Discontinuous Galerkin method for rotating shallow water equation on the sphere

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Shallow water model[1,2]

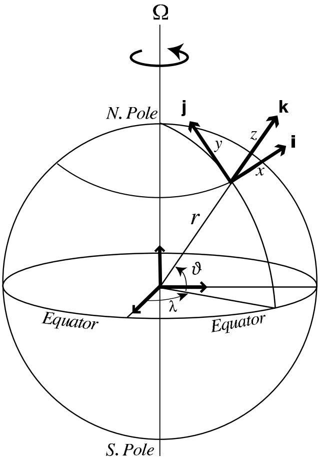
- Shallow atmosphere/ocean
  - $3/4$ of total mass within 11 km
  - Circumference = 40,000 km
- Assumptions
  - Incompressible flow
  - Hydrostatic equilibrium in radial direction

[1] Pedlosky, Geophysical Fluid Dynamics

[2] Vallis, Atmospheric and Oceanic Fluid Dynamics

Rotation, spherical coordinates

Fig. 2.3 The spherical coordinate system. The orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.



from *Vallis, Atmospheric and Oceanic Fluid Dynamics*

Radius: $R = 6.37122 \times 10^6\ m$, Rotation rate: $\Omega = 7.292 \times 10^{-5} s^{-1}$

Acc. due to gravity: $g = 9.80616\ m \cdot s^{-2}$

Rotating shallow water model

$$\boldsymbol{v} = \text{velocity}, \qquad H = \text{height of atm. rel. to mean surface}$$

$$H_s = \text{height of ground rel to mean surface}, \qquad D = H - H_s = \text{depth}$$

Vector invariant form

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + \nabla\Phi + (\omega + f)\boldsymbol{v}^{\perp} &= 0 \\ \frac{\partial D}{\partial t} + \nabla \cdot (\boldsymbol{v} D) &= 0 \end{aligned}$$

where

$$\Phi = gH + K, \qquad K = \frac{1}{2}|\boldsymbol{v}|^2, \qquad \omega = \mathbf{k} \cdot \nabla \times \boldsymbol{v}$$

$$\boldsymbol{v}^{\perp} = \mathbf{k} \times \boldsymbol{v}, \qquad f = 2\Omega \sin\theta$$

Energy is conserved

Conservation law

$$\frac{\partial}{\partial t}\left(\frac{1}{2}D|\boldsymbol{v}|^2+\frac{1}{2}gH^2\right)+\nabla\cdot\left[\left(gH+\frac{1}{2}|\boldsymbol{v}|^2\right)\boldsymbol{v}D\right]=0$$

Total energy is conserved

$$\int_S\left(\frac{1}{2}D|\boldsymbol{v}|^2+\frac{1}{2}gH^2\right)\mathrm{d}s=\mathrm{const.}$$

Vorticity dynamics

Absolute vorticity

$$\eta := \omega + f$$

Vorticity equation

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\boldsymbol{v}\eta) = 0$$

Potential vorticity

$$q := \frac{\eta}{D}$$

PV is advected by the flow

$$\frac{\partial q}{\partial t} + \boldsymbol{v} \cdot \nabla q = 0$$

q is constant along streamlines

Potential enstrophy is conserved

Another conservation law

$$\frac{\partial}{\partial t}\left(\frac{\eta^2}{D}\right) + \nabla \cdot \left(\frac{\eta^2}{D}\boldsymbol{v}\right) = 0$$

Potential enstrophy is conserved

$$\int_S \frac{\eta^2}{D}\mathrm{d}s = \int_S Dq^2\mathrm{d}s = \text{const.}$$

Infinite number of conserved quantities (Casimirs)

$$\int_S DF(q)\mathrm{d}s$$

for any function F, e.g.,

$$\int_S Dq^i\mathrm{d}s = \text{const.}, \qquad i = 0, 1, 2, \dots,$$

Conservation form

Momentum and mass conservation

$$
\begin{aligned}
\frac{\partial}{\partial t}(D\boldsymbol{v}) + \nabla \cdot \left(D\boldsymbol{v} \otimes \boldsymbol{v} + \frac{1}{2} g D^2 \boldsymbol{I} \right) &= -f(\mathbf{k} \times D\boldsymbol{v}) - gD\nabla H_s \\
\frac{\partial D}{\partial t} + \nabla \cdot (D\boldsymbol{v}) &= 0
\end{aligned}
$$

Finite volume, SEM, DG

Vorticity-divergence form (Z grid)

Tangential divergence: $\delta = \nabla \cdot \boldsymbol{v}$

$$
\begin{aligned}
\frac{\partial D}{\partial t} + \nabla \cdot (\boldsymbol{v} D) &= 0 \\
\frac{\partial \eta}{\partial t} + \nabla \cdot (\boldsymbol{v} \eta) &= 0 \\
\frac{\partial \delta}{\partial t} - \nabla \times (\boldsymbol{v} \eta) + \nabla^2 (gH + |\boldsymbol{v}|^2/2) &= 0 \\
-\nabla^2 \psi &= -(\eta - f) \\
-\nabla^2 \chi &= -\delta \\
\boldsymbol{v} &= \nabla \chi + \nabla^{\perp} \psi
\end{aligned}
$$

Usually solved with global pseudo-spectral methods[3] or finite volume method

[3] Hack & Jakob (1992)

Hyperbolic property

In Cartesian coordinates, with the z-axis along the axis of rotation

$$\frac{\partial \boldsymbol{U}}{\partial t} + A_1 \frac{\partial \boldsymbol{U}}{\partial x} + A_2 \frac{\partial \boldsymbol{U}}{\partial y} + \tilde{\boldsymbol{S}} = 0 \tag{1}$$

where

$$\boldsymbol{U} = \begin{bmatrix} v_1 \\ v_2 \\ D \end{bmatrix}, \quad A_1 = \begin{bmatrix} v_1 & 0 & g \\ 0 & v_1 & 0 \\ D & 0 & v_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} v_2 & 0 & 0 \\ 0 & v_2 & g \\ 0 & D & v_2 \end{bmatrix}$$

