

On the inversion formulas of Pestov and Uhlmann for the geodesic ray transform

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Abstract. We generalize the inversion formulas obtained by Pestov–Uhlmann for the geodesic ray transform of functions and vector fields on 2-dimensional manifolds with boundary of constant curvature. Our formulas hold for simple 2-dimensional manifolds whose curvatures are close to a constant.

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1 Introduction

Let (M, g) be a smooth compact Riemannian manifold with boundary ∂M . In this paper, we study the geodesic ray transform. This is defined as follows: Let $f \in L^2(M)$ be a symmetric tensor field of rank m written in local coordinates as,

$$f = f_{i_1 \dots i_m} dx^{i_1} \dots dx^{i_m}.$$

Here and below when repeating indices are encountered, Einstein summation convention is assumed. Let $\gamma : [0, l(\gamma)] \rightarrow M$ be a geodesic connecting boundary points, where $l(\gamma)$ is the length of the geodesic. The geodesic ray transform I_m of f along γ is defined as,

$$I_m f(\gamma) = \int_0^{l(\gamma)} f_{i_1 \dots i_m}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_m}(t) dt.$$

We will be interested only in the cases $m = 0$ (functions) and $m = 1$ (1-forms) here and denote their geodesic ray transforms by I_0 and I_1 respectively.

The geodesic ray transform is not injective in general. One needs additional restrictions on the metric and one such restriction is to assume that the Riemannian manifold (M, g) is *simple* [4] defined as follows:

Definition 1.1. A compact Riemannian manifold with boundary is simple if

- (a) The boundary ∂M is strictly convex: $\langle \nabla_\xi \nu, \xi \rangle < 0$ for $\xi \in T_x(\partial M)$ where ν is the unit inward normal to the boundary.
- (b) The map $\exp_x : \exp_x^{-1} M \rightarrow M$ is a diffeomorphism for each $x \in M$.

It is known that on a simple Riemannian manifold, $I_0 f$ uniquely determines f and $I_1 f$ uniquely determines the solenoidal component of f . See [4] and the references therein for these results. It is then natural to ask whether there exist explicit inversion formulas recovering a function or the solenoidal component of a vector field from their geodesic ray transforms. Several such inversion formulas are known in special cases [2].

In [3], Pestov and Uhlmann found Fredholm type inversion formulas for the geodesic ray transform on functions and vector fields on simple 2-dimensional manifolds. These formulas become exact inversion formulas for 2-dimensional manifolds of constant curvature. Moreover these inversion formulas hold even when conjugate points are present along geodesics.

The Fredholm-type inversion formulas of Pestov–Uhlmann are given by the following theorem:

Theorem 1.2 ([3, Theorem 5.4]). *Let (M, g) be a 2-dimensional Riemannian manifold. Then*

$$f + \mathcal{W}_g^2 f = \frac{1}{4\pi} \delta_\perp I_1^* (\alpha^* H(I_0 f)^- |_{\partial_+ SM}), \quad f \in L^2(M). \tag{1.1}$$

$$h + (\mathcal{W}_g^*)^2 h = \frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_\perp h)^+ |_{\partial_+ SM}), \quad h \in H_0^1(M). \tag{1.2}$$

Here \mathcal{W}_g is the operator (\mathcal{W}_g^* is its L^2 adjoint) on $L^2(M)$ defined by

$$\mathcal{W}_g f(x) = \frac{1}{2\pi} \int_{S_x} \mathcal{H}_\perp \left(\int_0^{\tau(x,\xi)} f(\gamma_{x,\xi}(t)) dt \right) dS_x(\xi),$$

where

$$\mathcal{H}_\perp u(x, \xi) = \xi_\perp^i \left(\frac{\partial u}{\partial x^i} - \Gamma_{ij}^k \xi^j \partial_k u \right),$$

where the derivative ∂_k on the unit sphere bundle is defined by

$$\partial_k u = \frac{\partial}{\partial \xi^k} u(x, \xi/|\xi|)|_{|\xi|=1}, \tag{1.3}$$

Γ_{ij}^k are the Christoffel symbols for the Levi–Civita connection on M and ξ_\perp is the 90° anticlockwise rotation of the unit vector ξ .

As shown in [3], for manifolds of constant curvature, $\mathcal{W}_g = \mathcal{W}_g^* = 0$ and hence these Fredholm-type formulas becomes exact inversion formulas. The right hand side of (1.1) and (1.2) can be determined from boundary measurements. We refer the reader to Pestov–Uhlmann’s paper [3] for the details.

