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A class of singular Fourier integral operators in Synthetic Aperture Radar imaging

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Abstract

In this article, we analyze the microlocal properties of the linearized forward scattering operator F and the normal operator F^*F (where F^* is the L^2 adjoint of F) which arises in Synthetic Aperture Radar imaging for the common midpoint acquisition geometry. When F^* is applied to the scattered data, artifacts appear. We show that F^*F can be decomposed as a sum of four operators, each belonging to a class of distributions associated to two cleanly intersecting Lagrangians, $I^{p,l}(\Lambda_0, \Lambda_1)$, thereby explaining the latter artifacts.

Keywords: Singular Fourier integral operators; Elliptical Radon transforms; Synthetic Aperture Radar; Fold and Blowdown singularities

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1. Introduction

In this article, we analyze the microlocal properties of a transform that appears in Synthetic Aperture Radar (SAR) imaging. In SAR imaging, a region on the surface of the earth is illuminated by an electromagnetic transmitter and an image of the region is reconstructed based on the measurement of scattered waves at a receiver. For in-depth treatments of SAR imaging, we refer the reader to [1,2]. The transform we study appears as a result of a common midpoint acquisition geometry: the transmitter and receiver move at equal speeds away from a common midpoint along a straight line. This geometry is of interest in bistatic imaging and in certain multiple scattering scenarios [21]. We first consider the linearized scattering operator F and show that it is a Fourier integral operator (FIO). Since the conventional method of reconstructing the image of an object involves "backprojecting" the scattered data, we next study the composition of F with its L^2 adjoint F^*. One of the main goals of this article is to understand the distribution class of the kernel of F^*F.

In general the composition of two FIOs is not an FIO. One needs additional geometric conditions such as the transverse intersection condition [16] or the clean intersection condition [4] to make the composition operator again an FIO. When these assumptions fail to be satisfied, it is very useful to study the canonical relation associated to an FIO by considering the left and the right projections. More precisely, let X and Y be manifolds and let $I^m(X, Y; C)$ be the class of FIOs $F : \mathcal{E}'(X) \to \mathcal{D}'(Y)$ of order m associated to the canonical relation $C \subset (T^*Y \times T^*X) \setminus \{0\}$ and denote by $\pi_L : C \to T^*Y$, $\pi_R : C \to T^*X$, the left and right projections respectively. Where and how these projections drop rank determine the nature of the normal operator F^*F.

Several authors have analyzed the nature of the canonical relation and the singularities of the left and right projections in many contexts including scattering theory, integral geometry and harmonic analysis [18,14,11,13,12,9,10,20,5–8,17]. The singularities which appear in previous work related to SAR [20,5,6,17] are folds and blowdowns, that is, π_L and π_R have both fold singularities or π_L has a fold singularity and π_R has a blowdown singularity. These singularities will be defined in Section 3. Then it is known that the corresponding normal operator belongs to a class of distributions $I^{2m,0}(\Delta, \widetilde{C})$ introduced in [15] (and defined in Section 3). This means that the adjoint operator F^* introduces an additional singularity given by $\widetilde{C}$ apart from the initial one given by Δ. For example, in the case of straight line acquisition geometry in monostatic radar – the transmitter and receiver are located at the same point and move along a straight line – the additional Lagrangian $\widetilde{C}$ is reflected in the fact that there is a natural left–right ambiguity in SAR; reflectors on one side of the flight path can give the same signal as reflectors on the other side. This implies that one can only recover the singularities of the even part of the target function. In other words, there is cancellation of certain singularities. Stefanov and Uhlmann prove that such cancellation of singularities can occur even with curved flight paths [22].

In this article, the linearized scattering operator F exhibits a new feature: both projections drop rank by one on a disjoint union of two smooth hypersurfaces $\Sigma_1 \cup \Sigma_2$. On each of them, π_L is a projection with fold singularities and π_R is a projection with blowdown singularities. Note that this is different from the situation in [11] where they study a class of geodesic X-ray transforms on manifolds in which the singularities of the left and right projections are in the reverse order. We then show that F^*F belongs to the class $I^{2m,0}(\Delta, C_1) + I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3) + I^{2m,0}(C_2, C_3)$ (where these classes are given in Definition 3.6). This means that the adjoint operator F^* adds three more singularities given by C_1, C_2, C_3 in addition to the true reconstructed singularity given by Δ. We clarify this in detail in Section 5. The main tool for proving our result is the iterated regularity property; a characteristic property of $I^{p,l}$ classes [13, Proposition 1.35].

2. Statement of the main results

2.1. The linearized scattering model

For simplicity, we assume that both the transmitter and receiver are at the same height $h > 0$ above the ground, $x_3 = 0$, at all times and move in opposite directions at equal speeds along the line parallel the x_1 axis and containing the common midpoint $(0, 0, h)$. Such a model arises when considering signals which have scattered from a wall within the vicinity of a scatterer and can be understood in the context of the method of images; see [21] for more details.

Let $\gamma_T(s) = (s, 0, h)$ and $\gamma_R(s) = (-s, 0, h)$ for $s \in (0, \infty)$ be the trajectories of the transmitter and receiver respectively.

The linearized model for the scattered signal we will use in this article is from [21]

$$d(s,t) := FV(s,t) = \int e^{-\mathrm{i}\omega(t - \frac{1}{c_0} R(s,x))} a(s,x,\omega) V(x)\, \mathrm{d}x\, \mathrm{d}\omega \tag{1}$$

for $(s,t) \in Y = (0,\infty) \times (0,\infty)$, where $V(x) = V(x_1, x_2)$ is the function modeling the object on the ground, $R(s,x)$ is the bistatic distance:

$$R(s,x) = \left|\gamma_T(s) - x\right| + \left|x - \gamma_R(s)\right|,$$

c_0 is the speed of electromagnetic wave in free-space and the amplitude term a is given by

$$a(s,x,\omega) = \frac{\omega^2 p(\omega)}{16\pi^2 |\gamma_T(s) - x||\gamma_R(s) - x|}, \tag{2}$$

where p is the Fourier transform of the transmitted waveform.

2.2. Preliminary modifications on the scattered data

For simplicity, from now on we will assume that $c_0 = 1$. To make the composition of F with its L^2 adjoint F^* to be well-defined, we multiply $d(s,t)$ by an infinitely differentiable function $f(s,t)$ identically equal to 1 in a compact subset of $(0,\infty) \times (0,\infty)$ and supported in a slightly bigger compact subset of $(0,\infty) \times (0,\infty)$. We rename $f \cdot d$ as d again.

As we will see below, our method cannot image a neighborhood of the common midpoint. That is, if the transmitter and receiver are at $(s, 0, h)$ and $(-s, 0, h)$ respectively, we cannot image a neighborhood of the origin on the horizontal plane of the earth, $x_3 = 0$. Therefore we modify d further by considering a smooth function $g(s,t)$ such that

$$g(s,t) = 0 \quad \text{for } (s,t)\text{: } |t - 2\sqrt{s^2 + h^2}| < 20\epsilon^2/h, \tag{3}$$

where $\epsilon > 0$ is given. Again we let $g \cdot d$ to be d and $g \cdot a$ to be a. The choice of constant on the right-hand side of (3) will be justified in Appendix B. Our forward operator is

$$d(s,t)=\int e^{-\mathrm{i}\varphi(s,t,x,\omega)}a(s,t,x,\omega)V(x)\,\mathrm{d}x\,\mathrm{d}\omega \tag{4}$$

where

$$\varphi(s,t,x,\omega)=\omega\Big(t-\sqrt{(x_1-s)^2+x_2^2+h^2}-\sqrt{(x_1+s)^2+x_2^2+h^2}\Big). \tag{5}$$

From now on, we will denote the ground (the plane $x_3=0$) by X, thus the points on X will be denoted $x=(x_1,x_2)$.

We assume that the amplitude function $a\in S^{m+\frac{1}{2}}$, that is, it satisfies the following estimate: For every compact set $K\subset Y\times X$, non-negative integer α, and 2-indexes $\beta=(\beta_1,\beta_2)$ and γ, there is a constant c such that

$$\big|\partial_\omega^\alpha\partial_s^{\beta_1}\partial_t^{\beta_2}\partial_x^\gamma a(s,t,x,\omega)\big|\leqslant c\big(1+|\omega|\big)^{m+(1/2)-\alpha}. \tag{6}$$

This assumption is satisfied if the transmitted waveform from the antenna is approximately a Dirac delta distribution.

With these modifications, we show that F is a Fourier integral operator of order m and study the properties of the natural projection maps from the canonical relation of F. Our first main result is the following:

Theorem 2.1. *Let F be as in (4). Then*:

(a) *F is an FIO of order m.*
(b) *The canonical relation C associated to F is given by*

$$\begin{aligned}C=\Bigg\{\Bigg(&s,t,-\omega\Bigg(\frac{x_1-s}{\sqrt{(x_1-s)^2+x_2^2+h^2}}-\frac{x_1+s}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\Bigg),-\omega;\\&x_1,x_2,-\omega\Bigg(\frac{x_1-s}{\sqrt{(x_1-s)^2+x_2^2+h^2}}+\frac{x_1+s}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\Bigg),\\&-\omega\Bigg(\frac{x_2}{\sqrt{(x_1-s)^2+x_2^2+h^2}}+\frac{x_2}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\Bigg)\Bigg):\\&s>0,\ t=\sqrt{(x_1-s)^2+x_2^2+h^2}+\sqrt{(x_1+s)^2+x_2^2+h^2},\\&x\neq 0,\ \text{and}\ \omega\neq 0\Bigg\},\end{aligned} \tag{7}$$

and C has global parameterization

$$(0,\infty)\times\big(\mathbb{R}^2\setminus 0\big)\times(\mathbb{R}\setminus 0)\ni(s,x_1,x_2,\omega)\mapsto C.$$

(c) *Let $\pi_L : C\to T^*Y$ and $\pi_R : C\to T^*X$ be the left and right projections respectively. Then π_L and π_R drop rank simply by one on a set $\Sigma=\Sigma_1\cup\Sigma_2$ where in the coordinates (s,x,ω),*

$\Sigma_1 = \{(s, x_1, 0, \omega) \mid s > 0,\ |x_1| > \epsilon',\ \omega \neq 0\}$ and $\Sigma_2 = \{(s, 0, x_2, \omega) \mid s > 0,\ |x_2| > \epsilon',\ \omega \neq 0\}$ for $0 < \epsilon'$ small enough.