For any unit vector $\boldsymbol{n} = (n_1, n_2)$

$$A_1 n_2 + A_2 n_2 = \begin{bmatrix} v_n & 0 & g n_1 \\ 0 & v_n & g n_2 \\ D n_1 & D n_2 & v_n \end{bmatrix}, \qquad R = \begin{bmatrix} -n_2 & -\sqrt{g} n_1 & \sqrt{g} n_1 \\ n_1 & -\sqrt{g} n_2 & \sqrt{g} n_2 \\ 0 & \sqrt{D} & \sqrt{D} \end{bmatrix}$$

$$\text{Eigenvalues:} \qquad v_n,\ v_n - \sqrt{gD},\ v_n + \sqrt{gD}$$

Riemann solver based numerical flux

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0$$

with initial condition

$$\boldsymbol{U}(x, 0) = \begin{cases} \boldsymbol{U}_l & x < 0 \\ \boldsymbol{U}_r & x > 0 \end{cases}$$

Linearized problem (Roe, 1981)

$$\frac{\partial \boldsymbol{U}}{\partial t} + A(\boldsymbol{U}_l, \boldsymbol{U}_r)\frac{\partial \boldsymbol{U}}{\partial x} = 0, \qquad A = R\Lambda R^{-1}$$

Solve Riemann problem exactly and compute flux at $x = 0$

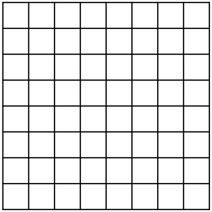
$$\boldsymbol{F}_{lr} = \frac{1}{2}[\boldsymbol{F}(\boldsymbol{U}_l) + \boldsymbol{F}(\boldsymbol{U}_r)] - \frac{1}{2}R|\Lambda|R^{-1}(\boldsymbol{U}_r - \boldsymbol{U}_l)$$

Previous works[4]

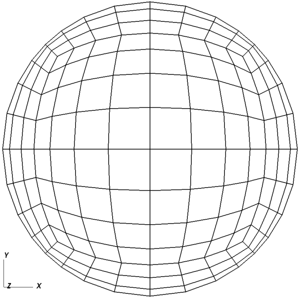
- Finite volume
  Arakawa & Lamb (1981), Salmon (2007), Eldred & Randall (2016)
- DG, conservation form
  Giraldo et al. (2002)
- DG, vector invariant form
  Nair et al. (2004)
- SEM
  Taylor et al. (1997), Giraldo (2001), Taylor & Fournier (2010)
- Compatible finite elements
  Natale et al. (2016)

[4] See review article by *Marras et al., Arch Comp Meth Eng, 2015*

Cubed-sphere grid[5]



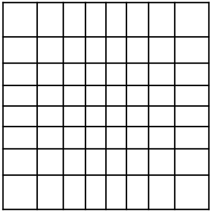
$(X_1, X_2) \in [-a, +a]^2$



$(x_1, x_2, x_3) \in S$

[5]Sadourny (1972), Ronchi et al. (1996)

Cubed-sphere grid



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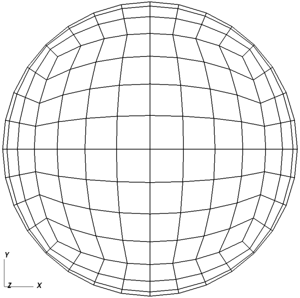
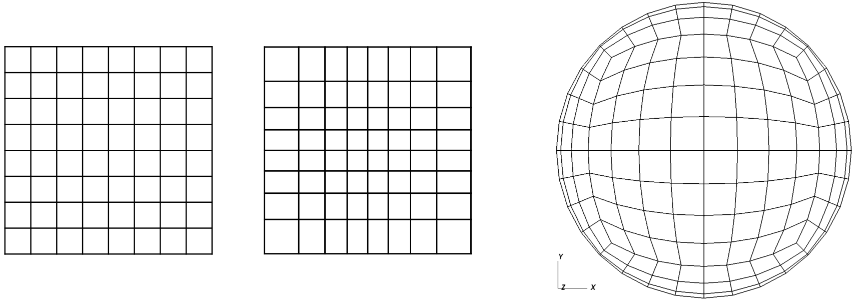


$(x_1, x_2, x_3) \in S$

Cubed-sphere grid

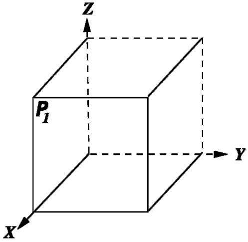


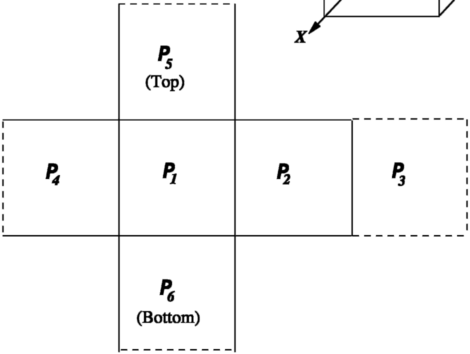
(χ_1, χ_2) $\qquad$ (X_1, X_2) $\qquad$ (x_1, x_2, x_3)

$$\chi_i = \arctan(X_i/a) \in [-\pi/4, \pi/4], \qquad i = 1, 2$$

Cubed-sphere map

From *Nair et al., MWR, 133, 2005*





Cube: $[-a, +a]^3$
$a = R/\sqrt{3}$

On P_1: $(x_1, x_2, x_3) = \frac{R}{r}(a, X_1, X_2)$, $(X_1, X_2) = a\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$

Finite element space

- $\mathcal{T}_h$ = cubed sphere grid
- $\hat{T} = [0,1] \times [0,1]$ is the reference cell
- Each cell $T \in \mathcal{T}_h$ obtained by mapping reference cell $\hat{T}$

$$F_T : \hat{T} \to T, \qquad (\xi_1, \xi_2) \to (\chi_1, \chi_2) \to (x_1, x_2, x_3)$$

