# Arbitrary (Almost) Lagrangian-Eulerian Discontinuous Galerkin method for 1-D Euler equations

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# Euler equations in 1-D

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}, \qquad \boldsymbol{f}(\boldsymbol{u}) = \begin{bmatrix} \rho v \\ p + \rho v^2 \\ \rho H v \end{bmatrix}$$

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2, \qquad H = (E + p)/\rho$$

# KH instability using fixed mesh finite volume

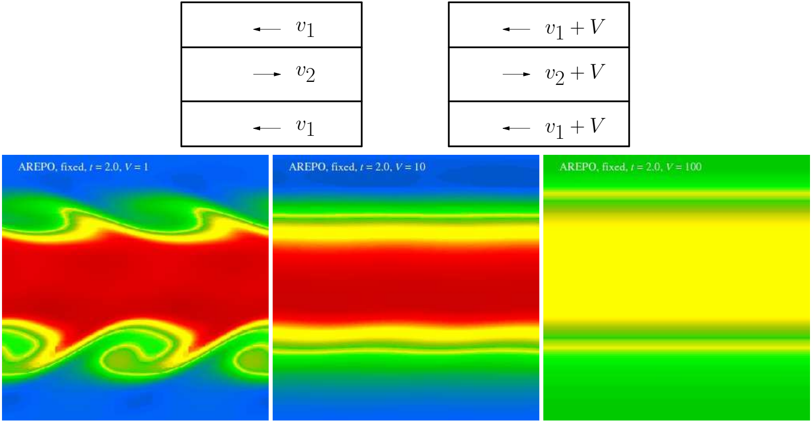



**Figure 33.** Kelvin Helmholtz instability test at time $t = 2.0$, computed with AREPO with a fixed mesh. In the three cases, different boost velocities along both the $x$- and $y$- directions have been applied. The fact that the results do not agree (and in particular not with the $V = 0$ result shown in the bottom of Figure 32) is direct evidence for a violation of Galilean invariance of the Eulerian approach. We note that we have obtained nearly identical results for this test when it is carried out with ATHENA instead of our code AREPO.

From Volker Springel, `https://arxiv.org/abs/0901.4107`

# Dissipation in upwind schemes

Upwind scheme for $u_t + au_x = 0$, modified PDE

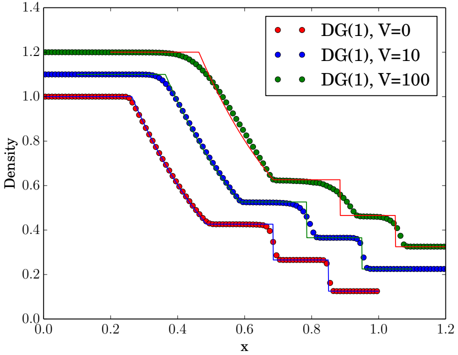
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{1}{2}|a|h(1-\nu)\frac{\partial^2 u}{\partial x^2} + O(h^2), \qquad \nu = \frac{|a|\Delta t}{h}$$

For Euler equations: $|v-c|$, $|v|$, $|v|+c$

Sod problem

$$(\rho, v, p) = \begin{cases} (1.000, V, 1.0) & x < 0.5 \\ (0.125, V, 0.1) & x > 0.5 \end{cases}$$

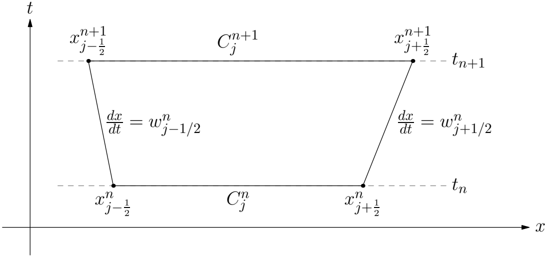
Roe flux, 100 cells, TVD limiter



# Mesh

Moving cell $C_j(t) = (x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t))$