(d) *π_L has a fold singularity along Σ.*

(e) *π_R has a blowdown singularity along Σ.*

Remark 2.2. Note that due to the function $g(s,t)$ of (3) in the amplitude, it is enough to consider only points in C that are strictly away from $\{(s, 0, \omega)\colon\ s > 0,\ \omega \neq 0\}$. This is reflected in the definitions of Σ_1 and Σ_2, where $|x_1|$ and $|x_2|$, respectively, are strictly positive.

Remark 2.3. Note that C is even with respect to both x_1 and x_2. In other words C is a four-to-one relation. This observation suggests that π_L (respectively π_R) has two fold (respectively blowdown) sets. See Proposition 4.3.

We then analyze the normal operator F^*F. Our next main result is the following:

Theorem 2.4. *Let F be as in (4) of order m. Then F^*F can be decomposed into a sum belonging to $I^{2m,0}(\Delta, C_1) + I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3) + I^{2m,0}(C_2, C_3)$ where these classes are given in Definition 3.6.*

In Remark 5.7, we will explain why the added singularities given by C_1, C_2, C_3 have the same strength as the object singularities given by Δ.

3. Preliminaries

3.1. Singularities and $I^{p,l}$ classes

In this section we will define fold and blowdown singularities and describe the $I^{p,l}$ class of distributions required for the analysis of the composition operator F^*F.

Definition 3.1. (See [14, pp. 109–111].) Let M and N be manifolds of dimension n and let $f : M \to N$ be C^∞. Let Ω be a non-vanishing volume form on N and define $\Sigma = \{\sigma \in M\colon\ f^*\Omega(\sigma) = 0\}$, that is, Σ is the set of critical points of f. Note that, equivalently, Σ is defined by the vanishing of the determinant of the Jacobian of f.

(a) If for all $\sigma \in \Sigma$, we have (i) the corank of f at σ is 1, (ii) $\ker(\mathrm{d}f_\sigma) \cap T_\sigma\Sigma = \{0\}$, (iii) $f^*\Omega$ vanishes exactly to first order on Σ, then we say that f is a *fold*.

(b) If for all $\sigma \in \Sigma$, we have (i) the rank of f is constant; let us call this constant k, (ii) $\ker(\mathrm{d}f_\sigma) \subset T_\sigma\Sigma$, (iii) $f^*\Omega$ vanishes exactly to order $n-k$ on Σ, then we say that f is a *blowdown*.

We now define $I^{p,l}$ classes. They were first introduced by Melrose and Uhlmann [19], Guillemin and Uhlmann [15] and Greenleaf and Uhlmann [13] and they were used in the context of radar imaging in [20,5,6].

Definition 3.2. Two submanifolds M and N intersect *cleanly* if $M \cap N$ is a smooth submanifold and $T(M \cap N) = TM \cap TN$.

Let us consider the following example:

Example 3.3. Let $\widetilde{\Lambda}_0 = \Delta_{T^*\mathbb{R}^n} = \{(x, \xi; x, \xi) \mid x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \setminus 0\}$ be the diagonal in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ and let $\widetilde{\Lambda}_1 = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) \mid x' \in \mathbb{R}^{n-1},\ \xi' \in \mathbb{R}^{n-1} \setminus 0\}$. Then, $\widetilde{\Lambda}_0$ intersects $\widetilde{\Lambda}_1$ cleanly in codimension 1.

Now we define the class of product-type symbols $S^{p,l}(m, n, k)$.

Definition 3.4. $S^{p,l}(m, n, k)$ is the set of all functions $a(z, \xi, \sigma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k)$ such that for every $K \subset \mathbb{R}^m$ and every $\alpha \in \mathbb{Z}^m_+$, $\beta \in \mathbb{Z}^n_+$, $\gamma \in \mathbb{Z}^k_+$ there is $c_{K,\alpha,\beta,\gamma}$ such that

$$\left|\partial_z^\alpha \partial_\xi^\beta \partial_\sigma^\gamma a(z, \xi, \sigma)\right| \leqslant c_{K,\alpha,\beta,\gamma}\big(1 + |\xi|\big)^{p-|\beta|}\big(1 + |\sigma|\big)^{l-|\gamma|}, \quad \forall (z, \xi, \tau) \in K \times \mathbb{R}^n \times \mathbb{R}^k.$$

Since any two sets of cleanly intersecting Lagrangians are equivalent [15], we first define $I^{p,l}$ classes for the case in Example 3.3.

Definition 3.5. (See [15].) Let $I^{p,l}(\widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_0^\infty$ and

$$u_2(x, y) = \int e^{i((x'-y')\cdot\xi' + (x_n - y_n - s)\cdot\xi_n + s\cdot\sigma)} a(x, y, s; \xi, \sigma)\, \mathrm{d}\xi\, \mathrm{d}\sigma\, \mathrm{d}s$$

with $a \in S^{p',l'}$ where $p' = p - \frac{n}{2} + \frac{1}{2}$ and $l' = l - \frac{1}{2}$.

Let (Λ_0, Λ_1) be a pair of cleanly intersection Lagrangians in codimension 1 and let χ be a canonical transformation which maps (Λ_0, Λ_1) into $(\widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$ and maps $\Lambda_0 \cap \Lambda_1$ to $\widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_1$, where $\widetilde{\Lambda}_j$ are from Example 3.3. Next we define the $I^{p,l}(\Lambda_0, \Lambda_1)$.

Definition 3.6. (See [15].) Let $I^{p,l}(\Lambda_0, \Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2 + \sum v_i$ where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i = A w_i$ where A is a zero order FIO associated to χ^{-1}, the canonical transformation from above, and $w_i \in I^{p,l}(\widetilde{\Lambda}_0, \widetilde{\Lambda}_1)$.

If u is the Schwartz kernel of the linear operator F, then we say $F \in I^{p,l}(\Lambda_0, \Lambda_1)$.

This class of distributions is invariant under FIOs associated to canonical transformations which map the pair (Λ_0, Λ_1) to itself and the intersection $\Lambda_0 \cap \Lambda_1$ to itself. If $F \in I^{p,l}(\Lambda_0, \Lambda_1)$ then $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ and $F \in I^p(\Lambda_1 \setminus \Lambda_0)$ [15]. Here by $F \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, we mean that the Schwartz kernel of F belongs to $I^{p+l}(\Lambda_0 \setminus \Lambda_1)$ microlocally away from Λ_1.

One way to show that a distribution belongs to $I^{p,l}$ class is by using the iterated regularity property:

Proposition 3.7. *(See [13, Proposition 1.35].) If $u \in \mathcal{D}'(X \times Y)$ then $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ if there is an $s_0 \in R$ such that for all first order pseudodifferential operators P_i with principal symbols vanishing on $\Lambda_0 \cup \Lambda_1$, we have $P_1 P_2 \ldots P_r u \in H^{s_0}_{\mathrm{loc}}$.*

4. Analysis of the operator F

In this section, we prove Theorem 2.1, as a result of Lemma 4.1 and Proposition 4.3.

Lemma 4.1. *F is an FIO of order m with the canonical relation C given by*

$$C = \Bigg\{\Bigg(s, t, -\omega\left(\frac{x_1 - s}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}} - \frac{x_1 + s}{\sqrt{(x_1+s)^2 + x_2^2 + h^2}}\right), -\omega;$$
$$x_1, x_2, -\omega\left(\frac{x_1 - s}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}} + \frac{x_1 + s}{\sqrt{(x_1+s)^2 + x_2^2 + h^2}}\right),$$
$$-\omega\left(\frac{x_2}{\sqrt{(x_1-s)^2 + x_2^2 + h^2}} + \frac{x_2}{\sqrt{(x_1+s)^2 + x_2^2 + h^2}}\right)\Bigg)\colon$$
$$s > 0,\ t = \sqrt{(x_1-s)^2 + x_2^2 + h^2} + \sqrt{(x_1+s)^2 + x_2^2 + h^2},$$
$$x \in \mathbb{R}^2 \setminus \{0\},\ \omega \neq 0\Bigg\}. \tag{8}$$

We note that $(0, \infty) \times (\mathbb{R}^2 \setminus 0) \times (\mathbb{R} \setminus 0) \ni (s, x_1, x_2, \omega) \mapsto C$ *is a global parametrization of* C.

We will use the coordinates (s, x, ω) in this lemma from now on to describe C and subsets of C.