- $\mathbb{Q}_N$ = 2-d tensor product polynomials of degree at most N
- Space of broken polynomials for scalar and vector functions

$$V_h^N = \{\psi \in L^2(S) : \psi|_T \circ F_T^{-1} \in \mathbb{Q}_N\}, \qquad \boldsymbol{W}_h^N = V_h^N \times V_h^N \times V_h^N$$

- Lagrange polynomials using Gauss-Legendre nodes
- Same nodes used for quadrature

Tangential gradient[6]

Coordinates

$$\boldsymbol{\xi} = (\xi_1, \xi_2) \in \hat{T}, \qquad \boldsymbol{x} = (x_1, x_2, x_3) \in S$$

First fundamental form

$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial \xi_i} \cdot \frac{\partial \boldsymbol{x}}{\partial \xi_j}, \qquad G = [g_{ij}]$$

and its inverse

$$g^{ij} = [G^{-1}]_{ij}$$

Tangential gradient

$$\frac{\partial \phi}{\partial x_d} = \sum_{i=1}^{2} \sum_{j=1}^{2} g^{ij} \frac{\partial x_d}{\partial \xi_i} \frac{\partial \phi}{\partial \xi_j}, \quad d = 1, 2, 3$$

[6] Dziuk & Elliott, Acta Numerica, 2013

DG for $\boldsymbol{v} - D$ model

For each cell $T \in \mathcal{T}_h$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_T \boldsymbol{v}_h \cdot \boldsymbol{\varphi}_h \mathrm{d}s - \int_T \Phi(\boldsymbol{v}_h, D_h, H_s)\nabla \cdot \boldsymbol{\varphi}_h \mathrm{d}s & \\
+ \int_{\partial T} \hat{\boldsymbol{F}}_v \cdot \boldsymbol{\varphi}_h^- \mathrm{d}\sigma + \int_T (\omega_h + f)\boldsymbol{v}_h^\perp \cdot \boldsymbol{\varphi}_h \mathrm{d}s &= 0, \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{W}_h^N \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_T D_h \psi_h \mathrm{d}s - \int_T \boldsymbol{v}_h D_h \cdot \nabla \psi_h \mathrm{d}s + \int_{\partial T} \hat{F}_D \psi_h^- \mathrm{d}\sigma &= 0, \quad \forall \psi_h \in V_h^N
\end{aligned}
$$

$\hat{\boldsymbol{F}}_v$, $\hat{F}_D$ are numerical fluxes obtained from Rusanov scheme

$$
\begin{aligned}
\hat{\boldsymbol{F}}_v &= \frac{1}{2}(\Phi^- + \Phi^+)\boldsymbol{n}^- - \frac{1}{2}\lambda(\boldsymbol{v}^+ - \boldsymbol{v}^-) \\
\hat{F}_D &= \frac{1}{2}(D^-\boldsymbol{v}^- + D^+\boldsymbol{v}^+)\boldsymbol{n}^- - \frac{1}{2}\lambda(D^+ - D^-)
\end{aligned}
$$

where

$$\lambda = \max\{|\boldsymbol{v}^- \cdot \boldsymbol{n}^-| + \sqrt{gD^-}, |\boldsymbol{v}^+ \cdot \boldsymbol{n}^+| + \sqrt{gD^+}\}$$

DG for $\boldsymbol{v} - D$ model

Vorticity computed locally on each cell

$$\int_T \omega_h \psi_h \mathrm{d}s = \int_T \nabla \psi_h \cdot \boldsymbol{v}_h^{\perp} \mathrm{d}s - \int_{\partial T} \psi_h^- \boldsymbol{n}^- \cdot \hat{\boldsymbol{v}}_h^{\perp} \mathrm{d}\sigma, \qquad \forall \psi_h \in V_h^N$$

where

$$\hat{\boldsymbol{v}} = \frac{1}{2}(\boldsymbol{v}^- + \boldsymbol{v}^+)$$

DG for $\boldsymbol{v} - D - \eta$ model

For each cell $T \in \mathcal{T}_h$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_T \boldsymbol{v}_h \cdot \boldsymbol{\varphi}_h \mathrm{d}s - \int_T \Phi(\boldsymbol{v}_h, D_h, H_s)\nabla \cdot \boldsymbol{\varphi}_h \mathrm{d}s & & \\
+ \int_{\partial T} \hat{\boldsymbol{F}}_v \cdot \boldsymbol{\varphi}_h^- \mathrm{d}\sigma + \int_T \eta_h \boldsymbol{v}_h^\perp \cdot \boldsymbol{\varphi}_h \mathrm{d}s &= 0, & \forall \boldsymbol{\varphi}_h \in \boldsymbol{W}_h^N \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_T D_h \psi_h \mathrm{d}s - \int_T \boldsymbol{v}_h D_h \cdot \nabla \psi_h \mathrm{d}s + \int_{\partial T} \hat{F}_D \psi_h^- \mathrm{d}\sigma &= 0, & \forall \psi_h \in V_h^N \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_T \eta_h \psi_h \mathrm{d}s - \int_T \boldsymbol{v}_h \eta_h \cdot \nabla \psi_h \mathrm{d}s + \int_{\partial T} \hat{F}_\eta \psi_h^- \mathrm{d}\sigma &= 0, & \forall \psi_h \in V_h^N
\end{aligned}
$$

$\hat{\boldsymbol{F}}_v$, $\hat{F}_D$, $\hat{F}_\eta$ are numerical fluxes obtained from a Riemann solver.