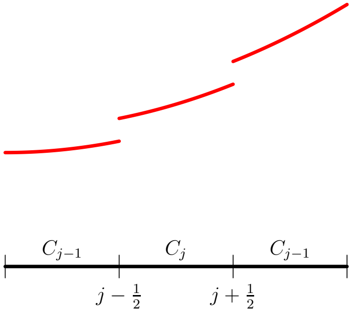
$$\text{Mesh velocity :} \quad \frac{\mathrm{d}}{\mathrm{d}t} x_{j+\frac{1}{2}}(t) = w_{j+\frac{1}{2}}(t) = w^n_{j+\frac{1}{2}}, \qquad t_n < t < t_{n+1}$$




$$x_j(t) = \frac{1}{2}(x_{j-\frac{1}{2}}(t) + x_{j+\frac{1}{2}}(t)), \qquad h_j(t) = x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t)$$

$$w(x,t) = \frac{x_{j+\frac{1}{2}}(t) - x}{h_j(t)} w^n_{j-\frac{1}{2}} + \frac{x - x_{j-\frac{1}{2}}(t)}{h_j(t)} w^n_{j+\frac{1}{2}}, \quad (x,t) \in C_j(t) \times (t_n, t_{n+1})$$

# Solution space




Degree $k$ piecewise polynomial solution

$$\boldsymbol{u}_h(x,t) = \sum_{m=0}^{k} \boldsymbol{u}_{j,m}(t)\varphi_m(x,t), \quad x \in C_j(t)$$

Legendre basis functions: $\xi = \frac{x - x_j(t)}{\frac{1}{2}h_j(t)}$

$$\varphi_m(x,t) = \hat{\varphi}_m(\xi) = \sqrt{2m+1}P_m(\xi)$$

# ALE-DG scheme

*ALE flux*

$$\boldsymbol{g}(\boldsymbol{u}, w) = \boldsymbol{f}(\boldsymbol{u}) - w\boldsymbol{u}$$

Numerical ALE flux

$$\hat{\boldsymbol{g}}_{j+\frac{1}{2}}(t) := \hat{\boldsymbol{g}}_{j+\frac{1}{2}}(\boldsymbol{u}_h(t)) := \hat{\boldsymbol{g}}(\boldsymbol{u}^-_{j+\frac{1}{2}}(t), \boldsymbol{u}^+_{j+\frac{1}{2}}(t), w_{j+\frac{1}{2}}(t))$$

$l$'th moment

$$\begin{aligned} h_j^{n+1}\boldsymbol{u}_{j,l}^{n+1} &= h_j^n \boldsymbol{u}_{j,l}^n + \int_{t_n}^{t_{n+1}} \int_{x_{j-\frac{1}{2}}(t)}^{x_{j+\frac{1}{2}}(t)} \boldsymbol{g}(\boldsymbol{u}_h, w) \frac{\partial}{\partial x} \varphi_l(x,t) \mathrm{d}x \mathrm{d}t \\ &\quad + \int_{t_n}^{t_{n+1}} [\hat{\boldsymbol{g}}_{j-\frac{1}{2}}(t) \varphi_l(x^+_{j-\frac{1}{2}}, t) - \hat{\boldsymbol{g}}_{j+\frac{1}{2}}(t) \varphi_l(x^-_{j+\frac{1}{2}}, t)] \mathrm{d}t \end{aligned}$$

Fully discrete scheme

- Replace $u_h$ with a locally predicted solution $\boldsymbol{U}_h$
- Quadrature in space and time

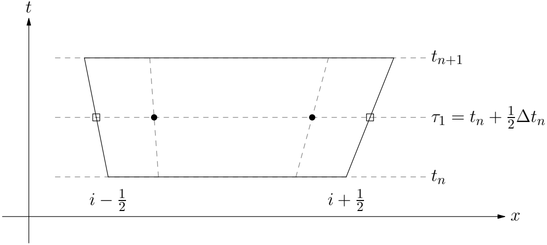
# Mesh velocity

$$\tilde{w}^n_{j+\frac{1}{2}} = \frac{1}{2}[v(x^-_{j+\frac{1}{2}}, t_n) + v(x^+_{j+\frac{1}{2}}, t_n)]$$