Proof of Lemma 4.1. The phase function φ is non-degenerate with $\partial_x\varphi$, $\partial_{s,t}\varphi$ nowhere 0 whenever $\partial_\omega\varphi = 0$. We should mention that $\nabla\partial_\omega\varphi \neq 0$. (Note that in order for $\partial_x\varphi$ to be nowhere 0, we require exclusion of the common midpoint from our analysis.) This observation is needed to show F is an FIO rather than just a Fourier integral distribution. Recalling that a satisfies amplitude estimates (6), we conclude that F is an FIO [23]. Also since a is of order $m + \frac{1}{2}$, the order of the FIO is m [3, Definition 3.2.2]. By definition [16, Eq. (3.1.2)]

$$C = \big\{\big((s, t, \partial_s\varphi, \partial_t\varphi); (x, -\partial_x\varphi)\big)\colon\ \partial_\omega\varphi = 0\big\}.$$

A calculation using this definition establishes (8). Furthermore, it is easy to see that (s, x_1, x_2, ω) is a global parametrization of Λ. □

Remark 4.2. In the SAR application, a has order 2 which makes operator F of order $\frac{3}{2}$. But from now on will consider that F has order m.

Proposition 4.3. *Denoting the restriction of the left and right projections to C by π_L and π_R respectively, we have*:

(a) *π_L and π_R drop rank by one on a set $\Sigma = \Sigma_1 \cup \Sigma_2$.*
Here we use the global coordinates from Lemma 4.1.

(b) π_L *has a fold singularity along* Σ.
(c) π_R *has a blowdown singularity along* Σ.

Proof. Let $A = \sqrt{(x_1 - s)^2 + x_2^2 + h^2}$ and $B = \sqrt{(x_1 + s)^2 + x_2^2 + h^2}$. We have

$$\pi_L(x_1, x_2, s, \omega) = \left(s, A + B, -\left(\frac{x_1 - s}{A} - \frac{x_1 + s}{B}\right)\omega, -\omega\right)$$

and

$$\mathrm{d}\pi_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{x_1-s}{A} + \frac{x_1+s}{B} & \frac{x_2}{A} + \frac{x_2}{B} & * & 0 \\ -\omega(\frac{x_2^2+h^2}{A^3} - \frac{x_2^2+h^2}{B^3}) & \omega(\frac{(x_1-s)x_2}{A^3} - \frac{(x_1+s)x_2}{B^3}) & * & * \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where $*$ denotes derivatives that are not needed for the calculation. The determinant is

$$\det \mathrm{d}\pi_L = \frac{4x_1x_2s\omega}{A^2B^2}\left(1 + \frac{x_1^2 - s^2 + x_2^2 + h^2}{AB}\right). \tag{9}$$

We have that $s > 0$ and the number in the parenthesis is a positive number by Lemma 4.4 below.

Therefore, π_L drops rank by one on $\Sigma = \Sigma_1 \cup \Sigma_2$. To show $\mathrm{d}(\det(\mathrm{d}\pi_L))$ is nowhere zero on Σ, one uses the product rule in (9) and the fact that the differential of $\frac{4x_1x_2s\omega}{A^2B^2}$ is never zero on Σ and the inequality in Lemma 4.4.

On Σ_1 the kernel of $\mathrm{d}\pi_L$ is $\frac{\partial}{\partial x_2}$ which is transversal to Σ_1 and on Σ_2 the kernel of $\mathrm{d}\pi_L$ is $\frac{\partial}{\partial x_1}$ which is transversal to Σ_2. This means that π_L has a fold singularity along Σ.

Similarly,

$$\pi_R(x_1, x_2, s, \omega) = \left(x_1, x_2, -\left(\frac{x_1 - s}{A} + \frac{x_1 + s}{B}\right)\omega, -\left(\frac{x_2}{A} + \frac{x_2}{B}\right)\omega\right).$$

Then

$$\mathrm{d}\pi_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & \omega(\frac{x_2^2+h^2}{A^3} - \frac{x_2^2+h^2}{B^3}) & -(\frac{x_1-s}{A} + \frac{x_1+s}{B}) \\ * & * & -\omega(\frac{(x_1-s)x_2}{A^3} - \frac{(x_1+s)x_2}{B^3}) & -(\frac{x_2}{A} + \frac{x_2}{B}) \end{pmatrix}$$

has the same determinant so π_R drops rank by one on Σ and the kernel of $\mathrm{d}\pi_R$ is a linear combination of $\frac{\partial}{\partial\omega}$ and $\frac{\partial}{\partial s}$ which are tangent to both Σ_1 and Σ_2. This means that π_R has a blowdown singularity along Σ. □

Lemma 4.4. *For all* $s \neq 0$,

$$1 + \frac{x_1^2 - s^2 + x_2^2 + h^2}{|x - \gamma_T(s)||x - \gamma_R(s)|} > 0.$$

Proof. Equivalently, we show that $(|x-\gamma_T(s)||x-\gamma_R(s)|)^2 > (x_1^2+x_2^2+h^2-s^2)^2$. Expanding out both sides and simplifying, we obtain $4s^2(x_2^2+h^2) > 0$ which holds for $s \neq 0$, since $h > 0$. Therefore the lemma is proved. □

5. Analysis of the normal operator F^*F

We have

$$F^*FV(x) = \int e^{\mathrm{i}\omega(t-(|x-\gamma_T(s)|+|x-\gamma_R(s)|))-\tilde{\omega}(t-(|y-\gamma_T(s)|+|y-\gamma_R(s)|))} \\ \times \overline{a(s,t,x,\omega)}a(s,t,y,\tilde{\omega})V(y)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}\tilde{\omega}\,\mathrm{d}y.$$

After an application of the method of stationary phase in t and $\tilde{\omega}$, the Schwartz kernel of this operator is

$$K(x,y) = \int e^{\mathrm{i}\omega(|y-\gamma_T(s)|+|y-\gamma_R(s)|-(|x-\gamma_T(s)|+|x-\gamma_R(s)|))}\tilde{a}(x,y,s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega. \tag{10}$$

Note that $\tilde{a} \in S^{2m+1}$ since we assume $a \in S^{m+1/2}$.

Let the phase function of the kernel K be denoted by

$$\Phi = \omega\big(\big|y-\gamma_T(s)\big| + \big|y-\gamma_R(s)\big| - \big(\big|x-\gamma_T(s)\big| + \big|x-\gamma_R(s)\big|\big)\big). \tag{11}$$

Proposition 5.1. *The wavefront set of the kernel K of F^*F satisfies,*

$$WF(K)' \subset \Delta \cup C_1 \cup C_2 \cup C_3,$$

where Δ is the diagonal in $T^*X \times T^*X$ and the Lagrangians C_i for $i = 1,2,3$ are the graphs of the following functions χ_i for $i = 1,2,3$ on T^*X:

$$\chi_1(x,\xi) = (x_1,-x_2,\xi_1,-\xi_2), \qquad \chi_2(x,\xi) = (-x_1,x_2,-\xi_1,\xi_2) \quad \textit{and} \quad \chi_3 = \chi_1 \circ \chi_2.$$

Furthermore we have:

(a) *Δ and C_1, Δ and C_2, C_1 and C_3, C_2 and C_3 intersect cleanly in codimension* 2.
(b) *$\Delta \cap C_3 = C_1 \cap C_2 = \emptyset$.*

Proof. In order to find the wavefront set of the kernel K, we consider the canonical relation $C^t \circ C$ of F^*F: $C^t \circ C = \{(x,\xi;y,\eta) \mid (x,\xi;s,t,\sigma,\tau) \in C^t;\ (s,t,\sigma,\tau;y,\eta) \in C\}$. We have that $(s,t,\sigma,\tau;y,\eta) \in C$ implies

$$t = \sqrt{(y_1-s)^2+y_2^2+h^2} + \sqrt{(y_1+s)^2+y_2^2+h^2},$$

$$\sigma = \tau\left(\frac{y_1-s}{\sqrt{(y_1-s)^2+y_2^2+h^2}} - \frac{y_1+s}{\sqrt{(y_1+s)^2+y_2^2+h^2}}\right),$$

$$\eta_1=\tau\left(\frac{y_1-s}{\sqrt{(y_1-s)^2+y_2^2+h^2}}+\frac{y_1+s}{\sqrt{(y_1+s)^2+y_2^2+h^2}}\right),$$
$$\eta_2=\tau\left(\frac{y_2}{\sqrt{(y_1-s)^2+y_2^2+h^2}}+\frac{y_2}{\sqrt{(y_1+s)^2+y_2^2+h^2}}\right) \tag{12}$$

and $(x,\xi;s,t,\sigma,\tau)\in C^t$ implies

$$t=\sqrt{(x_1-s)^2+x_2^2+h^2}+\sqrt{(x_1+s)^2+x_2^2+h^2},$$
$$\sigma=\tau\left(\frac{x_1-s}{\sqrt{(x_1-s)^2+x_2^2+h^2}}-\frac{x_1+s}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\right),$$
$$\xi_1=\tau\left(\frac{x_1-s}{\sqrt{(x_1-s)^2+x_2^2+h^2}}+\frac{x_1+s}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\right),$$
$$\xi_2=\tau\left(\frac{x_2}{\sqrt{(x_1-s)^2+x_2^2+h^2}}+\frac{x_2}{\sqrt{(x_1+s)^2+x_2^2+h^2}}\right). \tag{13}$$

From the first two relations in (12) and (13), we have

$$\sqrt{(y_1-s)^2+y_2^2+h^2}+\sqrt{(y_1+s)^2+y_2^2+h^2}$$
$$=\sqrt{(x_1-s)^2+x_2^2+h^2}+\sqrt{(x_1+s)^2+x_2^2+h^2} \tag{14}$$

and

$$\frac{y_1-s}{\sqrt{(y_1-s)^2+y_2^2+h^2}}-\frac{y_1+s}{\sqrt{(y_1+s)^2+y_2^2+h^2}}$$
$$=\frac{x_1-s}{\sqrt{(x_1-s)^2+x_2^2+h^2}}-\frac{x_1+s}{\sqrt{(x_1+s)^2+x_2^2+h^2}}. \tag{15}$$

We will use the prolate spheroidal coordinates to solve for x and y. We let

$$\begin{aligned} x_1&=s\cosh\rho\cos\phi, & y_1&=s\cosh\rho'\cos\phi',\\ x_2&=s\sinh\rho\sin\phi\cos\theta, & y_2&=s\sinh\rho'\sin\phi'\cos\theta',\\ x_3&=h+s\sinh\rho\sin\phi\sin\theta, & y_3&=h+s\sinh\rho'\sin\phi'\sin\theta' \end{aligned} \tag{16}$$

with $\rho>0$, $0\leqslant\phi\leqslant\pi$ and $0\leqslant\theta<2\pi$.