Numerical flux for $\boldsymbol{v} - D - \eta$ model

In Cartesian coordinates, with the z-axis along the axis of rotation

$$\frac{\partial \boldsymbol{U}}{\partial t} + A_1 \frac{\partial \boldsymbol{U}}{\partial x} + A_2 \frac{\partial \boldsymbol{U}}{\partial y} + \tilde{\boldsymbol{S}} = 0 \tag{2}$$

where

$$\boldsymbol{U} = \begin{bmatrix} v_1 \\ v_2 \\ D \\ \eta \end{bmatrix}, \quad A_1 = \begin{bmatrix} v_1 & 0 & g & 0 \\ 0 & v_1 & 0 & 0 \\ D & 0 & v_1 & 0 \\ \eta & 0 & 0 & v_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} v_2 & 0 & 0 & 0 \\ 0 & v_2 & g & 0 \\ 0 & D & v_2 & 0 \\ 0 & \eta & 0 & v_2 \end{bmatrix}$$

For any unit vector $\boldsymbol{n} = (n_1, n_2)$

$$A_1 n_2 + A_2 n_2 = \begin{bmatrix} v_n & 0 & g n_1 & 0 \\ 0 & v_n & g n_2 & 0 \\ D n_1 & D n_2 & v_n & 0 \\ \eta n_1 & \eta n_2 & 0 & v_n \end{bmatrix}$$

Numerical flux for $\boldsymbol{v} - D - \eta$ model

$$\text{Eigenvalues:} \qquad v_n,\ v_n,\ v_n - \sqrt{gD},\ v_n + \sqrt{gD}$$

Corresponding linearly independent eigenvectors

$$\boldsymbol{R} = \begin{bmatrix} 0 & -n_2 & -\sqrt{gD}n_1 & \sqrt{gD}n_1 \\ 0 & n_1 & -\sqrt{gD}n_2 & \sqrt{gD}n_2 \\ 0 & 0 & D & D \\ 1 & 0 & \eta & \eta \end{bmatrix}$$

Numerical flux

$$\hat{\boldsymbol{F}} = \frac{1}{2}\begin{bmatrix} (\Phi^- + \Phi^+)n_1^- \\ (\Phi^- + \Phi^+)n_2^- \\ (D^-\boldsymbol{v}^- + D^+\boldsymbol{v}^+)\cdot\boldsymbol{n}^- \\ (\eta^-\boldsymbol{v}^- + \eta^+\boldsymbol{v}^+)\cdot\boldsymbol{n}^- \end{bmatrix} - \frac{1}{2}\boldsymbol{R}|\Lambda|\boldsymbol{R}^{-1}\begin{bmatrix} v_1^+ - v_1^- \\ v_2^+ - v_2^- \\ D^+ - D^- \\ \eta^+ - \eta^- \end{bmatrix}$$

Rotate velocity flux to global (x_1, x_2, x_3) coordinates.

Time integration: $\dot{U} = R(U)$

3-stage strong stability preserving Runge-Kutta scheme

$$\begin{aligned}
U^{(1)} &= U^n - \Delta t \cdot R(U^n) \\
U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}[U^n + \Delta t \cdot R(U^{(1)})] \\
U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}[U^n + \Delta t \cdot R(U^{(2)})]
\end{aligned}$$

After each stage, project the velocity to tangent plane at all GL points.

CFL condition

$$\Delta t = \frac{\mathrm{CFL}}{2N^2+1} \min_{T \in \mathcal{T}_h} \frac{h_T}{\| |\boldsymbol{v}| + \sqrt{gD} \|_{L^\infty(T)}}$$

Numerical implementation

- Written in C++ and deal.II [7]
- Cube sphere mapping with MappingManifold
- Enough to provide these functions[8]
  - pull_back: $(x_1, x_2, x_3) \to (\chi_1, \chi_2)$
  - push_forward: $(\chi_1, \chi_2) \to (x_1, x_2, x_3)$
  - $\frac{\partial x_i}{\partial \chi_j}$
- Parallelized using MPI, p4est[9]

[7] http://www.dealii.org

[8] http://bitbucket.org/cpraveen/deal_ii/src/master/cubed_sphere

[9] http://www.p4est.org

Numerical Results

- Williamson et al., JCP, 102, 1992
- Galewsky et al., Tellus, 56A, 2004

Degree	N
Number of cells on each side of cube	N_e^2
Total number of cells	$6N_e^2$
Total number of grid points	$6N_e^2(N+1)^2$
Angular resolution	$360/(4N_eN)$

Discretization: $(6N_e^2) \times (N+1) \times (N+1)$

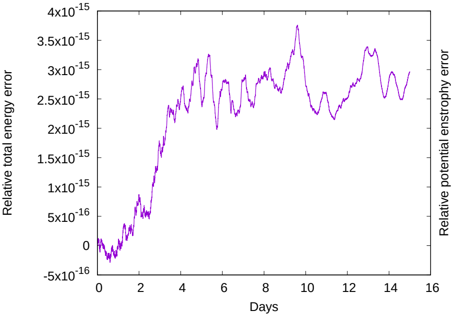
Steady zonal geostrophic flow (Test 2)

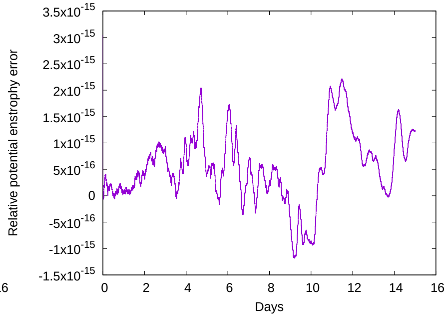
$$
\begin{aligned}
\bar{v}_1 &= u_0(\cos\theta\cos\alpha + \cos\lambda\sin\theta\sin\alpha) \\
\bar{v}_2 &= -u_0\sin\lambda\sin\alpha \\
gH &= gH_0 - \left(R\Omega u_0 + \frac{u_0^2}{2}\right)(-\cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha)
\end{aligned}
$$

$$
u_0 = \frac{2\pi R}{12 \text{ days}}, \qquad gH_0 = 29400\ m^2/s^2, \qquad \alpha = 0
$$