A bit of smoothing

$$w^n_{j+\frac{1}{2}} = \frac{1}{3}(\tilde{w}^n_{j-\frac{1}{2}} + \tilde{w}^n_{j+\frac{1}{2}} + \tilde{w}^n_{j+\frac{3}{2}})$$

# Predictor for $k = 1$: Taylor expansion in space-time

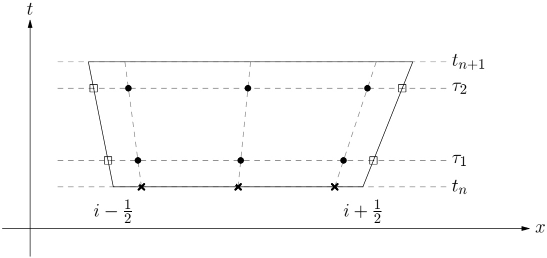



$$\boldsymbol{U}(x_q, \tau_1) = \boldsymbol{u}_h(x_j^n, t_n) + (\tau_1 - t_n)\frac{\partial \boldsymbol{u}_h}{\partial t}(x_j^n, t_n) + (x_q - x_j^n)\frac{\partial \boldsymbol{u}_h}{\partial x}(x_j^n, t_n)$$

Using conservation law, $\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{\partial \boldsymbol{f}}{\partial x} = -A\frac{\partial \boldsymbol{u}}{\partial x}$

$$\boldsymbol{U}_h(x_q, \tau_1) = \boldsymbol{u}_h^n(x_j^n) - (\tau_1 - t_n)\left[A(\boldsymbol{u}_h^n(x_j^n)) - w_q I\right]\frac{\partial \boldsymbol{u}_h^n}{\partial x}(x_j^n)$$

# Predictor for $k = 2$: Continuous expansion RK




$$\frac{\mathrm{d}\boldsymbol{U}_m}{\mathrm{d}t} = -[A(\boldsymbol{U}_m(t)) - w_m(t)I]\frac{\partial}{\partial x}\boldsymbol{U}_h(x_m, t) =: \boldsymbol{K}_m(t)$$

$$\boldsymbol{U}_m(t_n) = \boldsymbol{u}_h(x_m, t_n)$$

Using continuous expansion RK: for $m = 0, 1, \ldots, k$

$$\boldsymbol{U}_m(t) = \boldsymbol{u}_h(x_m, t_n) + \sum_{s=1}^{n_s} b_s((t - t_n)/\Delta t_n)\boldsymbol{K}_{m,s}, \quad t \in [t_n, t_{n+1})$$

where $\boldsymbol{K}_{m,s} = \boldsymbol{K}_m(t_n + \theta_s \Delta t_n)$, $\theta_s \Delta t_n$ is the stage time

# Numerical flux for $\boldsymbol{g}(\boldsymbol{u}, w) = \boldsymbol{f}(\boldsymbol{u}) - w\boldsymbol{u}$

**Rusanov flux**

$$\hat{\boldsymbol{g}}(\boldsymbol{u}_l, \boldsymbol{u}_r, w) = \frac{1}{2}[\boldsymbol{g}(\boldsymbol{u}_l, w) + \boldsymbol{g}(\boldsymbol{u}_r, w)] - \frac{1}{2}\lambda_{lr}(\boldsymbol{u}_r - \boldsymbol{u}_l)$$

$$\lambda_{lr} = \max\{|v_l - w| + c_l, |v_r - w| + c_r\}$$

**Roe flux**

$$\hat{\boldsymbol{g}}(\boldsymbol{u}_l, \boldsymbol{u}_r, w) = \frac{1}{2}[\boldsymbol{g}(\boldsymbol{u}_l, w) + \boldsymbol{g}(\boldsymbol{u}_r, w)] - \frac{1}{2}|A_w|(\boldsymbol{u}_r - \boldsymbol{u}_l)$$

