A UNIQUENESS RESULT FOR LIGHT RAY TRANSFORM ON SYMMETRIC 2-TENSOR FIELDS

VENKATESWARAN P KRISHNAN†, SOUMEN SENAPATI‡ AND MANMOHAN VASHISTH*

Abstract. We study light ray transform of symmetric 2-tensor fields defined on a bounded time-space domain in $\mathbb{R}^{1+n}$ for $n \geq 3$. We prove a uniqueness result for such light ray transforms. More precisely, we characterize the kernel of the light ray transform vanishing near a fixed direction at each point in the time-space domain.

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1. Introduction and statement of the main results

Let $S^2\mathbb{R}^{1+n}$ be the complex vector space of symmetric tensor fields of rank 2 in $\mathbb{R}^{1+n}$. Let Ω be a bounded domain in $\mathbb{R}^{1+n}$ with C^∞ boundary and $C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ be the space of $S^2\mathbb{R}^{1+n}$-valued C^∞ smooth symmetric 2-tensor fields on $\overline{\Omega}$. We represent points in Ω by (t,x). Any $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ will be denoted by

$$F(t,x) = (F_{ij}(t,x)) \text{ where } 0 \leq i,j \leq n \text{ with } F_{ij}(t,x) = F_{ji}(t,x) \text{ and } F_{ij}(t,x) \in C^\infty(\overline{\Omega}).$$

Note that we have used the 0-index to denote the time component of a symmetric 2-tensor field. Also note that a function $f \in C^\infty(\overline{\Omega})$ if it has a smooth extension to a slightly larger open set containing $\overline{\Omega}$.

The light ray transform of $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ is defined as follows. Consider a point $(t,x) \in \overline{\Omega}$ and fix a direction $\theta \in \mathbb{S}^{n-1}$. The light ray transform L of F is the usual ray transform of F through the point (t,x) in the direction $\widetilde{\theta} = (1,\theta)$. That is,

$$LF(t,x,\tilde{\theta}) = \int_{\mathbb{R}} \sum_{i,j=0}^{n} \tilde{\theta}^i \tilde{\theta}^j F_{ij}(t+s, x+s\theta)\, \mathrm{d}s. \tag{1.1}$$

We have assumed the Einstein summation convention and from now on, with repeating indices, this will be assumed, with the index varying from 0 to n. We also note that extending F to be 0 outside Ω, the definition of the light ray transform L can be extended to points $(t,x) \in \mathbb{R}^{1+n}$ and any $\tilde{\theta}$ as defined above. This will be assumed without comment from now on.

In this work, we address the question of characterizing the tensor fields $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ such that $LF(t,x,\tilde{\theta}) = 0$ for all $(t,x) \in \mathbb{R}^{1+n}$ and all $\tilde{\theta} = (1,\theta)$ with $\theta \in \mathbb{S}^{n-1}$ near some fixed $\theta_0 \in \mathbb{S}^{n-1}$.

Light ray transforms in Euclidean and manifold settings have been studied in several recent works; see [4, 9, 10, 11, 19, 22]. Most of these works analyze light ray transform from the view point of microlocal analysis. Light ray transforms arise in the study of inverse problems for hyperbolic PDEs with time-dependent coefficients as well; see references [1, 2, 3, 5, 6, 7, 8, 12, 13, 14, 17, 18, 20].

To the best of the authors' knowledge, an exact description of the kernel of the light ray transform on symmetric 2-tensor fields has not been precisely studied, and this is the main goal of the paper. In this work, we use Fourier transform techniques to prove the uniqueness result in the Minkowski setting. We should mention that the recent paper [4] also deals with uniqueness result for light ray transforms on symmetric tensor fields and a uniqueness result similar to ours is proven in the more general setting of Lorentzian manifolds. Our work was done independently, and the techniques employed here are different from theirs. Specifically, their uniqueness result for the light ray transform on certain Lorentzian manifolds relies on the uniqueness result for the corresponding geodesic ray transform on the base space; see [4, Theorem 2]. However, we work directly with the light ray transform, albeit in the Minkowski setting. Another distinction from the work of [4] is that our uniqueness result only assumes knowledge of the light ray transform in the neighborhood of a fixed direction. In other words, ours is a partial data result. The uniqueness result of [4], requires knowledge of the full light ray transform, even in the setting of Minkowski space.

We now state the main results. ults as well as in proofs below we use the following notation. By δ, we mean the Euclidean divergence and trace will refer to the Euclidean trace.

