Inversion of a class of circular and elliptical Radon transforms

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Abstract. The paper considers a class of elliptical and circular Radon transforms appearing in problems of ultrasound imaging. These transforms put into correspondence to an unknown image function f in 2D its integrals $\mathcal{R}f$ along a family of ellipses (or circles). From the imaging point of view, of particular interest is the circular geometry of data acquisition. Here the generalized Radon transform $\mathcal{R}$ integrates f along ellipses (circles) with their foci (centers) located on a fixed circle C. We prove that such transforms can be uniquely inverted from radially incomplete data to recover the image function in annular regions. Our results hold for cases when f is supported inside and/or outside of the data acquisition circle C.

1. Introduction

In various modalities of ultrasound imaging, the object under investigation is probed by sending through it acoustic waves and measuring the resulting wave reflections. In the bi-static setup of ultrasound reflection tomography (URT) one uses an emitter and a receiver separated from each other to send and receive acoustic signals at various locations around the body. In mono-static case the same device (a transducer) works both as an emitter and as a receiver.

The mathematical model considered in this paper uses two simplifying assumptions, which hold reasonably well in many applications. We assume that the speed of sound propagation in the medium is constant and the medium is weakly reflecting. The latter means that we ignore the signals that arrive at the receiver after reflecting more than once inside the object. Under these assumptions, the time delay t between sending and receiving the signal defines the distance $d_1 + d_2$ traveled by the wave from the emitter to the reflection location and from there to the receiver. Hence the measured signal is the superposition of all echoes generated by inclusions located at points x that have a constant sum $d_1 + d_2$ of distances to the emitter and the receiver. In bi-static setup this corresponds to the integral of the acoustic reflectivity function $f(x)$ along an ellipse with foci at the emitter and receiver locations. In mono-static setup it corresponds to the integral of $f(x)$ along a circle centered at the transducer location. The grayscale graph of $f(x)$ is used as an image of the medium, hence we need to find f from its corresponding integrals. By measuring the wave reflections for different time delays and different locations of the emitter and receiver one can generate a set of integrals $\mathcal{R}f$ of the function f along a 2D family of ellipses or circles. The problem of image reconstruction in URT is then mathematically equivalent to the inversion of the corresponding generalized Radon transform $\mathcal{R}$. For more details and rigorous derivation of the mathematical model we refer the reader to [**6, 13, 16**] and the references there.

From the imaging point of view of particular interest is the circular geometry of data acquisition. Here the emitter and the receiver travel along a circle C and are a fixed distance $2a$ apart from each other. In the mono-static case that fixed distance is simply 0. By making the signal measurements for various positions $\gamma_T(\phi)$ of the emitter and $\gamma_R(\phi)$ of the receiver and various time delays t, one can generate a 2D set of integrals $\mathcal{R}f(\phi, t)$ along ellipses (circles). These ellipses have foci at $\gamma_T(\phi)$ and $\gamma_R(\phi)$ (circles have centers at $\gamma_T(\phi)$) and the size of their major and minor semi-axes (or radii for circles) are defined by t (see Figure 1).

The problem of inversion of the circular Radon transform (CRT) has been extensively studied before by various authors (e.g. see [**1, 2, 4, 5, 7, 9, 10, 11, 12, 13, 15, 16, 17, 19**] and the references there). Most of these works deal with the inversion of CRT when the $\mathcal{R}f$ data is available for all possible radii. Few of them deal with the inversion of CRT from radially incomplete data and we refer the reader to [**4**] for a detailed discussion of those works. However very little is known about inversion of the elliptical Radon transforms (ERT). Some limited results