$A_w = A_w(\boldsymbol{u}_l, \boldsymbol{u}_r)$ satisfies $\boldsymbol{g}(\boldsymbol{u}_r, w) - \boldsymbol{g}(\boldsymbol{u}_l, w) = A_w(\boldsymbol{u}_r - \boldsymbol{u}_l)$

$$|A_w| = R|\Lambda - wI|R^{-1}$$

$R$, $\Lambda$ evaluated at average state $\boldsymbol{u}(\bar{\boldsymbol{q}})$, $\bar{\boldsymbol{q}} = \frac{1}{2}(\boldsymbol{q}_l + \boldsymbol{q}_r)$, where $\boldsymbol{q} = \sqrt{\rho}[1,\ v,\ H]^\top$ is the parameter vector introduced by Roe.

# Limiting

TVD and TVB limiters applied to characteristic variables

# Positivity property

First order scheme

$$h_j^{n+1}\bar{\boldsymbol{u}}_j^{n+1} = h_j^n\bar{\boldsymbol{u}}_j^n - \Delta t_n[\hat{\boldsymbol{g}}_{j+\frac{1}{2}}^n - \hat{\boldsymbol{g}}_{j-\frac{1}{2}}^n]$$

Restriction of time step to control change in cell size

$$|w_{j+\frac{1}{2}}^n - w_{j-\frac{1}{2}}^n|\Delta t_n \le \beta h_j^n, \qquad \text{e.g. } \beta = 0.1$$

## Positivity of first order scheme

The first order scheme with Rusanov flux is positivity preserving if the time step condition

$$\Delta t_n \le \Delta t_n^{(1)} := \min_j \left\{ \frac{(1-\frac{1}{2}\beta)h_j^n}{\frac{1}{2}(\lambda_{j-\frac{1}{2}}^n + \lambda_{j+\frac{1}{2}}^n)}, \frac{\beta h_j^n}{|w_{j+\frac{1}{2}}^n - w_{j-\frac{1}{2}}^n|} \right\}$$

is satisfied.

# Order of accuracy

Initial condition

$$\rho(x,0) = 1 + \exp(-10x^2), \qquad u(x,0) = 1, \qquad p(x,0) = 1$$

Exact solution

$$\rho(x,t) = \rho(x-t,0), \qquad u(x,t) = 1, \qquad p(x,t) = 1$$

The initial domain is $[-5, +5]$ and the final time is $t = 1$ units.

# Order of accuracy: Rusanov flux

| $N$ | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate |
| 100 | 4.370E-02 | - | 3.498E-03 | - | 3.883E-04 | - |
| 200 | 6.611E-03 | 2.725 | 4.766E-04 | 2.876 | 1.620E-05 | 4.583 |
| 400 | 1.332E-03 | 2.518 | 6.415E-05 | 2.885 | 9.376E-07 | 4.347 |
| 800 | 3.151E-04 | 2.372 | 8.246E-06 | 2.910 | 5.763E-08 | 4.239 |
| 1600 | 7.846E-05 | 2.280 | 1.031E-06 | 2.932 | 3.595E-09 | 4.180 |