In this paper, we prove the following result generalizing the above inversion formulas. Since simple Riemannian manifolds are diffeomorphic to balls in \mathbb{R}^n [4], from now on, we will fix M to be a closed ball in \mathbb{R}^n with Euclidean coordinates.

Theorem 1.3. *Let g_0 be a simple metric on M of constant curvature. There exists a positive ε with $\varepsilon \ll 1$ such that for all Riemannian metrics g on M with $\|g - g_0\|_{C^3} < \varepsilon$, the operators $I + \mathcal{W}_g^2$ and $I + (\mathcal{W}_g^*)^2$ are invertible on $L^2(M)$ and the inverses are given by Neumann series expansions.*

An immediate consequence of Theorem 1.3 is the following corollary.

Corollary 1.4. *Let g_0 be a simple Riemannian metric on M of constant curvature and let g be as in Theorem 1.3. Then*

$$f = \sum_{n=0}^{\infty} (-1)^n \mathcal{W}_g^{2n} \left(\frac{1}{4\pi} \delta_{\perp} I_1^* (\alpha^* H(I_0 f)^- |_{\partial_+ S M}) \right), \quad f \in L^2(M).$$

$$h = \sum_{n=0}^{\infty} (-1)^n (\mathcal{W}_g^*)^{2n} \left(\frac{1}{4\pi} I_0^* (\alpha^* H(I_1 \mathcal{H}_{\perp} h)^+ |_{\partial_+ S M}) \right), \quad h \in H_0^1(M).$$

As shown in [3], \mathcal{W}_g (\mathcal{W}_g^* is its L^2 adjoint) is an integral operator on $L^2(M)$ with kernel,

$$W_g(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{|\det(\exp_x^{-1})'(x, y)| \sqrt{\det(g(x))}}{\sqrt{\det(g(y))}}. \tag{1.4}$$

For the sake of completeness, and since we will use this kernel in the proof of Theorem 1.3, we sketch the proof of (1.4) given in [3].

We have

$$\begin{aligned} 2\pi \mathcal{W}_g f(x) &= \int_{S_x M} \mathcal{H}_{\perp} \int_0^{\tau(x, \xi)} f(\gamma(x, \xi, t)) dt dS_x \\ &= \int_{S_x M} \int_0^{\tau(x, \xi)} \frac{\partial f}{\partial x^j}(\gamma(x, \xi, t)) \mathcal{H}_{\perp} \gamma^j(x, \xi, t) dt dS_x. \end{aligned}$$

Let $x(s)$, $-\varepsilon < s < \varepsilon$ be a curve starting at x in the direction ξ_{\perp} . Now parallel translate the vector ξ along this curve, call it $\xi(s)$ and consider the variation by geodesics, $\gamma(x(s), \xi(s), t)$. The vector field

$$X(x, \xi, t) := \left. \frac{d}{ds} \right|_{s=0} \gamma(x(s), \xi(s), t), \tag{1.5}$$

is a Jacobi vector field along γ with the following initial conditions,

$$X(x, \xi, 0) = \xi_{\perp}, \quad D_t X(x, \xi, 0) = 0,$$

where D_t is the covariant derivative along γ . Differentiating the right-hand side of (1.5), we obtain,

$$X(x, \xi, t) = \mathcal{H}_{\perp} \gamma(x, \xi, t),$$

using the fact that $\xi(s)$ is the parallel translate of ξ along $x(s)$.

We now define another Jacobi field along $\gamma(x, \xi, t)$ by considering the variation by geodesics, $\gamma(x, \xi(s), t)$, where $\xi(s)$ is a smooth curve in $S_x M$ with initial tangent vector ξ_{\perp} . The Jacobi vector field

$$Y(x, \xi, t) := \left. \frac{d}{ds} \right|_{s=0} \gamma(x, \xi(s), t) \tag{1.6}$$

has initial conditions,

$$Y(x, \xi, 0) = 0, \quad D_t Y(x, \xi, 0) = \xi_{\perp}.$$

We denote Y by,

$$\partial_{\theta} \gamma(x, \xi, t) := Y(x, \xi, t) = \xi_{\perp}^i \partial_i \gamma(x, \xi, t),$$

where the operator ∂_i is defined in (1.3). Since X and Y are Jacobi fields normal to γ and since $\dim(M) = 2$, these two fields must be proportional to the parallel translate of the vector ξ_{\perp} along the geodesic γ . Let this parallel translate be denoted $\dot{\gamma}_{\perp}$. Then there exists two smooth functions $a(x, \xi, t)$ and $b(x, \xi, t)$ such that

$$X = a \dot{\gamma}_{\perp}, \quad Y = b \dot{\gamma}_{\perp}.$$

Denoting by prime ($'$), the derivatives with respect to t , the functions a and b satisfy the scalar Jacobi equations,

$$a''(x, \xi, t) + K(x, \xi, t)a(x, \xi, t) = b''(x, \xi, t) + K(x, \xi, t)b(x, \xi, t) = 0. \tag{1.7}$$