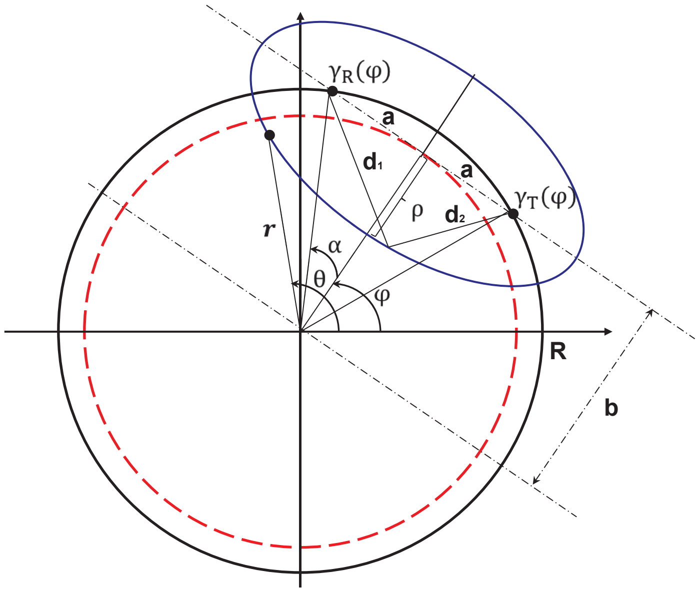

FIGURE 1. The sketch of the bi-static setup of URT and the variables.

were established in [**3, 12, 13, 19**], most of which deal with inversion of ERT from either full or half data in the "radial" variable.

In [**4**] the authors provided an exact inversion formula for CRT from radially partial data in the circular geometry of data acquisition. The current paper builds up on the results and techniques established in [**4**] and presents some further results both for CRT and ERT. We prove that such transforms can be uniquely inverted from radially incomplete data, that is, from data where t is limited to a small subset of $\mathbb{R}^+$. Our inversion formulas recover the image function f in annular regions defined by the smallest and largest available values of t. The results hold for cases when f is supported inside C (e.g. in mammography [**13, 14**]), outside of C (e.g. in intravascular imaging [**6**]), or simultaneously both inside and outside (e.g. in radar imaging [**3**]).

The rest of the paper is organized as follows. In Section 2, we introduce the notations and definitions. In Section 3, we present the main results in the form of three theorems. The proofs of these theorems are provided in Section 4. Section 5 includes additional remarks and acknowledgments.

2. Notations

Let us consider a generalized Radon transform integrating a function $f(x)$ of two variables along ellipses with the foci located on the circle $C(0,R)$ centered at 0 of radius R and $2a$ units apart (see Figure 1). We denote the fixed difference between the polar angles of the two foci by 2α, where $\alpha \in (0, \pi/2)$ and define

$$a = R\sin\alpha, \qquad b = R\cos\alpha.$$

We parameterize the location of the foci by

$$\begin{aligned}\gamma_T(\phi) &= R\,(\cos(\phi-\alpha), \sin(\phi-\alpha)),\\ \gamma_R(\phi) &= R\,(\cos(\phi+\alpha), \sin(\phi+\alpha)) \quad \text{for } \phi \in [0, 2\pi].\end{aligned}$$

Thus the foci move on the unit circle and are always $2a$ units apart. For $\phi \in [0, 2\pi]$ and $\rho > 0$, let

$$E(\rho,\phi) = \{x \in \mathbb{R}^2 : |x - \gamma_T(\phi)| + |x - \gamma_R(\phi)| = 2\sqrt{\rho^2 + a^2}\}.$$

Note that the center of the ellipse $E(\rho,\phi)$ is $(b\cos\phi, b\sin\phi)$ and ρ is the minor semiaxis of $E(\rho,\phi)$.

Consider a compactly supported function $f(r,\theta)$, where (r,θ) denote the polar coordinates in the plane. The elliptical Radon transform (ERT) of f on the ellipse parameterized by (ρ,ϕ) is denoted by

$$\mathcal{R}_E f(\rho,\phi) = \int\limits_{E(\rho,\phi)} f(r,\theta)\, ds,$$

where ds is the arc-length measure on the ellipse.