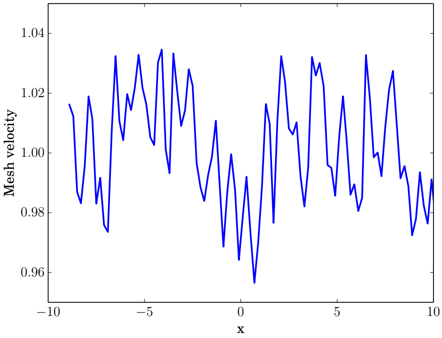
Static mesh

| $N$ | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate |
| 100 | 2.331E-02 | - | 3.979E-03 | - | 8.633E-04 | - |
| 200 | 6.139E-03 | 1.925 | 4.058E-04 | 3.294 | 1.185E-05 | 6.186 |
| 400 | 1.406E-03 | 2.0258 | 5.250E-05 | 3.122 | 7.079E-07 | 5.126 |
| 800 | 3.375E-04 | 2.0366 | 6.626E-06 | 3.077 | 4.340E-08 | 4.760 |
| 1600 | 8.278E-05 | 2.0344 | 8.304E-07 | 3.057 | 2.689E-09 | 4.573 |

Moving mesh

# Randomized mesh velocity: HLLC flux

Example of randomized velocity distribution for smooth test case

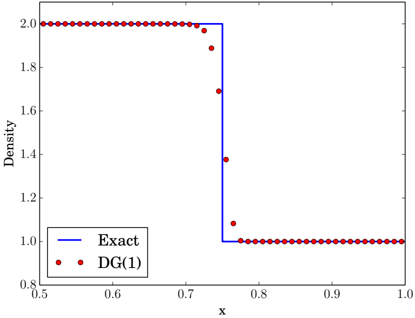



| $N$ | $k = 1$ | | $k = 2$ | | $k = 3$ | |
|---|---|---|---|---|---|---|
| | Error | Rate | Error | Rate | Error | Rate |
| 100 | 1.735E-02 | - | 1.798E-03 | - | 2.351E-04 | - |
| 200 | 4.179E-03 | 2.051 | 2.848E-04 | 2.676 | 1.416E-05 | 4.069 |
| 400 | 1.054E-03 | 2.035 | 4.301E-05 | 2.703 | 8.578E-07 | 4.041 |
| 800 | 2.615E-04 | 1.943 | 6.012E-06 | 2.838 | 5.476E-08 | 3.958 |
| 1600 | 7.279E-05 | 1.852 | 8.000E-07 | 2.909 | 3.505E-09 | 3.966 |

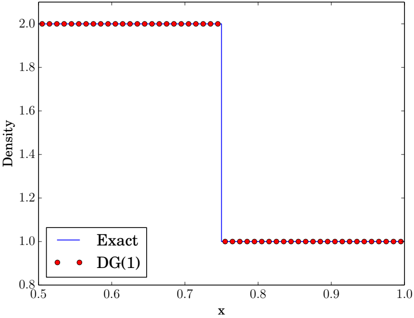
# Single contact wave

$$(\rho, v, p) = \begin{cases} (2.0, 1.0, 1.0) & \text{if } x < 0.5 \\ (1.0, 1.0, 1.0) & \text{if } x > 0.5 \end{cases}$$

Roe flux with 100 cells




static mesh

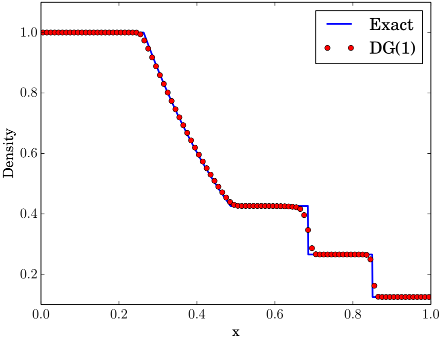



moving mesh

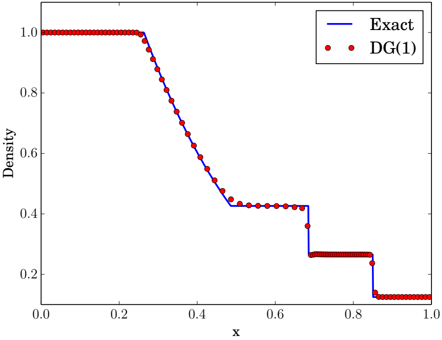
# Sod test: $x \in [0, 1]$, $T = 0.2$

$$(\rho, v, p) = \begin{cases} (1.000, 0.0, 1.0) & \text{if } x < 0.5 \\ (0.125, 0.0, 0.1) & \text{if } x > 0.5 \end{cases}$$