In this case $x_3=0$ and we use it to solve for h. Hence

$$(x_1-s)^2+x_2^2+h^2=s^2(\cosh\rho-\cos\phi)^2$$

and

$$(x_1+s)^2+x_2^2+h^2=s^2(\cosh\rho+\cos\phi)^2.$$

Noting that $s>0$ and $\cosh\rho\pm\cos\phi>0$, the first relation given by (14) in these coordinates becomes

$$s(\cosh\rho-\cos\phi)+s(\cosh\rho+\cos\phi)=s\bigl(\cosh\rho'-\cos\phi'\bigr)+s\bigl(\cosh\rho'+\cos\phi'\bigr)$$

from which we get

$$\cosh\rho=\cosh\rho' \quad\Rightarrow\quad \rho=\rho'.$$

The second relation given by (15) becomes

$$\frac{\cosh\rho\cos\phi-1}{\cosh\rho-\cos\phi}-\frac{\cosh\rho\cos\phi+1}{\cosh\rho+\cos\phi}=\frac{\cosh\rho\cos\phi'-1}{\cosh\rho-\cos\phi'}-\frac{\cosh\rho\cos\phi'+1}{\cosh\rho+\cos\phi'}.$$

After simplification we get

$$\frac{\sin^2\phi}{\cosh^2\rho-\cos^2\phi}=\frac{\sin^2\phi'}{\cosh^2\rho-\cos^2\phi'}$$

which implies

$$\bigl(\cosh^2\rho-1\bigr)\bigl(\sin^2\phi-\sin^2\phi'\bigr)=0.$$

Thus $\sin\phi=\pm\sin\phi'\Rightarrow\phi=\pm\phi',\pi\pm\phi'$.

We remark that $\cos\theta=\pm\sqrt{1-\frac{h^2}{s^2\sinh^2\rho\sin^2\phi}}=\pm\cos\theta'$ and note that $x_3=0$ implies that $\sin(\phi)\neq 0$, so that division by $\sin(\phi)$ is allowed here. We also remark that it is enough to consider $\cos\theta=\cos\theta'$ as no additional relations are introduced by considering $\cos\theta=-\cos\theta'$.

Now we go back to x and y coordinates.

If $\phi'=\phi$ then $x_1=y_1$, $x_2=y_2$, $\xi_i=\eta_i$ for $i=1,2$. For these points, the composition, $C^t\circ C\subset\Delta=\{(x,\xi;x,\xi)\}$.

If $\phi'=-\phi$ then $x_1=y_1$, $-x_2=y_2$, $\xi_1=\eta_1$, $-\xi_2=\eta_2$. For these points, the composition, $C^t\circ C$ is a subset of $C_1=\{(x_1,x_2,\xi_1,\xi_2;x_1,-x_2,\xi_1,-\xi_2)\}$ which is the graph of $\chi_1(x,\xi)=(x_1,-x_2,\xi_1,-\xi_2)$. This in the base space represents the reflection about the x_1 axis.

If $\phi'=\pi-\phi$ then $-x_1=y_1$, $x_2=y_2$, $-\xi_1=\eta_1$, $\xi_2=\eta_2$. For these points, the composition $C^t\circ C$ is a subset of $C_2=\{(x_1,x_2,\xi_1,\xi_2;-x_1,x_2,-\xi_1,\xi_2)\}$ which is the graph of $\chi_2(x,\xi)=(-x_1,x_2,-\xi_1,\xi_2)$. This in the base space represents the reflection about the x_2 axis.

If $\phi'=\pi+\phi$ then $-x_1=y_1$, $-x_2=y_2$, $-\xi_1=\eta_1$, $-\xi_2=\eta_2$. For these points, $C^t\circ C$ is a subset of $C_3=\{(x_1,x_2,\xi_1,\xi_2;-x_1,-x_2,-\xi_1,-\xi_2)\}$ which is the graph of $\chi_3(x,\xi)=(-x_1,-x_2,-\xi_1,-\xi_2)$. This in the base space represents the reflection about the origin.

Notice that $\chi_1\circ\chi_1=\mathrm{Id}$, $\chi_2\circ\chi_2=\mathrm{Id}$, $\chi_1\circ\chi_2=\chi_3$.

So far we have obtained that $C^t\circ C\subset\Delta\cup C_1\cup C_2\cup C_3$.

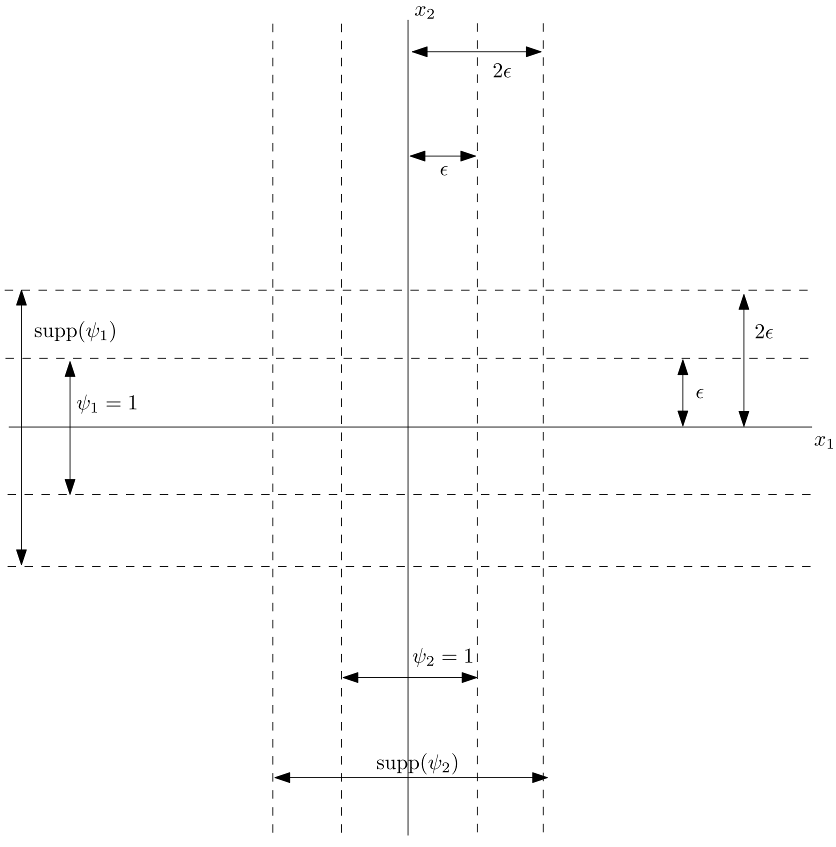

Fig. 1. Support of the cut-off functions ψ_1 and ψ_2.

Next we consider the intersections of any two of these Lagrangians. We have:

Δ intersects C_1 cleanly in codimension 2, $\Delta \cap C_1 = \{(x, \xi; y, \eta) \mid x_2 = 0 = \xi_2\}$.
Δ intersects C_2 cleanly in codimension 2, $\Delta \cap C_2 = \{(x, \xi; y, \eta) \mid x_1 = 0 = \xi_1\}$.
C_1 intersects C_3 cleanly in codimension 2, $C_1 \cap C_3 = \{(x, \xi; y, \eta) \mid x_1 = 0 = \xi_1\}$.
C_2 intersects C_3 cleanly in codimension 2, $C_2 \cap C_3 = \{(x, \xi; y, \eta) \mid x_2 = 0 = \xi_2\}$.
$\Delta \cap C_3 = \emptyset = C_1 \cap C_2$. $\square$

Theorem 5.2. *Let F be as in (4) with order m. Then F^*F can be decomposed as a sum of operators belonging in $I^{2m,0}(\Delta, C_1) + I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3) + I^{2m,0}(C_2, C_3)$.*

Proof. Recall from Theorem 2.1, that the canonical relation of F drops rank on the union of two sets, Σ_1 and Σ_2. Accordingly, we decompose F into components such that the canonical relation of each component is either supported near a subset of the union of these two sets, one of these two sets or away from both these sets. More precisely, we let ψ_1 and ψ_2 be two infinitely differentiable functions defined as follows (refer Fig. 1):

$$\psi_1(x) = \begin{cases} 1, & \text{on } \{(x_1, x_2)\colon\ |x_2| < \epsilon\}, \\ 0, & \text{on } \{(x_1, x_2)\colon\ |x_2| > 2\epsilon\} \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 1, & \text{on } \{(x_1, x_2)\colon\ |x_1| < \epsilon\}, \\ 0, & \text{on } \{(x_1, x_2)\colon\ |x_1| > 2\epsilon\}. \end{cases}$$