If we take $a = \alpha = 0$ then the foci coincide and the ellipses $E(\rho,\phi)$ become circles

$$C(\rho,\phi) = \{x \in \mathbb{R}^2 : |x - \gamma_T(\phi)| = \rho\}.$$

The resulting circular Radon transform (CRT) of f on the circle parameterized by (ρ,ϕ) is denoted by

$$\mathcal{R}_C f(\rho,\phi) = \int\limits_{C(\rho,\phi)} f(r,\theta)\, ds,$$

where ds is the arc-length measure on the circle.

In the rest of the text we will use the notation $g(\rho,\phi)$ to denote either $\mathcal{R}_E f(\rho,\phi)$, or (when working with CRT) $\mathcal{R}_C f(\rho,\phi)$.

Since both $f(r,\theta)$ and $g(\rho,\phi)$ are 2π-periodic in the second variable one can expand these functions into Fourier series.

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{in\theta} \tag{1}$$

and

$$g(\rho,\phi) = \sum_{n=-\infty}^{\infty} g_n(\rho)\, e^{in\phi}, \tag{2}$$

where

$$f_n(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(r,\theta)\, e^{-in\theta} d\theta$$

and

$$g_n(\rho) = \frac{1}{2\pi}\int\limits_0^{2\pi} g(\rho,\phi)\, e^{-in\phi} d\phi.$$

We will use Cormack-type [**8**] inversion strategy to recover Fourier coefficients of f from those of g for limited values of ρ both for CRT and ERT in various setups of the support of f.

In the statements below we denote by $A(r_1,r_2)$ the open annulus with radii $0 < r_1 < r_2$ centered at the origin

$$A(r_1,r_2) = \{(r,\theta) : r \in (r_1,r_2), \theta \in [0,2\pi]\}.$$

The disc of radius R centered at the origin is denoted by $D(0,R)$.

The k-th order Chebyshev polynomial of the first kind is denoted by T_k, i.e.

$$T_k(t) = \cos(k \arccos t).$$

3. Main Results

The first statement in this section is a generalization of Theorem 1 from [**4**], which was proved for CRT, to the case of ERT.

THEOREM 3.1. *Let $f(r,\theta)$ be a continuous function supported inside the annulus $A(\varepsilon,b)$. Suppose $\mathcal{R}_E f(\rho,\phi)$ is known for all $\phi \in [0,2\pi]$ and $\rho \in (0, b-\varepsilon)$, then $f(r,\theta)$ can be uniquely recovered.*

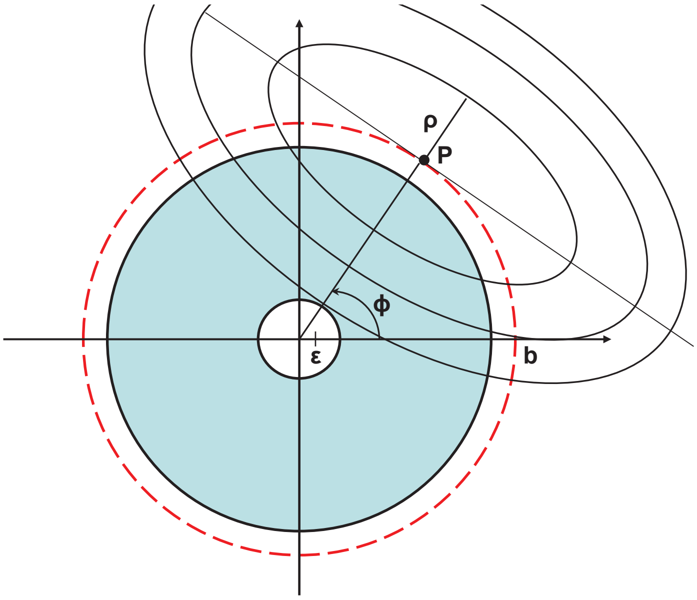

FIGURE 2. A sketch for Theorem 3.1. The shaded area denotes the support of f, the dashed circle is the set of centers of the integration ellipses.