Roe flux, 100 cells and TVD limiter




static mesh

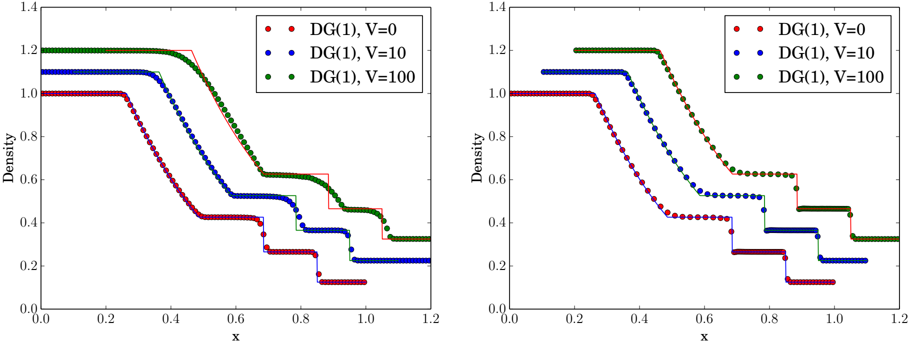



moving mesh

# Sod test: $x \in [0, 1]$, $T = 0.2$

$$(\rho, v, p) = \begin{cases} (1.000, V, 1.0) & \text{if } x < 0.5 \\ (0.125, V, 0.1) & \text{if } x > 0.5 \end{cases}$$

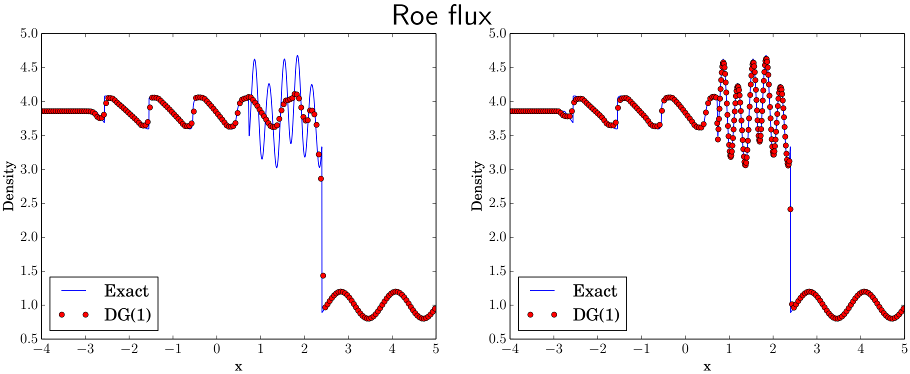
Roe flux, 100 cells and TVD limiter




static mesh | moving mesh

# Shu-Osher problem: $x \in [-5, +5]$, $T = 1.8$

$$(\rho, v, p) = \begin{cases} (3.857143, 2.629369, 10.333333) & \text{if } x < -4 \\ (1 + 0.2\sin(5x), 0.0, 1.0) & \text{if } x > -4 \end{cases}$$