In other words, for a symmetric 2-tensor field $F = (F_{ij})_{0\le i,j\le n}$:

$$(\delta F)_i = \frac{\partial F_{j0}}{\partial t} + \sum_{j=1}^{n} \frac{\partial F_{ij}}{\partial x_i} = \partial_i F_{ij}. \tag{1.2}$$

$$\operatorname{trace}(F) = \sum_{i=0}^{n} F_{ii}. \tag{1.3}$$

In the last equality in (1.2), we emphasize that the Einstein summation convention is assumed with the index varying between 0 and n and the $\frac{\partial}{\partial t}$ derivative is abbreviated as ∂_0 and the space derivatives $\frac{\partial}{\partial x_i}$ are denoted by ∂_i for $1 \le i \le n$.

Theorem 1.1. *Let $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ be such that $\delta F = 0$ and $\operatorname{trace}(F) = 0$. If for a fixed $\theta_0 \in \mathbb{S}^{n-1}$,*

$$LF(t,x,\tilde{\theta}) = 0 \text{ for all } (t,x) \in \mathbb{R}^{1+n} \quad \text{and} \quad \theta \text{ near } \theta_0,$$

then $F = 0$.

Theorem 1.2. *Let $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$. Then there exists an $\widetilde{F} \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ satisfying $\delta(\widetilde{F}) = trace(\widetilde{F}) = 0$, a function $\lambda \in C^\infty(\overline{\Omega})$, and a vector field v with components in $C^\infty(\overline{\Omega})$ satisfying $v|_{\partial\Omega} = 0$ such that F can be decomposed as*

$$F = \widetilde{F} + \lambda g + \mathrm{d}v. \tag{1.4}$$

Here g is the Minkowski metric with $(-1,1,1,\cdots,1)$ along the diagonal and d is the symmetrized derivative defined by

$$(\mathrm{d}v)_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right).$$

See also [16], where a decomposition result similar in spirit to the one above is shown in a Riemannian setting.

Combining the above two results, we get the following desired characterization.

Theorem 1.3. *Let $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$. If for a fixed $\theta_0 \in \mathbb{S}^{n-1}$,*

$$LF(t,x,\tilde{\theta}) = 0, \quad \text{for all } (t,x) \in \mathbb{R}^{1+n} \quad \text{and} \quad \theta \text{ near } \theta_0,$$

then $F = \lambda g + \mathrm{d}v$*, where* $\lambda \in C^\infty(\overline{\Omega})$*,* g *is the Minkowski metric and* v *is a vector field with components in* $C^\infty(\overline{\Omega})$ *with* $v|_{\partial\Omega} = 0$*.*

See [10, Lemma 9.1] as well, where a version of Theorem 1.3 is proven in the Euclidean setting in space dimensions $n = 3$.

2. Proofs

We prove two lemmas that would immediately give the proof of Theorem 1.1. As mentioned already, we will extend F as 0 outside $\overline{\Omega}$.

Lemma 2.1. *Under the assumptions of Theorem 1.1, we have the following equality:*

$$\tilde{\theta}^i\tilde{\theta}^j\widehat{F}_{ij}(\zeta) = 0 \text{ for all } \zeta \in (1,\theta)^\perp \text{ and } \theta \text{ near } \theta_0$$

where $\theta \in \mathbb{S}^{n-1}$ *and* $\perp$ *is with respect to the Euclidean metric.*

Proof. This is the Fourier slice theorem for the light ray transform. This result is standard; see [19]. We consider the Fourier transform of F_{ij}:

$$\widehat{F}_{ij}(\zeta) = \int_{\mathbb{R}^{1+n}} F_{ij}(t,x)e^{-\mathrm{i}(t,x)\cdot\zeta}\,\mathrm{d}t\mathrm{d}x. \tag{2.1}$$

Using the decomposition, $\mathbb{R}^{1+n} = \mathbb{R}(1,\theta) \oplus \ell$ with $\ell \in (1,\theta)^\perp$ combined with Fubini's theorem, we get

$$\widehat{F}_{ij}(\zeta) = \sqrt{2}\int_{(1,\theta)^\perp}\int_{\mathbb{R}} F_{ij}(\ell + s(1,\theta))e^{-\mathrm{i}(\ell+s(1,\theta))\cdot\zeta}\,\mathrm{d}s\,\mathrm{d}\ell.$$