In this and other theorems of this section we require f to be continuous, which guarantees the convergence of the Fourier series (1) and (2) almost everywhere. If one needs to ensure convergence everywhere, then some additional conditions on f (e.g. bounded variation) should be added. At the same time, if we assume that f is only piecewise continuous with respect to r for each fixed θ, then we will recover f correctly at points of continuity. As a result, if the function f is not identically zero in $D(0,\varepsilon_1)$, then one can consider a modified function $\tilde{f}$ such, that

$$\tilde{f}(r,\theta) = \begin{cases} 0, & r \le \varepsilon \\ f(r,\theta), & \varepsilon_1 < r < b \\ \text{smooth cutoff}\ , & \varepsilon < r < \varepsilon_1. \end{cases}$$

It is easy to notice that if $\varepsilon < \varepsilon_1$ then $\tilde{f}$ satisfies the hypothesis of the theorem, and by sending $\varepsilon_1 \to \varepsilon$ we also have $\mathcal{R}_E f(\rho,\phi) = \mathcal{R}_E \tilde{f}(\rho,\phi)$ for all $\phi \in [0,2\pi]$ and $\rho \in (0, b-\varepsilon_1)$. Hence we get the following statement.

REMARK 3.2. In order to reconstruct the function $f(r,\theta)$ in any subset Ω of the disc of its support $D(0,R)$, one needs to know $\mathcal{R}_E f(\rho,\phi)$ only for $\rho < b - R_0$, where $R_0 = \inf\{|x|,\ x \in \Omega\}$.

In other words, to image something at depth d from the surface of the disc, one only needs $\mathcal{R}_E f(\rho,\phi)$ data for $\rho \in [0,d]$, without making any assumptions about the shape of the support of f inside that disc.

The next two theorems demonstrate the possibility of inverting the CRT and ERT from radially partial data when the support of the function f lies on both sides of the data acquisition circle C (see Figures 3 and 4).

THEOREM 3.3. *Let $f(r,\theta)$ be a continuous function supported inside the disc $D(0,R_2)$ with $R_2 > 2R$. Suppose $\mathcal{R}_C(\rho,\phi)$ is known for all $\phi \in [0,2\pi]$ and $\rho \in [R_2 - R, R_2 + R]$, then $f(r,\theta)$ can be uniquely recovered in $A(R_1,R_2)$ where $R_1 = R_2 - 2R$.*

THEOREM 3.4. *Let $f(r,\theta)$ be a continuous function supported inside the disc $D(0,R_2)$ with $R_2 > 2b$. Suppose $\mathcal{R}_E(\rho,\phi)$ is known for all $\phi \in [0,2\pi]$ and $\rho \in [R_2 - b, R_2 + b]$, then $f(r,\theta)$ can be uniquely recovered in $A(R_1,R_2)$ where $R_1 = R_2 - 2b$.*

4. Proofs

The proofs of all three theorems are similar. The idea is to reduce the problem of inverting the generalized Radon transforms to solving an integral equation with a special kernel for Fourier coefficients of f. We will prove

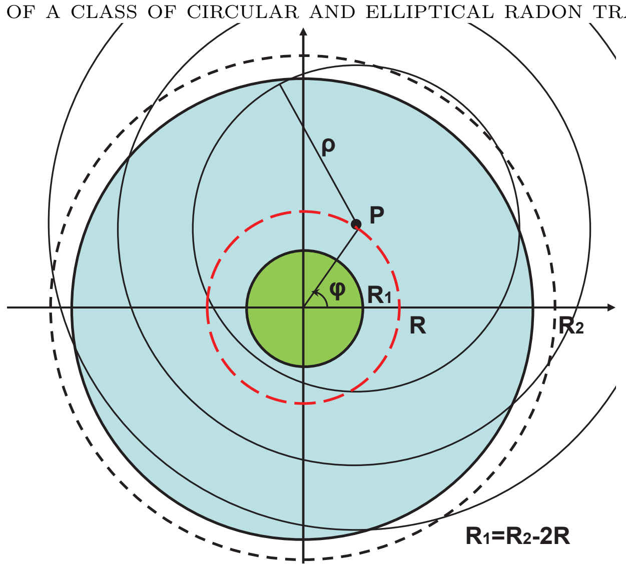

FIGURE 3. A sketch for Theorem 3.3. The shaded area denotes the support of f, the dashed inner circle is the set of transducer locations (centers of integration circles).