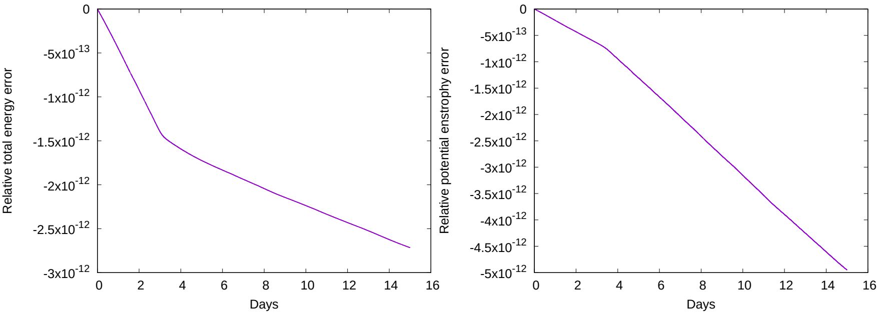
Degree: $N = 9$, Grid points: $96 \times 10 \times 10 = 9,600$
Total dofs: $48,000$

Test 2: $\boldsymbol{v} - D$ model





Test 2: $\boldsymbol{v} - D - \eta$ model



Zonal flow over mountain (Test 5)

Wind and height field same as in Test 2 but with

$$H_0 = 5960\ m, \qquad u_0 = 20\ m/s, \qquad \alpha = 0$$

Mountain height

$$H_s = h_{s0}(1 - r/a)$$

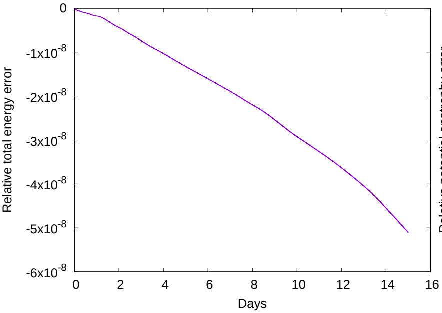
$$h_{s0} = 2000\ m, \quad a = \frac{\pi}{9}, \quad r^2 = \min[a^2, (\lambda - \lambda_c)^2 + (\theta - \theta_c)^2]$$

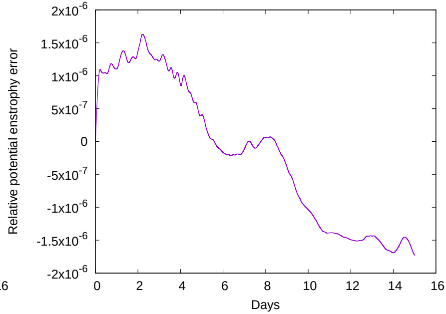
where

$$\lambda_c = \frac{3\pi}{2}, \qquad \theta_c = \frac{\pi}{6}$$

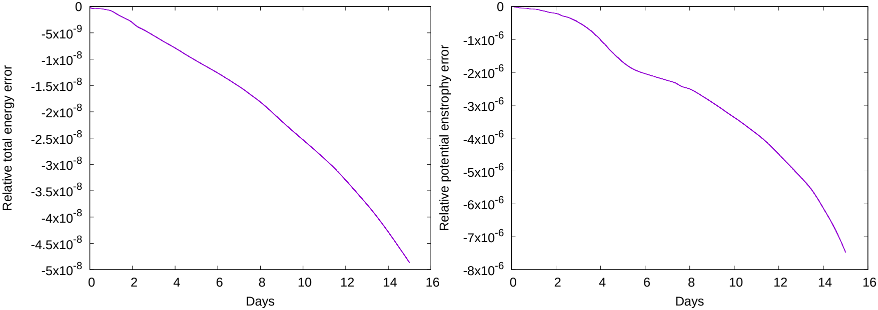
Degree: $N = 5$, Grid points: $384 \times 6 \times 6 = 13,824$
Total dofs: $69,120$

Test 5: $\boldsymbol{v} - D$ model





Test 5: $\boldsymbol{v} - D - \eta$ model



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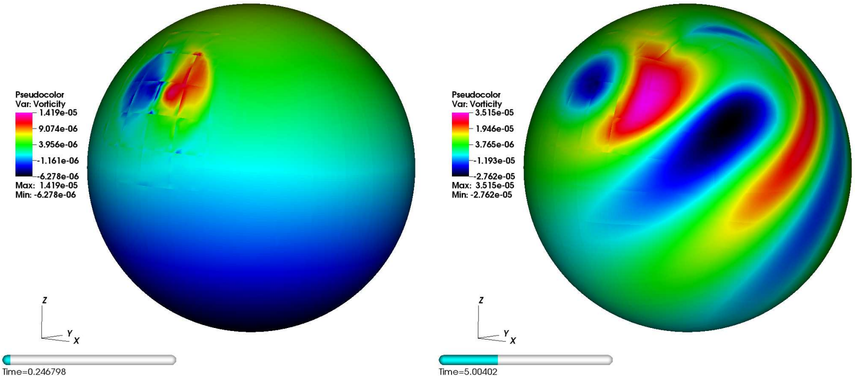

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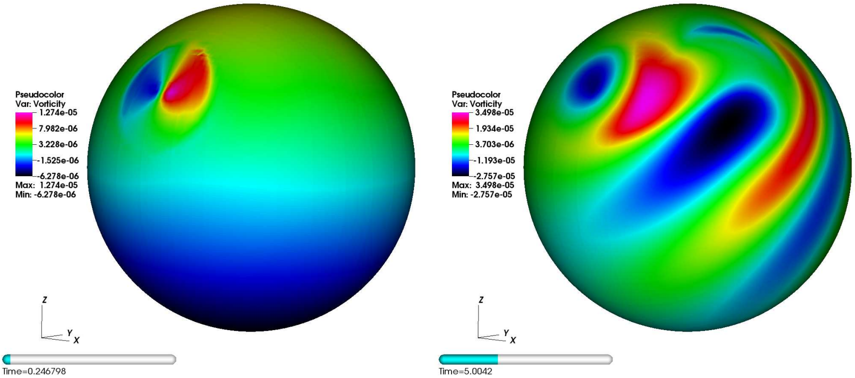


