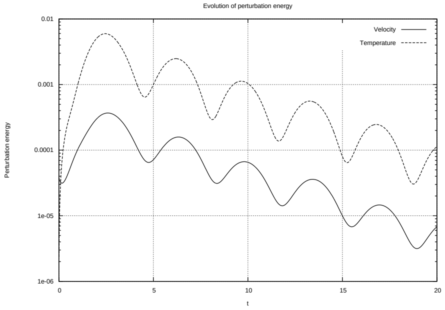
$$u = -Kz = -B^\top Z_y \pi z$$

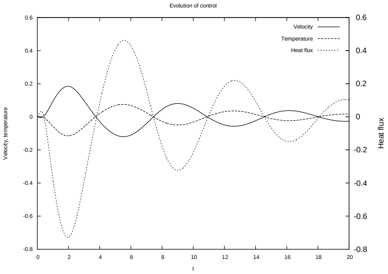
or

$$u = -(B^\top Z_y \pi Z_y^\top E_{11}) y =: -\tilde{K} y$$

Numerical example

$$y(0) = y_s + 0.01 y_e, \qquad n_u = 2, \qquad s = 1.0, \qquad r_1 = r_2 = r_3 = -\frac{1}{2}$$





Summary

- Applied the feedback stabilization approach to Boussinesq
  - Dirichlet controls
  - Proper handling of divergence constraint
  - Feedback law computed efficiently
  - Extended system approach
- Ongoing/future work
  - Theoretical analysis of stabilizability
  - More realistic model/boundary conditions
  - Stabilization using partial information
  - Better numerics