If $\zeta \in (1,\theta)^\perp$, then

$$\tilde{\theta}^i\tilde{\theta}^j\widehat{F}_{ij}(\zeta) = \sqrt{2}\int_{(1,\theta)^\perp}\int_{\mathbb{R}} \tilde{\theta}^i\tilde{\theta}^jF_{ij}(s(1,\theta)+\ell)e^{-i\ell\cdot\zeta}\,\mathrm{d}s\,\mathrm{d}\ell.$$

Using the fact that

$$\int_{\mathbb{R}} \tilde{\theta}^i\tilde{\theta}^jF_{ij}(t+s, x+s\theta)\,\mathrm{d}s = 0, \text{ for all } (t,x) \in \mathbb{R}^{1+n}, \text{ and } \theta \text{ near } \theta_0, \tag{2.2}$$

we get

$$\tilde{\theta}^i\tilde{\theta}^j\widehat{F}_{ij}(\zeta) = 0 \text{ for all } \zeta \in (1,\theta)^\perp \text{ with } \theta \text{ near } \theta_0. \tag{2.3}$$

□

In the following lemma, without loss of generality, we fix $\theta_0 = (1, 0\cdots,0) \in \mathbb{S}^{n-1}$.

Lemma 2.2. *Let* $F \in C^\infty(\overline{\Omega}; S^2\mathbb{R}^{1+n})$ *be such that* $\delta F = 0$ *and* $\mathrm{trace}(F) = 0$*. Suppose also that* $\tilde{\theta}^i\tilde{\theta}^j\widehat{F}_{ij}(\zeta) = 0$ *for* $\zeta \in (1,\theta)^\perp$ *and* θ *near* θ_0*. Then*

$$\widehat{F}_{ij}(\zeta) = 0$$

in a small conical neighborhood of the space-like vector $\zeta_0 = (0,0,1,0,\cdots,0) \in \mathbb{R}^{1+n}$*.*

Proof. In order to make the presentation clear, we first give the proof in $\mathbb{R}^{1+3}$ and then generalize it to $\mathbb{R}^{1+n}$ when $n \geq 4$.

First let us show that $\widehat{F}_{ij}(\zeta_0) = 0$ for all $0 \leq i, j \leq 3$. We fix $\theta_0 = (1, 0, 0)$. Note that $(1, \theta_0) \cdot \zeta_0 = 0$. Consider

$$\theta_0(a) = (\cos a, 0, \sin a). \tag{2.4}$$

If a is near 0, then $\theta_0(a)$ is near θ_0. Also note that $(1, \theta_0(a)) \cdot \zeta_0 = 0$. Substituting $\zeta_0 = (0, 0, 1, 0)$ and $\theta_0(a)$ as above into (2.3), we get,

$$\left(\widehat{F}_{00} + 2\cos a\widehat{F}_{01} + 2\sin a\widehat{F}_{03} + \cos^2 a\widehat{F}_{11} + 2\sin a\cos a\widehat{F}_{13} + \sin^2 a\widehat{F}_{33}\right)(\zeta_0) = 0. \tag{2.5}$$

Let us differentiate this equation with respect to a four times. We get,

$$\left(-2\sin a\widehat{F}_{01} + 2\cos a\widehat{F}_{03} - \sin 2a\widehat{F}_{11} + 2\cos 2a\widehat{F}_{13} + \sin 2a\widehat{F}_{33}\right)(\zeta_0) = 0 \tag{2.6}$$

$$\left(-2\cos a\widehat{F}_{01} - 2\sin a\widehat{F}_{03} - 2\cos 2a\widehat{F}_{11} - 4\sin 2a\widehat{F}_{13} + 2\cos 2a\widehat{F}_{33}\right)(\zeta_0) = 0 \tag{2.7}$$

$$\left(2\sin a\widehat{F}_{01} - 2\cos a\widehat{F}_{03} + 4\sin 2a\widehat{F}_{11} - 8\cos 2a\widehat{F}_{13} - 4\sin 2a\widehat{F}_{33}\right)(\zeta_0) = 0 \tag{2.8}$$

$$\left(2\cos a\widehat{F}_{01} + 2\sin a\widehat{F}_{03} + 8\cos 2a\widehat{F}_{11} + 16\sin 2a\widehat{F}_{13} - 8\cos 2a\widehat{F}_{33}\right)(\zeta_0) = 0. \tag{2.9}$$