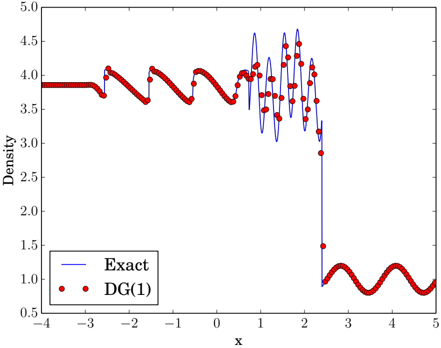



static mesh, 200 cells, $M = 0$

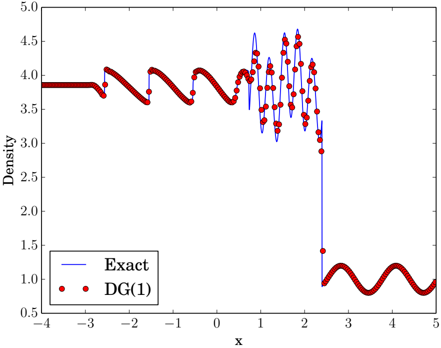
moving mesh, 200 cells, $M = 0$

# Shu-Osher problem: $x \in [-5, +5]$, $T = 1.8$

Roe flux



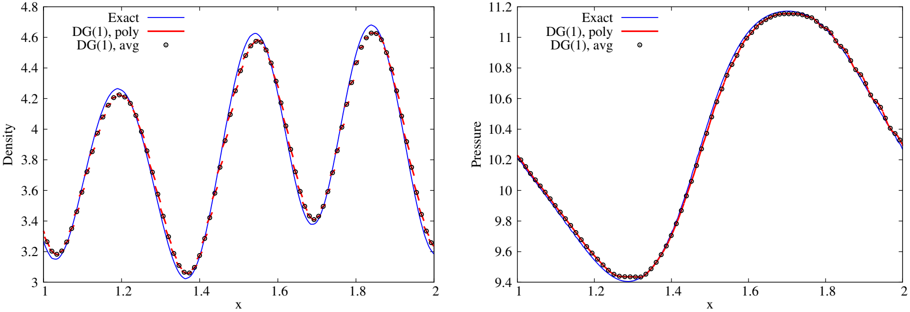

static mesh, 200 cells, $M = 100$




static mesh, 300 cells, $M = 100$

# Shu-Osher problem: $x \in [-5, +5]$, $T = 1.8$

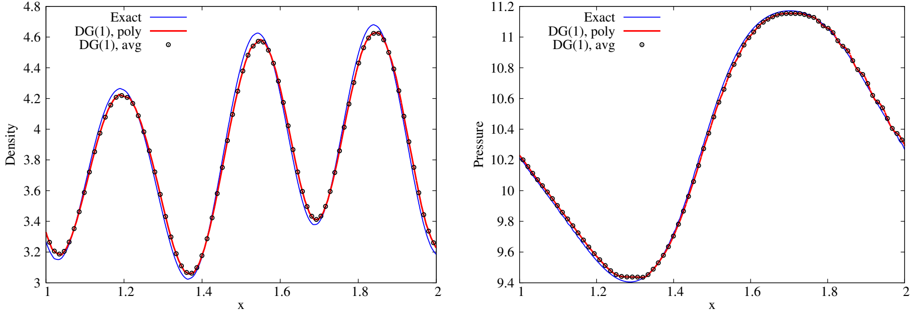
Roe flux on moving mesh



# Shu-Osher problem: $x \in [-5, +5]$, $T = 1.8$

$$|\lambda_2| = \begin{cases} |v - w| & \text{if } |v - w| > \delta = \alpha c \\ \frac{1}{2}(\delta + |v - w|^2/\delta) & \text{otherwise} \end{cases}$$