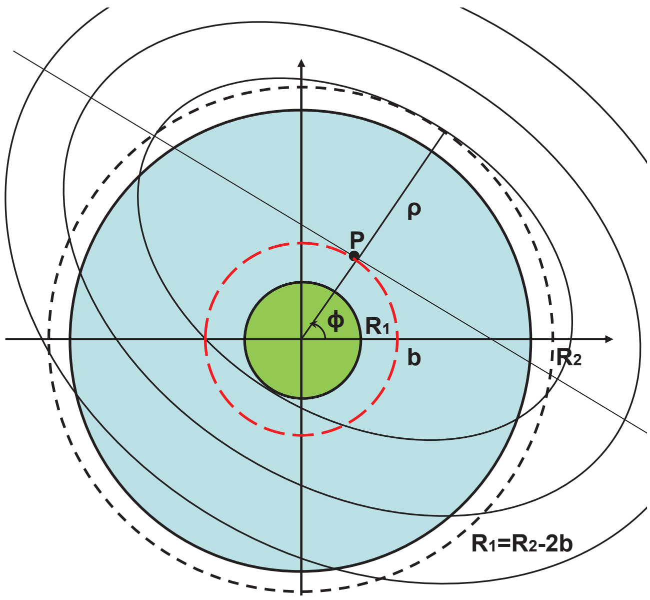

FIGURE 4. A sketch for Theorem 3.4. The shaded area denotes the support of f, the dashed inner circle is the set of the centers of integration ellipses.

Theorem 3.1 in detail, then indicate the arising differences in the proofs of the other two theorems, and the strategy of dealing with those differences.

PROOF OF THEOREM 3.1. By using the definition of ERT and the Fourier series expansion of f we get

$$g(\rho,\phi) = \int\limits_{E(\rho,\phi)} f(r,\theta)\, ds = \sum_{n=-\infty}^{+\infty} \int\limits_{E(\rho,\phi)} f_n(r)\, e^{in\theta}\, ds = \sum_{n=-\infty}^{+\infty} \int\limits_{E^+(\rho,\phi)} f_n(r) \left(e^{in\theta} + e^{in(2\phi-\theta)}\right) ds$$

where $E^+(\rho,\phi)$ denotes the part of the ellipse $E(\rho,\phi)$ corresponding to $\theta \geq \phi$ (see Figure 1). Simplifying further we get

$$g(\rho,\phi)=2\sum_{n=-\infty}^{+\infty}\int\limits_{E^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]\, e^{in\phi}ds.$$

Comparing this equation with (2) we obtain

$$g_n(\rho)=2\int\limits_{E^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]ds. \tag{3}$$

As we explicitly show below (see formulas (4) and (5)) the Fourier series expansion diagonalizes the Radon transform, i.e. g_n depends only on f_n and vice versa for all integer values of n. Hence the problem of inverting ERT in this setup is reduced to solving the one-dimensional integral equation (3). Let us discuss this in detail.

Since the left-hand side of equation (3) does not depend on ϕ, it should hold for any choice of ϕ in the right-hand side. Without loss of generality, we will assume from now on that $\phi=\pi/2$. Then the points of $E^+(\rho,\phi)$ for $\rho\in(0,b-\varepsilon)$ will be limited to the second quadrant. We parameterize the points on $E^+(\rho,\phi)$ as follows:

$$x(t)=-\sqrt{\rho^2+a^2}\sin t,\quad y(t)=b-\rho\cos t,\quad t\in[0,\pi/2].$$

For brevity denote $A=\sqrt{\rho^2+a^2}$ and $B=\rho$. Then simple calculations show that

$$\begin{aligned}
ds&=\sqrt{A^2\cos^2 t+B^2\sin^2 t}\ dt,\\
\theta-\phi&=\arccos\left(\frac{b-B\cos t}{r}\right),\\
\cos t&=\frac{-b\rho+\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2},\ \text{and}\\
\sin^2 t&=\frac{\left[a^2+\left(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho\right)\right]\left[a^2-\left(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho\right)\right]}{a^4}.
\end{aligned}$$