Letting $a \to 0$ in (2.5), (2.6), (2.7), (2.8) and (2.9), we have the following 5 equations:

$$\left(\widehat{F}_{00} + 2\widehat{F}_{01} + \widehat{F}_{11}\right)(\zeta_0) = 0 \tag{2.10}$$

$$\left(\widehat{F}_{03} + \widehat{F}_{13}\right)(\zeta_0) = 0 \tag{2.11}$$

$$\left(\widehat{F}_{01} + \widehat{F}_{11} - \widehat{F}_{33}\right)(\zeta_0) = 0 \tag{2.12}$$

$$\left(\widehat{F}_{03} + 4\widehat{F}_{13}\right)(\zeta_0) = 0 \tag{2.13}$$

$$\left(\widehat{F}_{01} + 4\widehat{F}_{11} - 4\widehat{F}_{33}\right)(\zeta_0) = 0. \tag{2.14}$$

Since $\delta(F) = \mathrm{trace}(F) = 0$, we have

$$\widehat{F}_{02}(\zeta_0) = \widehat{F}_{12}(\zeta_0) = \widehat{F}_{22}(\zeta_0) = \widehat{F}_{32}(\zeta_0) = 0, \tag{2.15}$$

$$\left(\widehat{F}_{00} + \widehat{F}_{11} + \widehat{F}_{22} + \widehat{F}_{33}\right)(\zeta_0) = 0. \tag{2.16}$$

From (2.11) and (2.13), we get that $\widehat{F}_{03}(\zeta_0) = \widehat{F}_{13}(\zeta_0) = 0$. Subtracting (2.12) from (2.14), we get that $\widehat{F}_{11}(\zeta_0) = \widehat{F}_{33}(\zeta_0)$. Therefore (2.12) gives that $\widehat{F}_{01}(\zeta_0) = 0$. Substituting $\widehat{F}_{11}(\zeta_0) = \widehat{F}_{33}(\zeta_0)$ into (2.16), and using the fact that $\widehat{F}_{22}(\zeta_0) = 0$ from (2.15), we get that $\widehat{F}_{00}(\zeta_0) + 2\widehat{F}_{11}(\zeta_0) = 0$. Combining this with (2.10), we get that $\widehat{F}_{00}(\zeta_0) = \widehat{F}_{11}(\zeta_0) = 0$. Combining all these, we have now shown that $\widehat{F}_{ij}(\zeta_0) = 0$ for all $0 \leq i, j \leq 3$.

Next our goal is to show that if ζ is any non-zero space-like vector in a small enough conical neighborhood (in the Euclidean sense) of ζ_0, then $\widehat{F}_{ij}(\zeta) = 0$, for $0 \leq i, j \leq 3$ as well. We recall that a non-zero vector $\zeta = (\zeta^0, \zeta^1, \zeta^2, \zeta^3)$ is space-like if $|\zeta^0| < \|(\zeta^1, \zeta^2, \zeta^3)\|$, where the norm $\|\cdot\|$ refers to the Euclidean norm.

We start with a unit vector $(\zeta^1, \zeta^2, \zeta^3)$ in $\mathbb{R}^3$, and we choose $\zeta^0 = -\sin\varphi$. Then $(-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$ is a space-like vector for $-\pi/2 < \varphi < \pi/2$.

Let us recall that in showing $\widehat{F}_{ij}(\zeta_0) = 0$, we considered a perturbation $\theta_0(a)$ (see (2.4)) of the vector $\theta_0 = (1, 0, 0)$. Note that we required that $\theta_0(a)$ was close enough to θ_0 and $(1, \theta_0(a)) \cdot \zeta_0 = 0$. The following calculations are motivated by having these same requirements for the vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$.

Since we are interested in a non-zero space-like vector in a small enough conical neighborhood of ζ_0, let us choose

$$\zeta^1 = \sin\alpha\cos\beta,\ \zeta^2 = \cos\alpha \text{ and } \zeta^3 = \sin\alpha\sin\beta.$$

Then clearly ζ is close to $(0, 1, 0)$ whenever α and β are close enough to 0, and choosing φ close to 0, we get that the space-like vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$ is close enough to $(0, 0, 1, 0)$.