Then we write $F = F_0 + F_1 + F_2 + F_3$ where F_i are given in terms of their kernels

$$K_{F_0} = \int e^{-i\varphi} a\psi_1\psi_2 \,\mathrm{d}\omega, \qquad K_{F_1} = \int e^{-i\varphi} a\psi_1(1-\psi_2)\,\mathrm{d}\omega,$$
$$K_{F_2} = \int e^{-i\varphi} a(1-\psi_1)\psi_2 \,\mathrm{d}\omega, \qquad K_{F_3} = \int e^{-i\varphi} a(1-\psi_1)(1-\psi_2)\,\mathrm{d}\omega,$$

where φ is the phase function of F in (4). Now we consider F^*F, which using the decomposition of F as above can be written as

$$\begin{aligned} F^*F = F_0^*F + (F_1+F_2)^*F_0 + F_1^*F_1 + F_2^*F_2 + F_1^*F_2 \\ + F_2^*F_1 + F_1^*F_3 + F_2^*F_3 + F_3^*F. \end{aligned} \tag{17}$$

The theorem now follows from Lemmas 5.3, 5.4 and Theorem 5.5 below, where we analyze each of the compositions above. □

Lemma 5.3. *F_0, $F_1^*F_2$ and $F_2^*F_1$ are smoothing operators.*

Proof. We will only prove that F_0 and $F_1^*F_2$ are smoothing. The proof for $F_2^*F_1$ is similar to that of $F_1^*F_2$. Let $\tilde{\varphi} = \frac{1}{\omega}\varphi$, where φ is the phase function in (4).

For $\delta = 18\epsilon^2/h$, we analyze F_0 according to the following cases:

(a) $\{(s,t)\colon\ s>0,\ 0<t<2\sqrt{s^2+h^2}-\delta\}$.
For this case, we show that K_{F_0} is smoothing. For, on $\{(s,t)\colon\ s>0,\ t<2\sqrt{s^2+h^2}-\delta\}$, $\tilde{\varphi}$ is bounded away from 0 and hence is a smooth function. Therefore for any $m \geqslant 0$

$$\left(\frac{\mathrm{i}}{\tilde{\varphi}}\right)^m K_{F_0}(s,t,x) = \int \partial_\omega^m \left(e^{-\mathrm{i}\omega\tilde{\varphi}(s,t,x)}\right)\psi_1(x)\psi_2(x)a(s,t,x,\omega)\,\mathrm{d}\omega.$$

Now by integration by parts, the order of the amplitude can be made smaller than any negative number. Therefore K_{F_0} is smoothing.

(b) $\{(s,t)\colon\ s>0,\ |t-2\sqrt{s^2+h^2}| \leqslant \delta\}$.
For (s,t) in this set, the kernel K_{F_0} is identically 0 due to our choice of the function $g(s,t)$ in (3).

(c) $\{(s,t)\colon\ s>0,\ t>2\sqrt{s^2+h^2}+\delta\}$.
In this case, we have that depending on our choice of x, the kernel K_{F_0} is either identically 0 or smoothing. For, if we consider x in the complement of the set $(-2\epsilon, 2\epsilon)^2$, then due to the fact that $\operatorname{supp}(\psi_1(x)\psi_2(x)) \subset [-2\epsilon, 2\epsilon]^2$, we have that the kernel is identically 0. Now

if we consider $x \in (-2\epsilon, 2\epsilon)^2$, then $\tilde{\varphi}$ is never vanishing. Then by an integration by parts argument as in Case (a) above, we have that K_{F_0} is smoothing.

Now we consider $F_2^* F_1$. We have

$$K_{F_1}(s,t,x) = \int e^{-\mathrm{i}\omega\tilde{\varphi}(s,t,x)} \psi_1(x)\big(1-\psi_2(x)\big) a(s,t,x,\omega)\,\mathrm{d}\omega$$

and

$$K_{F_2}^*(x,s,t) = \int e^{\mathrm{i}\omega\tilde{\varphi}(s,t,x)} \big(1-\psi_1(x)\big)\psi_2(x)\overline{a(s,t,x,\omega)}\,\mathrm{d}\omega.$$

Due to the cut-off functions ψ_1 and ψ_2 in these kernels, we are only interested in those singularities lying above a small neighborhood of the rectangles with vertices $(\pm\epsilon, \pm\epsilon)$, $(\pm\epsilon, \pm 2\epsilon)$, $(\pm 2\epsilon, \pm\epsilon)$, $(\pm 2\epsilon, \pm 2\epsilon)$.

We have that K_{F_1} is smoothing when x values are restricted to a small neighborhood of these rectangles. For, as in the previous case, we consider the three cases: For Cases (a) and (c), the kernel K_{F_1} is smoothing and the proof is identical as before. For Case (b), due to the choice of the function $g(s,t)$, the kernel $K_{F_1} = 0$. Therefore $F_2^* F_1$ is smoothing. □

Lemma 5.4. *$F_1^* F_3$, $F_2^* F_3$ and $F_3^* F$ can be decomposed as a sum of operators belonging to the space $I^{2m}(\Delta) + I^{2m}(C_1 \setminus \Delta) + I^{2m}(C_2 \setminus \Delta) + I^{2m}(C_3 \setminus (C_1 \cup C_2))$.*

Proof. Each of these compositions is covered by the transverse intersection calculus. Below we will prove for the case of $F_1^* F_3$. For the other operators, the proofs are similar.

Let us decompose

$$F_3 = F_3^1 + F_3^2 + F_3^3 + F_3^4 \quad \text{and} \quad F_1^* = \big(F_1^1 + F_1^2 + F_1^3 + F_1^4\big)^*,$$

where the superscripts in both these sums denote restriction of F_3 and F_1, respectively, to each of the four quadrants. Note that in the decomposition of F_1^*, we stay away from Σ_1 by introducing a microlocal cut-off. This is valid because the support of the canonical relation of F_3 stays away from $\Sigma_1 \cup \Sigma_2$. Then we have

$$\big(F_1^1\big)^* F_3^1 \in I^{2m}(\Delta), \qquad \big(F_1^1\big)^* F_3^4 \in I^{2m}(C_1 \setminus \Delta),$$
$$\big(F_1^1\big)^* F_3^2 \in I^{2m}(C_2 \setminus \Delta) \quad \text{and} \quad \big(F_1^1\big)^* F_3^3 \in I^{2m}\big(C_3 \setminus (C_1 \cup C_2)\big).$$

The other compositions can be considered similarly. □

We are left with the analysis of the compositions $F_1^* F_1$ and $F_2^* F_2$. This is the content of the next theorem:

Theorem 5.5. *Let F_1 and F_2 be as above. Then*:

(a) $F_1^* F_1 \in I^{2m,0}(\Delta, C_1) + I^{2m,0}(C_2, C_3)$.
(b) $F_2^* F_2 \in I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3)$.

Proof. We consider $F_1^* F_1$. The proof for $F_2^* F_2$ is similar.

We decompose F_1 by introducing a smooth cut-off function $\psi_3(x)$ such that $\psi_3(x) = 1$ for $x_1 > \epsilon/2$ and supported on the right-half plane $x_1 \geqslant \epsilon/4$. That is, we write F_1 as

$$F_1 = F_1^+ + F_1^-,$$

where

$$F_1^+ V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)} \psi_1(x)\big(1-\psi_2(x)\big)\psi_3(x) a(s,t,x,\omega) V(x)\,\mathrm{d}x$$

and

$$F_1^- V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)} \psi_1(x)\big(1-\psi_2(x)\big)\big(1-\psi_3(x)\big) a(s,t,x,\omega) V(x)\,\mathrm{d}x.$$

Now

$$F_1^* F_1 = \big(F_1^+\big)^* F_1^+ + \big(F_1^-\big)^* F_1^+ + \big(F_1^+\big)^* F_1^- + \big(F_1^-\big)^* F_1^-. \tag{18}$$

The canonical relation of F_1^+ is a subset of (8) with the additional condition that $x_1 > \epsilon$. Then we have that $WF((F_1^+)^* F_1^+)' \subset \Delta \cup C_1$. For, we already saw in Proposition 5.1 that $WF(F^*F) \subset \Delta \cup C_1 \cup C_2 \cup C_3$. In our case, imposing the additional restriction that $x_1 > \epsilon$, the only contributions are in Δ and C_1. By a similar argument, we have that $WF((F_1^-)^* F_1^-)' \subset \Delta \cup C_1$.

Now let us consider the compositions $(F_1^-)^* F_1^+$ and $(F_1^+)^* F_1^-$. The wavefront sets of these operators are of the form (x,ξ,y,η) such that x_1 and y_1 have opposite signs. We have already established in Proposition 5.1 that $|x_i| = |y_i|$ and $|\xi_i| = |\eta_i|$ for $i = 1, 2$. Now with the additional restriction that x_1 and y_1 have opposite signs (and therefore ξ_1 and η_1 have different signs as well), we have contributions contained in only C_2 and C_3.

The Lagrangian pairs Δ, C_1 and C_2, C_3 intersect cleanly. Therefore there is a well-defined $I^{p,l}$ class – which we will identify shortly – in which each of the summands in (18) lie.