Let us now rewrite the integral in (3) in the variable r. We have $g_n(\rho)=$

$$=2a\int\limits_{b-\rho}^{b}\frac{r\,f_n(r)\,T_n\left(\frac{b(\rho^2+a^2)-\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2r}\right)\sqrt{2R^2\rho^2+a^2(R^2-r^2)-2b\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}\,dr}{\sqrt{R^2\rho^2+a^2(R^2-r^2)}\sqrt{(a^2+(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))(a^2-(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))}}$$

$$=\int\limits_{b-\rho}^{b}\frac{\widetilde{K}_n(\rho,r)}{\sqrt{a^2+b\rho-\sqrt{R^2\rho^2+a^2(R^2-r^2)}}}\,f_n(r)\,dr$$

where

$$\widetilde{K}_n(\rho,r)=\frac{2arT_n\left(\frac{b(\rho^2+a^2)-\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2r}\right)\sqrt{2R^2\rho^2+a^2(R^2-r^2)-2b\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}}{\sqrt{R^2\rho^2+a^2(R^2-r^2)}\sqrt{a^2+(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho)}}$$

Making the change of variables $u=b-r$, we get

$$g_n(\rho)=\int\limits_0^{\rho}\frac{\widetilde{K}_n(\rho,b-u)}{\sqrt{a^2+b\rho-\sqrt{R^2\rho^2+a^2(R^2-(b-u)^2)}}}\,f_n(b-u)\,du=\int\limits_0^{\rho}\frac{K_n(\rho,u)}{\sqrt{\rho-u}}\,F_n(u)\,du, \tag{4}$$

where

$$F_n(u)=f_n(b-u)\quad\text{and}\quad K_n(\rho,u)=\frac{\widetilde{K}_n(\rho,b-u)\sqrt{\rho-u}}{\sqrt{a^2+b\rho-\sqrt{R^2\rho^2+a^2(R^2-(b-u)^2)}}}. \tag{5}$$

A simple calculation shows that $a^2+b\rho-\sqrt{R^2\rho^2+a^2(R^2-(b-u)^2)}=0$ if and only if $u=\rho$ and its derivative at $u=\rho$ does not vanish. Therefore, $K_n(\rho,u)$ is a C^1 function.

Hence (4) is a Volterra integral equation of the first kind with a weakly singular kernel (e.g. see [**18, 20**]). This type of equations have a unique solution and there is a standard approach for writing down that solution through a resolvent kernel using Picard's process of successive approximations (e.g. see [**18**]). The details of this part follow exactly as in [**4**, Theorem 1], and this finishes the proof. $\square$

Proof of Theorem 3.3. Similar to the proof of the previous theorem we have

$$g(\rho,\phi)=\int\limits_{C(\rho,\phi)} f(r,\theta)\,ds=\sum_{n=-\infty}^{\infty}\int\limits_{C(\rho,\phi)} f_n(r)\,e^{in\theta}ds=\sum_{n=-\infty}^{\infty}\int\limits_{C^+(\rho,\phi)} f_n(r)\left(e^{in\theta}+e^{in(2\phi-\theta)}\right)ds$$

where $C^+(\rho,\phi)$ denotes the part of the circle $C(\rho,\phi)$ corresponding to $\theta\geq\phi$.

$$g(\rho,\phi)=2\sum_{n=-\infty}^{\infty}\int\limits_{C^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]\ e^{in\phi}ds.$$