Here K is the Gaussian curvature and

$$a(x, \xi, 0) = 1, \quad a'(x, \xi, 0) = 0, \quad b(x, \xi, 0) = 0, \quad b'(x, \xi, 0) = 1.$$

We now write,

$$\begin{aligned} \mathcal{W}_g f(x) &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \langle \nabla f, \mathcal{H}_{\perp} \gamma \rangle dt dS_x \\ &= \frac{1}{2\pi} \int_{S_x M} \int_0^{\tau(x, \xi)} \frac{a}{b} \langle \nabla f, Y \rangle dt dS_x \\ &= -\frac{1}{2\pi} \int_0^{\tau(x, \xi)} \int_{S_x M} \partial_{\theta} \left(\frac{a}{b} \right) (f \circ \gamma) dS_x dt. \end{aligned}$$

Let

$$q(x, \xi, t) = \partial_{\theta} \left(\frac{a}{b} \right).$$

This function is initially not defined when $t = 0$, but by the initial conditions of Lemma 2.1, there exists a function $Q(x, t\xi) \in C^{\infty}(TM)$ such that

$$q(x, \xi, t) = tQ(x, t\xi).$$

Now by a change of variables involving the inverse of the exponential map, we have the expression for W_g in (1.4).

2 The proof

We prove the following lemma. Here, as before, the derivatives indicated by prime ($'$) are with respect to time.

Lemma 2.1. *Let $\varphi = b\partial_{\theta}a - a\partial_{\theta}b$ and γ be the geodesic starting at a point x with unit tangent vector ξ . Then φ satisfies the following ordinary differential equation,*

$$\varphi'''(x, \xi, t) + 4K(x, \xi, t)\varphi'(x, \xi, t) + 2K'(x, \xi, t)\varphi(x, \xi, t) = -2\partial_{\theta}(K(x, \xi, t)),$$

with initial conditions,

$$\varphi(x, \xi, 0) = \varphi'(x, \xi, 0) = \varphi''(x, \xi, 0) = 0.$$

Proof. First of all we have

$$ab' - a'b \equiv 1. \quad (2.1)$$

For, let $\phi = ab' - ba'$. Then $\phi' = ab'' - a''b$. From equation (1.7) we get that $\phi' = 0$ and so ϕ is a constant. Since $\phi(0) = 1$, we have the claim. With this we now show that $\varphi = b\partial_{\theta}a - a\partial_{\theta}b$ satisfies the ODE above.

$$\varphi' = b'\partial_{\theta}a + b\partial_{\theta}a' - a'\partial_{\theta}b - a\partial_{\theta}b'.$$

From (2.1) we get

$$b'\partial_{\theta}a + a\partial b' - b\partial_{\theta}a' - a'\partial_{\theta}b = 0.$$

This gives

$$\varphi' = 2(b'\partial_{\theta}a - a'\partial_{\theta}b) = 2(b\partial_{\theta}a' - a\partial_{\theta}b').$$

Differentiating again, we get

$$\varphi'' = 2(b''\partial_{\theta}a + b'\partial_{\theta}a' - a''\partial_{\theta}b - a'\partial_{\theta}b').$$

Using equation (1.7), this becomes

$$\varphi'' = 2(-K\varphi + b'\partial_\theta a' - a'\partial_\theta b').$$

Differentiating yet again, and as in the steps above, we finally get,

$$\varphi''' + 4K\varphi' + 2K'\varphi = -2\partial_\theta K. \tag{2.2}$$

It now follows from these equations that

$$\varphi(x, \xi, 0) = \varphi'(x, \xi, 0) = \varphi''(x, \xi, 0) = 0. \quad \square$$

Proof. We now prove Theorem 1.3. We rewrite (2.2) as a first order differential equation:

$$\begin{pmatrix} \varphi'_1 \\ \varphi'_2 \\ \varphi'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2K' & -4K & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2\partial_\theta K \end{pmatrix}, \tag{2.3}$$

where

$$\varphi_1 = \varphi, \quad \varphi_2 = \varphi' \quad \text{and} \quad \varphi_3 = \varphi''.$$