We now show that $(F_1^+)^* F_1^+, (F_1^-)^* F_1^- \in I^{2m,0}(\Delta, C_1)$ and that $(F_1^-)^* F_1^+,\ (F_1^+)^* F_1^- \in I^{2m,0}(C_2, C_3)$. We follow the ideas of [5], where the iterated regularity theorem of was used to prove an analogous result. The ideas of [5] were recently employed to prove a similar result for a common-offset geometry in [17]. The proof we give is similar to the one given in [17], but the phase function we work with is different.

We first consider the generator of the ideal of functions that vanish on $\Delta \cup C_1$ [5]

$$\tilde p_1 = x_1 - y_1, \qquad \tilde p_2 = x_2^2 - y_2^2, \qquad \tilde p_3 = \xi_1 - \eta_1, \qquad \tilde p_4 = (x_2 + y_2)(\xi_2 - \eta_2),$$
$$\tilde p_5 = (x_2 - y_2)(\xi_2 + \eta_2), \qquad \tilde p_6 = \xi_2^2 - \eta_2^2.$$

Let $p_i = q_i \tilde p_i$, for $1 \leqslant i \leqslant 6$, where q_1, q_2 are homogeneous of degree 1 in (ξ,η), q_3, q_4 and q_5 are homogeneous of degree 0 in (ξ,η) and q_6 is homogeneous of degree -1 in (ξ,η). Let P_i be pseudodifferential operators with principal symbols p_i for $1 \leqslant i \leqslant 6$.

We show in Appendix A that each $\tilde{p}_i$ can be expressed in the following forms:

$$\tilde{p}_1 = \frac{f_{11}(x,y,s)}{\omega}\partial_s\Phi + f_{12}(x,y,s)\partial_\omega\Phi, \tag{19}$$

$$\tilde{p}_2 = \frac{f_{21}(x,y,s)}{\omega}\partial_s\Phi + f_{22}(x,y,s)\partial_\omega\Phi, \tag{20}$$

$$\tilde{p}_3 = f_{31}(x,y,s)\partial_s\Phi + \omega f_{32}(x,y,s)\partial_\omega\Phi, \tag{21}$$

$$\tilde{p}_4 = f_{41}(x,y,s)\partial_s\Phi + \omega f_{42}(x,y,s)\partial_\omega\Phi, \tag{22}$$

$$\tilde{p}_5 = f_{51}(x,y,s)\partial_s\Phi + \omega f_{52}(x,y,s)\partial_\omega\Phi, \tag{23}$$

$$\tilde{p}_6 = \omega f_{61}(x,y,s)\partial_s\Phi + \omega^2 f_{62}(x,y,s)\partial_\omega\Phi, \tag{24}$$

where f_{ij} for $1 \leqslant i \leqslant 6$ and $j = 1, 2$ are smooth functions.

Now the rest of the proof is the same as in [5, Theorem 1.6]. We give it for completeness.

Let K_1^+ be the kernel of $(F_1^+)^* F_1^+$. This is the kernel in (10), but with $\tilde{a}$ there replaced by $\psi_1(x)(1-\psi_2(x))\psi_3(x)\psi_1(y)(1-\psi_2(y))\psi_3(y)\tilde{a}(x,y,s,\omega)$. For simplicity, we rename this as $\tilde{a}$ again.

We then have that $\tilde{a} \in S^{2m+1}$ and

$$\begin{aligned}
P_1 K_1^+(x,y) &= \int e^{\mathrm{i}\Phi(x,y,s,\omega)}\tilde{a}(x,y,s,\omega)q_1\left[\frac{f_{11}(x,y,s)}{\omega}\partial_s\Phi + f_{12}(x,y,s)\partial_\omega\Phi\right]\mathrm{d}s\,\mathrm{d}\omega \\
&= \int \partial_s\left[e^{\mathrm{i}\Phi(x,y,s,\omega)}\right]\frac{q_1}{\mathrm{i}\omega}\tilde{a}(x,y,s,\omega)f_{11}(x,y,s)\,\mathrm{d}s\,\mathrm{d}\omega \\
&\quad + \int \partial_\omega\left[e^{\mathrm{i}\Phi(x,y,s,\omega)}\right]\frac{q_1}{\mathrm{i}}\tilde{a}(x,y,s,\omega)f_{12}(x,y,s)\,\mathrm{d}s\,\mathrm{d}\omega.
\end{aligned}$$

By integration by parts

$$\begin{aligned}
&= -\left\{\int e^{\mathrm{i}\Phi(x,y,s,\omega)}\partial_s\left[\frac{q_1}{\mathrm{i}\omega}\tilde{a}(x,y,s,\omega)f_{11}(x,y,s)\right]\mathrm{d}s\,\mathrm{d}\omega\right. \\
&\quad \left. + \int e^{\mathrm{i}\Phi(x,y,s,\omega)}\partial_\omega\left[\frac{q_1}{\mathrm{i}}\tilde{a}(x,y,s,\omega)f_{12}(x,y,s)\right]\mathrm{d}s\,\mathrm{d}\omega\right\}.
\end{aligned}$$

Note that q_1 is homogeneous of degree 1 in ω, and $\tilde{a}$ is a symbol of order $2m+1$, hence each amplitude term in the sum above is of order $2m+1$.

Therefore by Proposition 3.7, we have that $P_1 K_1^+ \in H_{\mathrm{loc}}^{s_0}$ for some s_0.

A similar argument works for each of the other five pseudodifferential operators. Hence by Proposition 3.7, we have that $(F_1^+)^* F_1^+ \in I^{p,l}(\Delta, C_1)$. Because C is a local canonical graph away from Σ, the transverse intersection calculus applies for the composition $(F_1^+)^* F_1^+$ away from Σ. Hence $(F_1^+)^* F_1^+$ is of order $2m$ on $\Delta \setminus C_1$ and $C_1 \setminus \Sigma$. Since $(F_1^+)^* F_1^+$ is of order $p+l$ on $\Delta \setminus \Sigma$ and is of order p on $C_1 \setminus \Sigma$, we have that $p = 2m$ and $l = 0$. Therefore $(F_1^+)^* F_1^+ \in I^{2m,0}(\Delta, C_1)$. Similarly $(F_1^-)^* F_1^- \in I^{2m,0}(\Delta, C_1)$.

To show that $(F_1^-)^* F_1^+, (F_1^+)^* F_1^- \in I^{2m,0}(C_2, C_3)$ we can use the iterated regularity result as above.

The generators of the ideal of functions that vanish on $C_2 \cup C_3$ are:

$\tilde{r}_1 = x_1 + y_1$, $\tilde{r}_2 = \xi_1 + \eta_1$ and $\tilde{p}_2$, $\tilde{p}_4$, $\tilde{p}_5$, $\tilde{p}_6$ are the same as in (20), (22), (23), and (24) respectively. Four of the functions in the ideal are the same as in the proof above and we can find similar expressions for the first two.

However we will also give an alternate proof below. □

Proposition 5.6. $(F_1^-)^* F_1^+, (F_1^+)^* F_1^- \in I^{2m,0}(C_2, C_3)$.

Proof. We show for $(F_1^-)^* F_1^+$. The proof for the other case is similar. Consider the operator R defined as follows:

$$RV(x_1, x_2) = V(-x_1, x_2).$$

This is a Fourier integral operator of order 0 with the canonical relation C_2. This is because,

$$\begin{aligned} RV(x_1, x_2) &= \int e^{\mathrm{i}(x-y)\cdot\xi} R_2 V(y_1, y_2)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \int e^{\mathrm{i}(x-y)\cdot\xi} V(-y_1, y_2)\,\mathrm{d}y\,\mathrm{d}\xi \\ &= \int e^{\mathrm{i}[(x_1+y_1)\xi_1+(x_2-y_2)\xi_2]} V(y_1, y_2)\,\mathrm{d}y\,\mathrm{d}\xi. \end{aligned}$$

It is easy to check that canonical relation is C_2.

Now consider the operator $\widetilde{F} = F_1^- \circ R$. This is given by

$$\widetilde{F} V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)} b(s,t,x,\omega) V(x)\,\mathrm{d}x\,\mathrm{d}\omega$$

where

$$b(s,t,x,\omega) = \big[\psi_1(1-\psi_2)(1-\psi_3)\big](-x_1, x_2) a(s,t,-x_1,x_2,\omega).$$

Note that $(1-\psi_3)(-x_1, x_2) = \psi_3(x_1, x_2)$ except in a small neighborhood of the origin. Since $\psi_1(1-\psi_2)$ is 0 in a neighborhood of the origin, and noting that we can arrange ψ_1 and ψ_2 to be symmetric with respect to x_1, we have

$$\widetilde{F} V(s,t) = \int e^{-\mathrm{i}\varphi(s,t,x,\omega)} \big[\psi_1(1-\psi_2)\psi_3\big](x) a(s,t,-x_1,x_2,\omega) V(x)\,\mathrm{d}x\,\mathrm{d}\omega.$$

Now we have that $\widetilde{F}^* F_1^+ \in I^{2m,0}(\Delta, C_1)$. In fact the kernel of this operator has the same form as in (10) and the same proof as in Theorem 5.5 applies. Next we use [15, Proposition 4.1] to show that $R^* \widetilde{F}^* F_1^+ \in I^{2m,0}(C_2, C_3)$. It is straightforward to check that $C_2 \circ \Lambda = C_2$, $C_2 \circ C_1 = C_3$ and $C_2 \times \Delta$ (as well as $C_2 \times C_1$) intersects $T^*X \times \Delta_{T^*X} \times T^*X$ transversally. Hence the hypotheses of [15, Proposition 4.1] are verified and we conclude that $R^* \widetilde{F}^* F_1^+ \in I^{2m,0}(C_2, C_3)$. Since $\widetilde{F}^* = R^*(F_1^-)^*$ and $(R^*)^2 = \mathrm{Id}$ we have $(F_1^-)^* F_1^+ \in I^{2m,0}(C_2, C_3)$. □

Since $I^{2m}(\Delta) \in I^{2m,0}(\Delta, C_1)$, $I^{2m}(C_i \setminus \Delta) \in I^{2m,0}(\Delta, C_i)$ for $i = 1, 2$ and $I^{2m}(C_3 \setminus (C_1 \cup C_2)) \in I^{2m,0}(C_1, C_3)$, Theorem 5.2 follows using Lemmas 5.3, 5.4, Theorem 5.5 and Proposition 5.6.