Therefore we have,

$$g_n(\rho)=2\int\limits_{C^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]ds. \tag{6}$$

Then

$$\theta-\phi=\arccos\left(\frac{r^2+R^2-\rho^2}{2rR}\right)$$

and

$$\begin{aligned}
g_n(\rho)&=2\int\limits_{C^+(\rho,\phi)} f_n(r)\,T_n\left(\frac{r^2+R^2-\rho^2}{2rR}\right)ds\\
&=\frac{2}{R}\int\limits_{\rho-R}^{R_2}\frac{r\,T_n\left(\frac{r^2+R^2-\rho^2}{2rR}\right)}{\sqrt{1-\left(\frac{\rho^2+R^2-r^2}{2\rho R}\right)^2}}\,f_n(r)\,dr
\end{aligned}$$

Let us make the substitution, $u=R_2-r$. We get

$$g_n(\rho)=\frac{2}{R}\int\limits_0^{R_2+R-\rho}\frac{(R_2-u)\,T_n\left(\frac{(R_2-u)^2+R^2-\rho^2}{2(R_2-u)R}\right)}{\sqrt{1-\left(\frac{\rho^2+R^2-(R_2-u)^2}{2\rho R}\right)^2}}\,f_n(R_2-u)\,du$$

Setting $\widetilde{\rho}=R_2+R-\rho$ for simplicity and renaming $\widetilde{\rho}$ as ρ, we have

$$\begin{aligned}
g_n(R_2+R-\rho)&=\frac{2}{R}\int\limits_0^{\rho}\frac{(R_2-u)\;T_n\left(\frac{(R_2-u)^2+R^2-(R_2+R-\rho)^2}{2(R_2-u)R}\right)}{\sqrt{1-\left(\frac{(R_2+R-\rho)^2+R^2-(R_2-u)^2}{2(R_2+R-\rho)R}\right)^2}}\,f_n(R_2-u)\;du\\
&=4\int\limits_0^{\rho}\frac{(R_2+R-\rho)(R_2-u)\,T_n\left(\frac{(R_2-u)^2+R^2-(R_2+R-\rho)^2}{2(R_2-u)R}\right)}{\sqrt{(\rho-u)(2R_2-\rho-u)(2R+u-\rho)(2R+2R_2-\rho-u)}}\,f_n(R_2-u)\;du\\
&=\int\limits_0^{\rho}\frac{K_n(\rho,u)}{\sqrt{\rho-u}}\,F_n(u)\;du.
\end{aligned}$$

Here K_n and F_n are

$$K_n(\rho,u)=\frac{4(R_2+R-\rho)(R_2-u)\,T_n\left(\frac{(R_2-u)^2+R^2-(R_2+R-\rho)^2}{2(R_2-u)R}\right)}{\sqrt{(\rho-u)(2R_2-\rho-u)(2R+u-\rho)(2R+2R_2-\rho-u)}}$$

and

$$F_n(u)=f_n(R_2-u).$$

Note that $K_n(\rho,u)$ is a C^1 function. Rest of the proof follows exactly as in [**4**, Theorem 1]. This completes the proof. □

Proof of Theorem 3.4. We have

$$g(\rho,\phi)=\int\limits_{E(\rho,\phi)} f(r,\theta)\,ds=\sum_{n=-\infty}^{\infty}\int\limits_{E(\rho,\phi)} f_n(r)\,e^{in\theta}\,ds=\sum_{n=-\infty}^{\infty}\int\limits_{E^+(\rho,\phi)} f_n(r)\left(e^{in\theta}+e^{in(2\phi-\theta)}\right)ds$$

where $E^+(\rho,\phi)$ denotes the part of the ellipse $E(\rho,\phi)$ corresponding to $\theta\geq\phi$.

$$g(\rho,\phi)=2\sum_{n=-\infty}^{\infty}\int\limits_{E^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]\,e^{in\phi}ds.$$