Rossby-Haurwitz waves (Test 6)

Analytic solution of nonlinear barotropic vorticity equation
Initial velocity field is nondivergent, with stream function

$$\psi = -R^2\omega \sin\theta + R^2 K \cos^r\theta \sin\theta \cos(r\lambda)$$

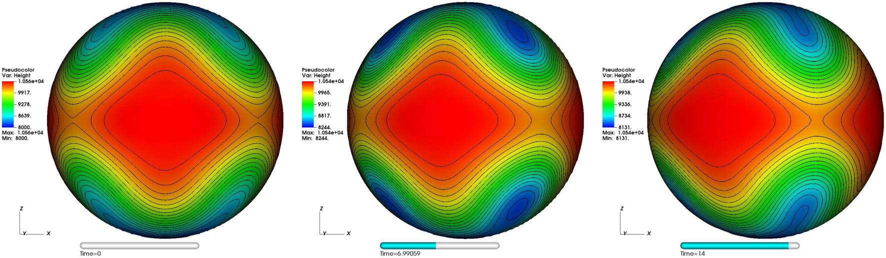
Pattern moves without change of shape with angular velocity[10]

$$\nu = \frac{r(3+r)\omega - 2\Omega}{(1+r)(2+r)}$$

Degree: $N = 5$, Grid points: $384 \times 6 \times 6 = 13,824$
Total dofs: $69,120$

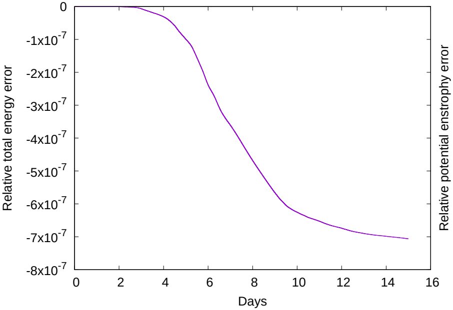
[10] Haurwitz (1940)

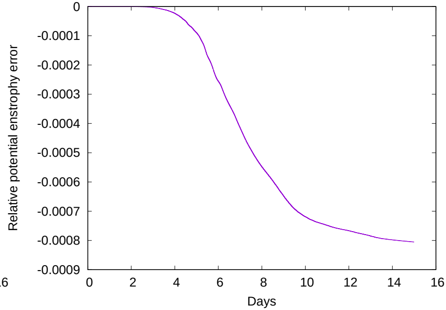
Rossby-Haurwitz waves (Test 6)



Animation

Rossby-Haurwitz waves (Test 6)





Barotropic instability (Galewsky et al.)

Balanced, barotropically unstable, mid-latitude jet

Zonal wind

$$\bar{v}_1(\theta) = \begin{cases} 0 & \theta \le \theta_0 \\ \frac{u_{max}}{e_n} \exp\left[\frac{1}{(\theta-\theta_0)(\theta-\theta_1)}\right] & \theta_0 \le \theta \le \theta_1 \\ 0 & \theta \ge \theta_1 \end{cases}$$

$$u_{max} = 80m/s, \quad \theta_0 = \frac{\pi}{7}, \quad \theta_1 = \frac{\pi}{2} - \theta_0, \quad e_n = \exp\left[-\frac{4}{(\theta_1 - \theta_0)^2}\right]$$

Height obtained by integrating velocity equation

$$gH = gH_0 - R\int^{\theta} \bar{v}_1(\theta)\left[f + \frac{\tan\theta'}{R}\bar{v}_1(\theta')\right] \mathrm{d}\theta'$$

H_0 chosen so that mean layer depth is 10 Km.

Barotropic instability (Galewsky et al.)

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Height perturbation

$$H' = \hat{H} \cos\theta \exp\left(-\frac{\lambda^2}{\alpha^2} - \frac{(\theta_2 - \theta)^2}{\beta^2}\right), \quad -\pi \le \lambda \le +\pi$$

$$\hat{H} = 120m, \quad \theta_2 = \frac{\pi}{4}, \quad \alpha = \frac{1}{3}, \quad \beta = \frac{1}{15}$$

Barotropic instability (Galewsky et al.)

Balanced, barotropically unstable, mid-latitude jet

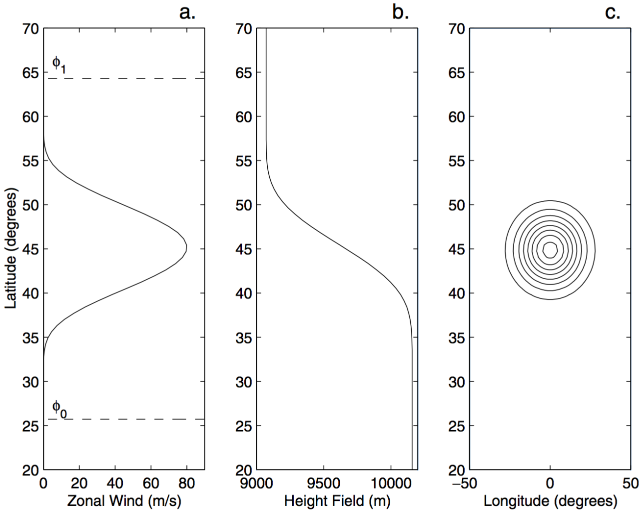
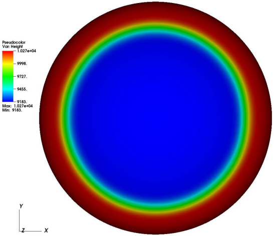


Fig 1. The initial conditions for the new test case. (a) The zonal wind, as defined in eq. (2); (b) the corresponding, balanced height field, calculated using eq. (3); (c) the height field perturbation, as defined in eq. (4), with a contour interval of 10 m; the outermost contour is at 10 m.