Remark 5.7. Using the properties of the $I^{p,l}$ classes, $F^*F \in I^{2m,0}(\Delta, C_1)$ implies that $F^*F \in I^{2m}(\Delta \setminus C_1)$ and $F^*F \in I^{2m}(C_1 \setminus \Delta)$. This means that F^*F has the same order on both Δ and C_1 which implies that the artifact C_1 has the same strength as the initial singularities given by Δ. A similar statement can be applied to C_2 and C_3. Note that C_1 gives an artifact that is a reflection in the x_1 axis, C_2 gives an artifact that is a reflection in the x_2 axis, and C_3 gives an artifact that is a reflection in the origin.

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Appendix A

Here we give the derivations of (19)–(24). We will work in the coordinate system defined in (16). We extend the phase function in (11) to $\mathbb{R}^3$ by letting

$$\widetilde{\Phi} = \omega\Big\{\sqrt{(y_1 - s)^2 + y_2^2 + (y_3 - h)^2} + \sqrt{(y_1 + s)^2 + y_2^2 + (y_3 - h)^2} \\ - \Big(\sqrt{(x_1 - s)^2 + x_2^2 + (x_3 - h)^2} + \sqrt{(x_1 + s)^2 + x_2^2 + (x_3 - h)^2}\Big)\Big\}.$$

Then note that

$$\partial_\omega \widetilde{\Phi}|_{x_3=y_3=0} = \partial_\omega \Phi \quad \text{and} \quad \partial_s \widetilde{\Phi}|_{x_3=y_3=0} = \partial_s \Phi.$$

A.1. Expression for $x_1 - y_1$

We obtain an expression for $x_1 - y_1$ in the form

$$A_1 := x_1 - y_1 = \frac{f_{11}(x, y, s)}{\omega} \partial_s \Phi + f_{12}(x, y, s) \partial_\omega \Phi,$$

where f_{11} and f_{12} are smooth functions. In the coordinate system (16),

$$A_1 = s\bigl(\cosh\rho\cos\phi - \cosh\rho'\cos\phi'\bigr),$$
$$\partial_\omega\tilde{\phi} = 2s\bigl(\cosh\rho' - \cosh\rho\bigr),$$
$$\begin{aligned}\partial_s\widetilde{\Phi} &= \omega\left\{\left(\frac{\cosh\rho'\cos\phi' + 1}{\cosh\rho' + \cos\phi'} - \frac{\cosh\rho'\cos\phi' - 1}{\cosh\rho' - \cos\phi'}\right)\right.\\ &\quad \left.- \left(\frac{\cosh\rho\cos\phi + 1}{\cosh\rho + \cos\phi} - \frac{\cosh\rho\cos\phi - 1}{\cosh\rho - \cos\phi}\right)\right\}\\ &= 2\omega\left\{\frac{\cosh\rho' - \cosh\rho'\cos^2\phi'}{\cosh^2\rho' - \cos^2\phi'} - \frac{\cosh\rho - \cosh\rho\cos^2\phi}{\cosh^2\rho - \cos^2\phi}\right\}.\end{aligned}$$

After simplifying, we get,

$$\begin{aligned}&= 2\omega\left\{\frac{(\cosh\rho - \cosh\rho')(\cosh\rho\cosh\rho' - \cos^2\phi\cos^2\phi')}{(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}\right.\\ &\quad \left.+ \frac{(\cosh\rho'\cos^2\phi - \cosh\rho\cos^2\phi')(\cosh\rho\cosh\rho' - 1)}{(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}\right\}.\end{aligned}$$

Now observing that $\cosh\rho' - \cosh\rho = \frac{\partial_\omega\widetilde{\Phi}}{2s}$ and adding and subtracting $\cosh\rho\cos^2\phi$ to the second term on the right above, we have,

$$\begin{aligned}&\partial_s\widetilde{\Phi} + \frac{\omega}{s}\frac{(\cosh\rho\cosh\rho' - \cos^2\phi\cos^2\phi')}{(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}\partial_\omega\widetilde{\Phi}\\ &\quad = 2\omega\frac{\{(\cosh\rho' - \cosh\rho)\cos^2\phi + \cosh\rho(\cos^2\phi - \cos^2\phi')\}(\cosh\rho\cosh\rho' - 1)}{(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}.\end{aligned}$$

From this we get

$$\begin{aligned}\cos\phi - \cos\phi' &= \frac{\partial_s\widetilde{\Phi}(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}{2\omega\cosh\rho(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')}\\ &\quad + \frac{\frac{\omega}{s}\partial_\omega\widetilde{\Phi}((\cosh\rho\cosh\rho' - \cos^2\phi\cos^2\phi') - \cos^2\phi(\cosh\rho\cosh\rho' - 1))}{2\omega\cosh\rho(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')}.\end{aligned} \tag{A.1}$$

Now note that

$$A_1 = \frac{s\partial_s \widetilde{\Phi}(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}{2\omega(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')} - \frac{\cos\phi'}{2}\partial_\omega \widetilde{\Phi}$$
$$+ \frac{\omega\partial_\omega \widetilde{\Phi}((\cosh\rho\cosh\rho' - \cos^2\phi\cos^2\phi') - \cos^2\phi(\cosh\rho\cosh\rho' - 1))}{2\omega(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')}.$$

Now letting $x_3 = y_3 = 0$, we see that we have written $x_1 - y_1$ as a combination in terms of $\partial_\omega \Phi$ and $\partial_s \Phi$ as follows:

$$= \frac{s(\cosh^2\rho' - \cos^2\phi')(\cosh^2\rho - \cos^2\phi)}{2(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')}\frac{\partial_s \Phi}{\omega}$$
$$+ \left\{\frac{((\cosh\rho\cosh\rho' - \cos^2\phi\cos^2\phi') - \cos^2\phi(\cosh\rho\cosh\rho' - 1))}{2(\cosh\rho'\cosh\rho - 1)(\cos\phi + \cos\phi')} - \frac{\cos\phi'}{2}\right\}\partial_\omega \Phi.$$

We can write the above expression in the Cartesian coordinate system. First, for simplicity, let

$$X_1 = \sqrt{(x_1 - s)^2 + x_2^2 + h^2},$$
$$X_2 = \sqrt{(x_1 + s)^2 + x_2^2 + h^2}$$

with Y_1 and Y_2 being similarly defined with x replaced by y. Then we have

$$x_1 - y_1 = \frac{s(\frac{Y_1Y_2}{s^2})(\frac{X_1X_2}{s^2})}{2((\frac{Y_1+Y_2}{2s})(\frac{X_1+X_2}{2s}) - 1)(\frac{x_1}{\frac{X_1+X_2}{2}} + \frac{y_1}{\frac{Y_1+Y_2}{2}})}\frac{\partial_s \Phi}{\omega}$$
$$+ \left\{\frac{(\frac{X_1+X_2}{2s})(\frac{Y_1+Y_2}{2s}) - \frac{x_1^2y_1^2}{(\frac{X_1+X_2}{2})^2(\frac{Y_1+Y_2}{2})^2} - (\frac{x_1^2}{(\frac{X_1+X_2}{2})^2}((\frac{Y_1+Y_2}{2s})(\frac{X_1+X_2}{2s}) - 1))}{2((\frac{Y_1+Y_2}{2s})(\frac{X_1+X_2}{2s}) - 1)(\frac{x_1}{\frac{X_1+X_2}{2}} + \frac{y_1}{\frac{Y_1+Y_2}{2}})}\right.$$
$$\left. - \frac{y_1}{Y_1 + Y_2}\right\}\partial_\omega \Phi.$$

A.2. *Expression for $x_2^2 - y_2^2$*

Now we write $x_2^2 - y_2^2$ in the form

$$A_2 := x_2^2 - y_2^2 = \frac{f_{21}(x, y, s)}{\omega}\partial_s \Phi + f_{22}(x, y, s)\partial_\omega \Phi, \tag{A.2}$$

where f_{21} and f_{22} are smooth functions. A_2 in the coordinate system (16) is

$$A_2 = s^2(\sinh^2\rho\sin^2\phi\cos^2\theta - \sinh^2\rho'\sin^2\phi'\cos^2\theta')$$
$$= s^2(\sinh^2\rho\sin^2\phi - \sinh^2\rho'\sin^2\phi') \tag{A.3}$$
$$+ s^2(\sinh^2\rho'\sin^2\phi'\sin^2\theta' - \sinh^2\rho\sin^2\phi\sin^2\theta). \tag{A.4}$$

For $x_3 = y_3 = 0$, (A.4) is 0. Therefore we focus only on the term (A.3), which we still denote as A_2, and obtain an expression of the form (A.2) for this term.