Therefore we have,

$$g_n(\rho)=2\int\limits_{E^+(\rho,\phi)} f_n(r)\cos\left[n(\theta-\phi)\right]\,ds. \tag{7}$$

We first observe that, without loss of generality, we may assume from now on that $\phi=\pi/2$. We introduce the following elliptic coordinates on the ellipse determined by ρ:

$$x(t)=-\sqrt{\rho^2+a^2}\sin t,\quad y(t)=b-\rho\cos t.$$

For simplicity denote $A=\sqrt{\rho^2+a^2}$ and $B=\rho$.

$$\begin{aligned}
ds&=\sqrt{A^2\cos^2 t+B^2\sin^2 t}\,dt\\
\theta-\phi&=\arccos\left(\frac{b-B\cos t}{r}\right)\\
\cos t&=\frac{-b\rho+\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2}\\
\sin^2 t&=\frac{(a^2+(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))(a^2-(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))}{a^4}
\end{aligned}$$

Let us now rewrite the integral in (7) in the variable r. We have $g_n(\rho)=$

$$\begin{aligned}
&=2a\int\limits_{\rho-b}^{R_2}\frac{rf_n(r)T_n\left(\frac{b(\rho^2+a^2)-\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2r}\right)\sqrt{2R^2\rho^2+a^2(R^2-r^2)-2b\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}}{\sqrt{R^2\rho^2+a^2(R^2-r^2)}\sqrt{(a^2+(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))(a^2-(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho))}}\,dr\\
&=\int\limits_{\rho-b}^{R_2}\frac{\widetilde{K}_n(\rho,r)}{\sqrt{a^2+b\rho-\sqrt{R^2\rho^2+a^2(R^2-r^2)}}}f_n(r)\,dr
\end{aligned}$$

where

$$\widetilde{K}_n(\rho,r)=\frac{2arT_n\left(\frac{b(\rho^2+a^2)-\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}{a^2r}\right)\sqrt{2R^2\rho^2+a^2(R^2-r^2)-2b\rho\sqrt{R^2\rho^2+a^2(R^2-r^2)}}}{\sqrt{R^2\rho^2+a^2(R^2-r^2)}\sqrt{a^2+(\sqrt{R^2\rho^2+a^2(R^2-r^2)}-b\rho)}}$$

Making the change of variable $u=R_2-r$, $\widetilde{\rho}=R_2+b-\rho$, and replacing $\widetilde{\rho}$ by ρ, we get

$$\begin{aligned}
g_n(R_2+b-\rho)&=\int\limits_0^{\rho}\frac{\widetilde{K}_n(R_2+b-\rho,R_2-u)}{\sqrt{a^2+b(R_2+b-\rho)-\sqrt{R^2(R_2+b-\rho)^2+a^2(R^2-(R_2-u)^2)}}}f_n(R_2-u)\,du\\
&=\int\limits_0^{\rho}\frac{K_n(\rho,u)}{\sqrt{\rho-u}}F_n(u)\,du,
\end{aligned}$$

where $F_n(u) = f_n(R_2 - u)$ and

$$K_n(\rho, u) = \frac{\widetilde{K}_n(R_2 + b - \rho, R_2 - u)\sqrt{\rho - u}}{\sqrt{a^2 + b(R_2 + b - \rho) - \sqrt{R^2(R_2 + b - \rho)^2 + a^2(R^2 - (R_2 - u)^2)}}}.$$

A simple calculation would show that $a^2 + b(R_2 + b - \rho) - \sqrt{R^2(R_2 + b - \rho)^2 + a^2(R^2 - (R_2 - u)^2} = 0$ if and only if $u = \rho$ and its derivative at $u = \rho$ does not vanish. Therefore, $K_n(\rho, u)$ is a C^1 function.

Now rest of the proof follows exactly as in [**4**, Theorem 1].

□

5. Additional Remarks and Acknowledgments

The second author and his collaborators have numerically implemented the inversion formulas obtained in [**4**] and in this paper. The work has been submitted for publication.

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