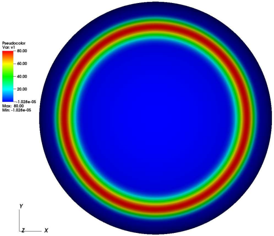
Barotropic instability: Initial condition

Degree: $N = 6$, Grid points: $6144 \times 7 \times 7 = 3,01,056$

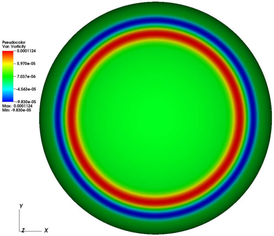
Total dofs: 1,505,280



Height

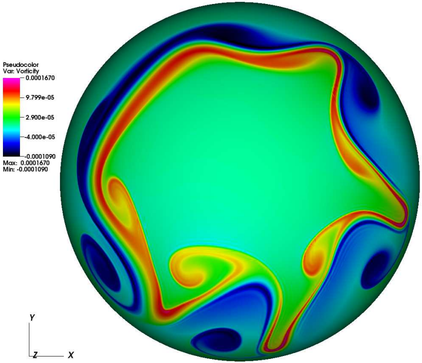


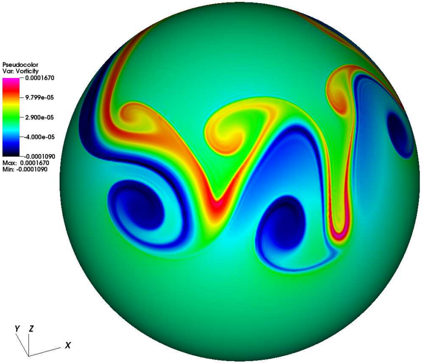
Zonal wind



Relative vorticity

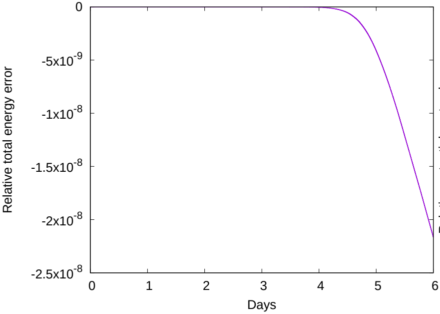
Barotropic instability: Vorticity after 6 days

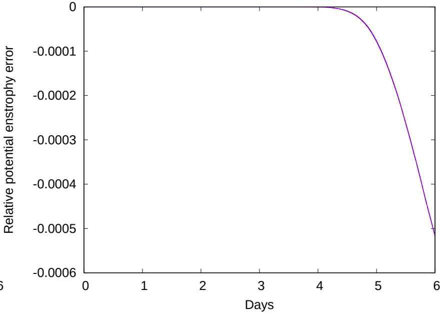




Animation

Barotropic instability





Spectral Element Method

Space of continuous piecewise polynomials

$$V_h^N = \{\psi \in C(S) : \psi|_T \circ F_T^{-1} \in \mathbb{Q}_N\}, \qquad \boldsymbol{W}_h^N = V_h^N \times V_h^N \times V_h^N$$

Basis functions are Lagrange polynomials on Gauss-Lobatto-Legendre (GLL) nodes.
Define the interpolation operator

$$\Pi_h : C(S) \to V_h^N$$

and the discrete inner product

$$(\phi, \psi)_h = \sum_{T \in \mathcal{T}_h} (\phi, \psi)_{T,N}$$

where $(\cdot, \cdot)_{T,N}$ denotes the GLL quadrature on element T.

Spectral Element Method

Let

$$\Phi_h = \Pi_h \Phi(\boldsymbol{v}_h, D_h, H_s), \qquad \boldsymbol{v}_h^{\perp} = \Pi_h(\mathbf{k} \times \boldsymbol{v}_h)$$

The semi-discrete spectral element scheme is given by

$$\begin{aligned}
(\partial_t u_h, \psi_h)_h + (\partial_x \Phi_h, \psi_h)_h + (\eta_h u_h^{\perp}, \psi_h)_h &= 0 \\
(\partial_t v_h, \psi_h)_h + (\partial_y \Phi_h, \psi_h)_h + (\eta_h v_h^{\perp}, \psi_h)_h &= 0 \\
(\partial_t w_h, \psi_h)_h + (\partial_z \Phi_h, \psi_h)_h + (\eta_h w_h^{\perp}, \psi_h)_h &= 0 \\
(\partial_t D_h, \psi_h)_h - (D_h u_h, \partial_x \psi_h)_h - (D_h v_h, \partial_y \psi_h)_h - (D_h w_h, \partial_z \psi_h)_h &= 0
\end{aligned}$$

Vorticity

$$\int_S \omega_h \psi_h \mathrm{d}s = \int_S \nabla \psi_h \cdot \boldsymbol{v}^{\perp} \mathrm{d}s \qquad \forall \psi_h \in V_h^N$$

Mass matrix is diagonal

SEM: Energy conservation

Let

$$P_h = \Pi_h(D_h u_h), \quad Q_h = \Pi_h(D_h v_h), \quad R_h = \Pi_h(D_h w_h)$$

Take the test functions to be P_h, Q_h, R_h, Φ_h respectively

$$\begin{aligned}
(\partial_t u_h, P_h)_h + (\partial_x \Phi_h, P_h)_h + (\eta_h u_h^\perp, P_h)_h &= 0 \\
(\partial_t v_h, Q_h)_h + (\partial_y \Phi_h, Q_h)_h + (\eta_h v_h^\perp, Q_h)_h &= 0 \\
(\partial_t w_h, R_h)_h + (\partial_z \Phi_h, R_h)_h + (\eta_h w_h^\perp, R_h)_h &= 0 \\
(\partial_t D_h, \Phi_h)_h - (D_h u_h, \partial_x \Phi_h)_h - (D_h v_h, \partial_y \Phi_h)_h - (D_h w_h, \partial_z \Phi_h)_h &= 0
\end{aligned}$$

and add the four equations. Note that

$$(D_h u_h, \partial_x \psi_h)_h = (P_h, \partial_x \psi_h)_h, \quad \text{etc.}$$