For simplicity, let us write (2.3) as a system of the form

$$\boldsymbol{\varphi}(x, \xi, t) = A(x, \xi, t)\boldsymbol{\varphi}(x, \xi, t) + B(x, \xi, t), \tag{2.4}$$

where $\boldsymbol{\varphi}$ is the column vector on the l.h.s. of (2.3), A and B are the 3×3 matrix and the column vector on the r.h.s. of (2.3) respectively. From [1], since $\boldsymbol{\varphi}(0) = 0$, we have a solution to this differential equation to be

$$\boldsymbol{\varphi}(x, \xi, t) = \Phi(x, \xi, t) \int_0^t \Phi^{-1}(x, \xi, s)B(x, \xi, s) ds,$$

where Φ is the fundamental matrix of the homogeneous system,

$$\boldsymbol{\varphi}'(x, \xi, t) = A(x, \xi, t)\boldsymbol{\varphi}(x, \xi, t).$$

Since $\|g - g_0\|_{C^3} < \varepsilon$ with $\varepsilon \ll 1$ and M is compact, there exists a $C_1 = C_1(g_0) > 0$ such that $\|\Phi\|_{C^0} < C_1$, $\|\Phi^{-1}\|_{C^0} < C_1$ and

$$|\partial_\theta K| \leq C_1 \|\nabla K\|_{C^0}.$$

Combining these inequalities we get

$$|\varphi''(x, \xi, t)| \leq |\boldsymbol{\varphi}(x, \xi, t)| \leq C_2 t \|\nabla K\|_{C^0}$$

for some $C_2 = C_2(g_0) > 0$.

Since

$$\varphi(x, \xi, t) = \int_0^t \int_0^s \varphi''(x, \xi, u) du ds, \quad \text{for } 0 \leq s \leq t,$$

we have

$$|\varphi(x, \xi, t)| \leq C_2 t^3 \|\nabla K\|_{C^0}.$$

Since g is simple for all g such that $\|g - g_0\|_{C^3} < \varepsilon$, we have $b \neq 0$ for $t \neq 0$, since $b(0) = 0$. Now we write $b(x, y, t) = t\tilde{b}(x, y, t)$ with $\tilde{b} \neq 0$. Therefore there exists $C_3 = C_3(g_0) > 0$ and

$$|q(x, \xi, t)| = \left| \partial_\theta \frac{a}{b} \right| \leq C_3 t \|\nabla K\|_{C^0}.$$

Since $tQ(x, t\xi) = q(x, \xi, t)$, we have

$$|Q(x, t\xi)| \leq C_3 \|\nabla K\|_{C^0}.$$

The remaining terms in

$$W_g(x, y) = -Q(x, \exp_x^{-1}(y)) \frac{|\det(\exp_x^{-1})'(x, y)| \sqrt{\det(g(x))}}{\sqrt{\det(g(y))}}$$

are bounded by a constant that depends on g_0 and so we have a $C_4 = C_4(g_0) > 0$ such that

$$\|W_g\|_{C^0} \leq C_4 \|\nabla K\|_{C^0}.$$

Finally, there exists a $C > 0$ uniformly constant in a C^3 neighborhood of g_0 such that,

$$\|\mathcal{W}_g\|_{L^2(M) \rightarrow L^2(M)} \leq C \|\nabla K\|_{C^0}.$$

So now choosing ε to be small enough, we can make $\|\nabla K\|_{C^0} < 1/C$. Hence for all g with $\|g - g_0\|_{C^3} < \varepsilon$, we have $\|\mathcal{W}_g\|_{L^2 \rightarrow L^2} < 1$. Therefore the operator $I + \mathcal{W}_g^2$ is invertible and the inverse is given by a Neumann series expansion. A similar argument works for the operator $I + (\mathcal{W}_g^*)^2$. This completes the proof of Theorem 1.3. \square

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Bibliography

- [1] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [2] S. Helgason, *The Radon transform*, Progress in Mathematics 5, Birkhäuser Boston, Mass., 1980.
- [3] L. Pestov and G. Uhlmann, On characterization of the range and inversion formulas for the geodesic X-ray transform, *Int. Math. Res. Not.* (2004), 4331–4347.
- [4] V. A. Sharafutdinov, *Integral geometry of tensor fields*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1994.

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