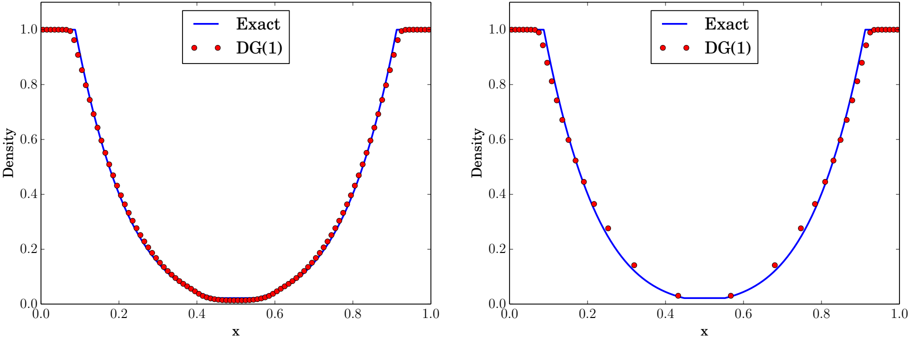
Modified Roe flux on moving mesh



# 123 problem: $x \in [0, 1]$, $T = 0.15$

$$(\rho, v, p) = \begin{cases} (1.0, -2.0, 0.4) & \text{if } x < 0.5 \\ (1.0, +2.0, 0.4) & \text{if } x > 0.5 \end{cases}$$

HLLC flux using 100 cells

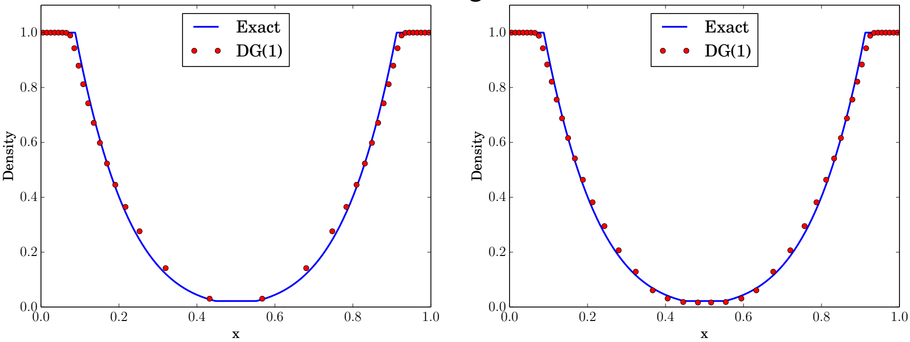



static mesh

moving mesh

# 123 problem: $x \in [0,1]$, $T = 0.15$, $h_{\max} = 0.05$

HLLC flux using 100 cells



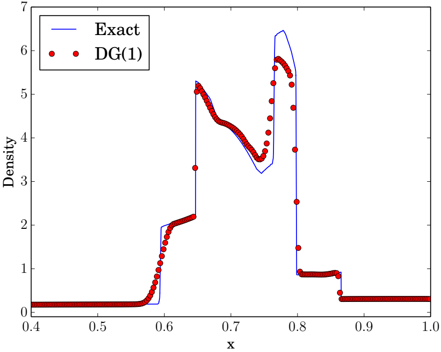

moving mesh

moving mesh + adaptation
108 cells at final time

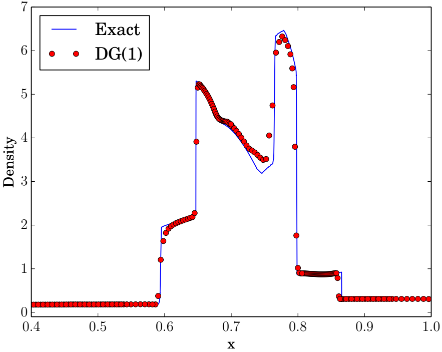
# Blast test: $x \in [0,1]$, $T = 0.038$, $h_{\min} = 10^{-3}$

$$(\rho, v, p) = \begin{cases} (1.0, 0.0, 10^{+3}) & \text{if } x < 0.1 \\ (1.0, 0.0, 10^{-2}) & \text{if } 0.1 < x < 0.9 \\ (1.0, 0.0, 10^{+2}) & \text{if } x > 0.9 \end{cases}$$

HLLC flux using 400 cells




static mesh




moving mesh

303 cells at final time

# Summary

- ALE-DG scheme with almost Lagrangian nature
- Simple, local specification of mesh velocity
- Galilean invariant solutions
- Very low dissipation in contact waves
- Single step time integration
- Satisfies geometric conservation law
- Ongoing work
  - Extension to 2-D on triangular grids

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# Thank You