The time rate of change of energy density E can be written as

$$Du\frac{\partial u}{\partial t} + Dv\frac{\partial v}{\partial t} + Dw\frac{\partial w}{\partial t} + \Phi\frac{\partial D}{\partial t} = \frac{\partial E}{\partial t}$$

SEM: Energy conservation

Hence we have

$$(\partial_t u_h, P_h)_h + (\partial_t v_h, Q_h)_h + (\partial_t w_h, R_h)_h + (\partial_t D_h, \Phi_h)_h = (\partial_t E_h, 1)_h$$

Moreover, since $\boldsymbol{v} \cdot (\mathbf{k} \times \boldsymbol{v}) = 0$,

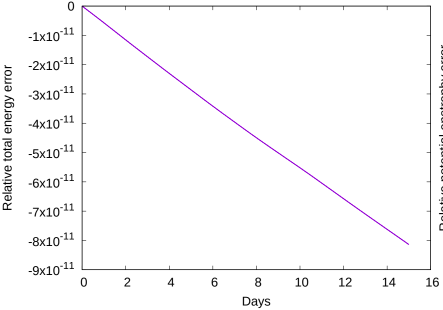
$$(\eta_h u_h^{\perp}, P_h)_h + (\eta_h v_h^{\perp}, Q_h)_h + (\eta_h w_h^{\perp}, R_h)_h = 0$$

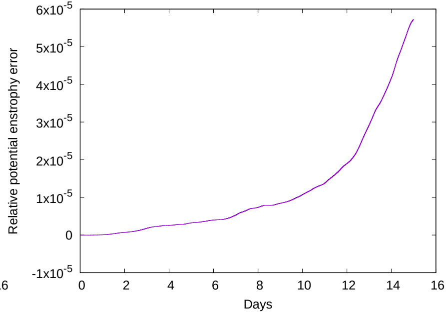
Hence the result of adding all the equations is

$$(\partial_t E_h, 1)_h = 0$$

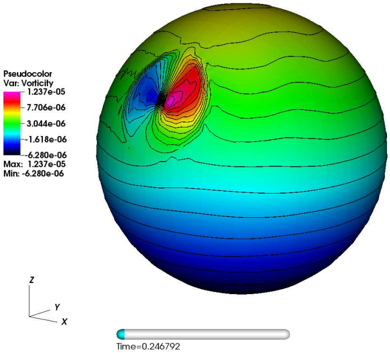
which shows that the total energy is conserved.

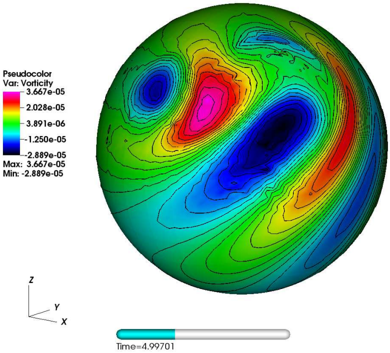
Test 5


Relative total energy error
0
-1x10^-11
-2x10^-11
-3x10^-11
-4x10^-11
-5x10^-11
-6x10^-11
-7x10^-11
-8x10^-11
-9x10^-11
0 2 4 6 8 10 12 14 16
Days


Relative potential enstrophy error
6x10^-5
5x10^-5
4x10^-5
3x10^-5
2x10^-5
1x10^-5
0
-1x10^-5
0 2 4 6 8 10 12 14 16
Days

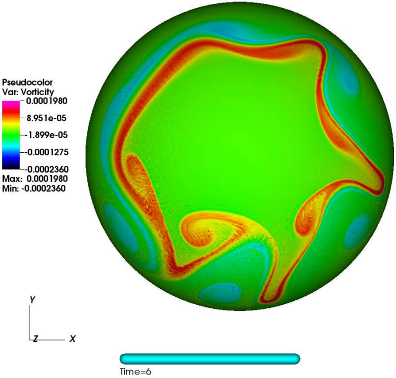
Test 5



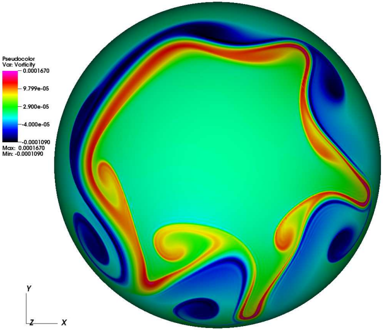


Animation

Barotropic instability

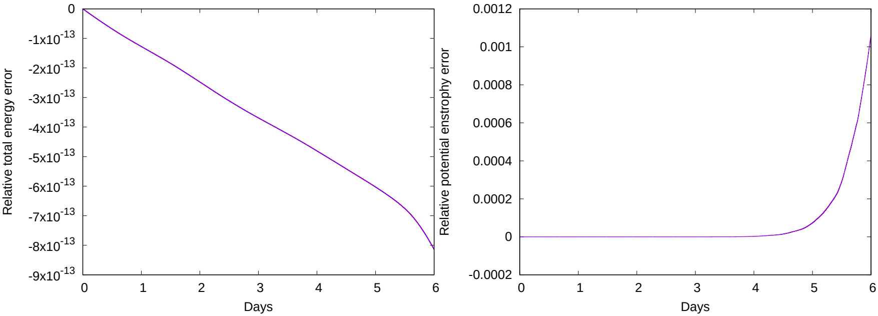


SEM



DG

Barotropic instability



Summary

- DG method for $\boldsymbol{v} - D - \eta$ model
  - Update vorticity as an independent variable
- In Cartesian coordinates – no spherical transformations
- Riemann solver based numerical flux
- Good energy, potential enstrophy conservation
- Further work
  - Add dissipation or limiter into vorticity equation
  - Local grid adaptation
- Spectral element method
  - Good energy conservation
  - Need to add some numerical diffusion
- DG scheme which conserves/dissipates total energy

Acknowledgements

- Ramachandran Nair
  National Center for Atmospheric Research, Boulder
- Compact course planned in summer 2017 on at TIFR-CAM
  High order methods for weather prediction

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