Using the formulas $\sinh^2\rho = \cosh^2\rho - 1$, $\sin^2\phi = 1 - \cos^2\phi$, and simplifying, we have

$$\begin{aligned} A_2 &= s^2\big(\big(\cosh^2\rho - \cosh^2\rho'\big)\sin^2\phi - \big(\cos^2\phi - \cos^2\phi'\big)\sinh^2\rho'\big) \\ &= s^2\big(\big(\cosh\rho - \cosh\rho'\big)\big(\cosh\rho + \cosh\rho'\big) \\ &\quad - \big(\cos\phi - \cos\phi'\big)\big(\cos\phi + \cos\phi'\big)\sinh^2\rho'\big). \end{aligned}$$

Recall that $\cosh\rho - \cos\rho' = \frac{\partial_\omega\tilde{\phi}}{2s}$ and using the expression for $\cos\phi - \cos\phi'$ in (A.1), and setting $x_3 = y_3 = 0$, we see that $x_2^2 - y_2^2$ can be written in the form (A.2).

A.3. Expression for $\xi_1 - \eta_1$

Note that $\xi_1 = \partial_{x_1}\Phi$ and $\eta_1 = -\partial_{y_1}\Phi$ and so $A_3 := \xi_1 - \eta_1 = \partial_{x_1}\Phi + \partial_{y_1}\Phi$.

We have

$$A_3 = \omega\left\{\left(\frac{y_1 - s}{|y - \gamma_T(s)|} + \frac{y_1 + s}{|y - \gamma_R(s)|}\right) - \left(\frac{x_1 - s}{|x - \gamma_T(s)|} + \frac{x_1 + s}{|x - \gamma_R(s)|}\right)\right\}.$$

In the coordinate system (16) this is

$$= 2\omega\left(\frac{\sinh^2\rho'\cos\phi'}{\cosh^2\rho' - \cos^2\phi'} - \frac{\sinh^2\rho\cos\phi}{\cosh^2\rho - \cos^2\phi}\right).$$

Simplifying this, we get

$$\begin{aligned} &= 2\omega\left\{\frac{(\cos\phi - \cos\phi')(-\sinh^2\rho\cosh^2\rho' - \sinh^2\rho'\cos\phi\cos\phi')}{(\cosh^2\rho - \cos^2\phi)(\cosh^2\rho' - \cos^2\phi')}\right. \\ &\quad \left. + \frac{(\cosh\rho - \cosh\rho')(\cosh\rho + \cosh\rho')\cos\phi'(1 - \cos\phi\cos\phi')}{(\cosh^2\rho - \cos^2\phi)(\cosh^2\rho' - \cos^2\phi')}\right\}. \end{aligned}$$

Now noting that $\cosh\rho - \cosh\rho' = \frac{\partial_\omega\widetilde{\Phi}}{2s}$ and using the formula (A.1) for $\cos\phi - \cos\phi'$ and setting $x_3 = y_3 = 0$, we can write A_3 in the form (21).

A.4. Expression for $(x_2 - y_2)(\xi_2 + \eta_2)$

Using the coordinate system (16), we can write $A_4 := (x_2 + y_2)(\xi_2 - \eta_2)$ (up to a negative sign) as

$$\begin{aligned} &(x_2 - y_2)(\xi_2 + \eta_2) \\ &\quad = \omega(x_2 - y_2)\left(\frac{x_2}{|x - \gamma_T|} + \frac{x_2}{|x - \gamma_R|} + \frac{y_2}{|y - \gamma_T|} + \frac{y_2}{|y - \gamma_R|}\right) \\ &\quad = \frac{2\omega}{s}\left(\frac{x_2^2\cosh\rho}{\cosh^2\rho - \cos^2\theta} - \frac{y^2\cosh\rho'}{\cosh^2\rho' - \cos^2\theta'} + \frac{x_2 y_2\cosh\rho'}{\cosh^2\rho' - \cos^2\theta'} - \frac{x_2 y_2\cosh\rho}{\cosh^2\rho - \cos^2\theta}\right) \end{aligned}$$

$$= \frac{2\omega}{s}\left(\frac{x_2^2\cosh\rho}{\cosh^2\rho-\cos^2\theta} - \frac{x_2^2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'} + \left(x_2^2-y_2^2\right)\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\right.$$
$$\left. + \frac{x_2y_2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'} - \frac{x_2y_2\cosh\rho}{\cosh^2\rho-\cos^2\theta}\right).$$

Here we have added and subtracted $\frac{x_2^2\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}$ in the previous equation. Simplifying this we get,

$$(x_2-y_2)(\xi_2+\eta_2)$$
$$= \frac{2\omega}{s}\left(\left(x_2^2-x_2y_2\right)\left[\frac{(\cosh\rho\cosh\rho'+\cos^2\theta)(\cosh\rho'-\cosh\rho)}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\right.\right.$$
$$\left.\left. + \frac{\cosh\rho(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}\right] + \left(x_2^2-y_2^2\right)\frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}\right).$$

Now note that $\cosh\rho'-\cosh\rho = \frac{\partial_\omega\Phi}{2s}$ and we already have expressions for $\cos\theta-\cos\theta'$ (Eq. (A.1)) and for $x_2^2-y_2^2$ involving combinations of $\partial_\omega\Phi$ and $\partial_s\Phi$.

Hence we can write $(x_2-y_2)(\xi_2+\eta_2)$ in the form of (22). Note that our calculation in this section shows that

$$\frac{\cosh\rho}{\cosh^2\rho-\cos^2\theta} - \frac{\cosh\rho'}{\cosh^2\rho'-\cos^2\theta'}$$
$$= \frac{(\cosh\rho\cosh\rho'+\cos^2\theta)(\cosh\rho'-\cosh\rho)}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}$$
$$+ \frac{\cosh\rho(\cos\theta+\cos\theta')(\cos\theta-\cos\theta')}{(\cosh^2\rho-\cos^2\theta)(\cosh^2\rho'-\cos^2\theta')}. \tag{A.5}$$

This will be useful in the derivation of (24) in Appendix A.6 below.

A.5. *Expression for* $(x_2+y_2)(\xi_2-\eta_2)$

This is very similar to the derivation of the expression we obtained for $(x_2-y_2)(\xi_2+\eta_2)$.

A.6. *Expression for* $\xi_2^2-\eta_2^2$

We have

$$\xi_2^2-\eta_2^2 = \omega^2\left(\left(\frac{x_2}{|x-\gamma_T|}+\frac{x_2}{|x-\gamma_R|}\right)^2 - \left(\frac{y_2}{|y-\gamma_T|}+\frac{y_2}{|y-\gamma_R|}\right)^2\right)$$
$$= 4\omega^2\left(x_2^2\frac{\cosh^2\rho}{(\cosh^2\rho-\cos^2\theta)^2} - y_2^2\frac{\cosh^2\rho'}{(\cosh^2\rho'-\cos^2\theta')^2}\right)$$

$$= 4\omega^2 \left\{ x_2^2 \left(\frac{\cosh^2 \rho}{(\cosh^2 \rho - \cos^2 \theta)^2} - \frac{\cosh^2 \rho'}{(\cosh^2 \rho' - \cos^2 \theta')^2} \right) \right.$$
$$\left. + \left(x_2^2 - y_2^2\right) \frac{\cosh^2 \rho'}{(\cosh^2 \rho' - \cos^2 \theta')^2} \right\}.$$

Now using the computations for $x_2^2 - y_2^2$ and $(x_2 - y_2)(\xi_2 + \eta_2)$, in particular (A.5), we can write $\xi_2^2 - \eta_2^2$ in the form

$$\xi_2^2 - \eta_2^2 = \omega f_{61}(x, y, s)\partial_s \Phi + \omega^2 f_{62}(x, y, s)\partial_\omega \Phi$$

for smooth functions f_{61}, f_{62}.

Appendix B

Here we explain the reason for setting $g(s, t) = 0$ for $|t - 2\sqrt{s^2 + h^2}| < 20\epsilon^2/h$ in (3).

In the proof of Theorem 5.2 – more precisely Lemma 5.3 – recall that we consider four squares with vertices $(\pm\epsilon, \pm\epsilon)$, $(\pm\epsilon, \pm 2\epsilon)$, $(\pm 2\epsilon, \pm\epsilon)$, $(\pm 2\epsilon, \pm 2\epsilon)$. The motivation to choose $g = 0$ as above comes from the fact that we want the amplitude term a of F to be 0 for those (s, t) such that the ellipse defined by it is contained in a small neighborhood containing these squares.

One way to find this is as follows:

Given (s, t), the ellipse $\sqrt{(x_1 - s)^2 + x_2^2 + h^2} + \sqrt{(x_1 + s)^2 + x_2^2 + h^2} = t$ can be written in the form

$$\left(4t^2 - 16s^2\right)x_1^2 + 4t^2 x_2^2 = t^4 - 4t^2\left(s^2 + h^2\right).$$

Note that for this ellipse, the length of the semi-minor axis is always smaller that the length of the semi-major axis. The point $(2\epsilon, 2\epsilon)$ is $2\sqrt{2}\epsilon$ away from the origin. Therefore let us choose a t for which the ellipse passes through the point $(0, 3\epsilon)$. The time t is such that

$$t^2 - 4\left(s^2 + h^2\right) = 36\epsilon^2.$$

Hence

$$t - 2\sqrt{s^2 + h^2} = 36\epsilon^2 / \left(t + 2\sqrt{s^2 + h^2}\right).$$

Since $t > 0$ and $s > 0$, we have

$$t - 2\sqrt{s^2 + h^2} < 18\epsilon^2 / h.$$

This explains the factor 18 in Lemma 5.3. Now choosing 20 (any number bigger than 18 would do) explains our choice of the constant in (3).

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