Next choose $\theta_0(\varphi) = (\cos\varphi, \sin\varphi, 0)$ for φ close to 0 and the perturbation of $\theta_0(\varphi)$ for a close to 0 by

$$\theta_0(\varphi, a) = (\cos a\cos\varphi, \sin\varphi, \sin a\cos\varphi)\,.$$

Our goal is next to modify $\theta_0(\varphi, a)$ to $\Theta_0(\varphi, a)$ such that $(1, \Theta_0(\varphi, a)) \cdot \zeta = 0$. To this end, let us consider the orthogonal matrix A:

$$A = \begin{bmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Define Θ_0, $\Theta_0(\varphi)$ and $\Theta_0(a, \varphi)$ by

$$\Theta_0 = A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta \\ -\sin\alpha \\ \cos\alpha\sin\beta \end{bmatrix},$$

$$\Theta_0(\varphi) = A^T\theta_0(\varphi) = A^T \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11}\cos\varphi + a_{21}\sin\varphi \\ a_{12}\cos\varphi + a_{22}\sin\varphi \\ a_{13}\cos\varphi + a_{23}\sin\varphi \end{bmatrix}$$

and

$$\Theta_0(\varphi, a) = A^T\theta_0(\varphi, a) = \begin{bmatrix} a_{11}\cos a\cos\varphi + a_{21}\sin\varphi + a_{31}\sin a\cos\varphi \\ a_{12}\cos a\cos\varphi + a_{22}\sin\varphi + a_{32}\sin a\cos\varphi \\ a_{13}\cos a\cos\varphi + a_{23}\sin\varphi + a_{33}\sin a\cos\varphi \end{bmatrix} = \begin{bmatrix} A_1(a) \\ A_2(a) \\ A_3(a) \end{bmatrix}.$$

We first note that if a, φ, α and β are close enough to 0, then $\Theta_0(\varphi, a)$ is close enough to θ_0. As before, defining $\widetilde{\Theta}(\varphi, a) = (1, \Theta(\varphi, a))$, we have $LF(t, x, \widetilde{\Theta}_0(\varphi, a))$ is 0.

Next we show that for all φ, a, α and β close enough to 0, $(1, \Theta_0(\varphi, a)) \cdot \zeta = 0$. To see this, consider

$$\begin{aligned}(-\sin\varphi, \sin\alpha\cos\beta, \cos\alpha, \sin\alpha\sin\beta) \cdot (1, \Theta_0(\varphi)) &= -\sin\varphi + \langle \left(\zeta^1, \zeta^2, \zeta^3\right), A^T\left(\theta_0(\varphi, a)\right)\rangle \\ &= -\sin\varphi + \langle A\left(\zeta^1, \zeta^2, \zeta^3\right), \theta_0(\varphi, a)\rangle.\end{aligned}$$

The matrix A is such that $A\left(\zeta^1, \zeta^2, \zeta^3\right) = (0, 1, 0)$.

Since $\theta_0(\varphi, a) = (\cos a\cos\varphi, \sin\varphi, \sin a\cos\varphi)$, we now get that $(1, \Theta_0(\varphi, a)) \cdot \zeta = 0$. Using this choice of $\Theta_0(\varphi, a)$ in (2.3), we get

$$\begin{aligned}\Big(&\widehat{F}_{00}+2A_1(a)\widehat{F}_{01}+2A_2(a)\widehat{F}_{02}+2A_3(a)\widehat{F}_{03}+(A_1(a))^2\widehat{F}_{11}+2A_1(a)A_2(a)\widehat{F}_{12}\\ &+2A_1(a)A_3(a)\widehat{F}_{13}+(A_2(a))^2\widehat{F}_{22}+2A_2(a)A_3(a)\widehat{F}_{23}+(A_3(a))^2\widehat{F}_{33}\Big)(\zeta)=0. \end{aligned} \tag{2.17}$$

As before, we consider (2.17) and differentiate it 4 times and let $a \to 0$. These would give 5 equations. Also since F is divergence free and trace free, we have the following 5 equations:

$$\begin{aligned}&\left(-\sin\varphi\widehat{F}_{00}+a_{21}\widehat{F}_{01}+a_{22}\widehat{F}_{02}+a_{23}\widehat{F}_{03}\right)(\zeta)=0\\ &\left(-\sin\varphi\widehat{F}_{10}+a_{21}\widehat{F}_{11}+a_{22}\widehat{F}_{12}+a_{23}\widehat{F}_{13}\right)(\zeta)=0\\ &\left(-\sin\varphi\widehat{F}_{20}+a_{21}\widehat{F}_{21}+a_{22}\widehat{F}_{22}+a_{23}\widehat{F}_{23}\right)(\zeta)=0\\ &\left(-\sin\varphi\widehat{F}_{30}+a_{21}\widehat{F}_{31}+a_{22}\widehat{F}_{32}+a_{23}\widehat{F}_{33}\right)(\zeta)=0\\ &\left(\widehat{F}_{00}+\widehat{F}_{11}+\widehat{F}_{22}+\widehat{F}_{33}\right)(\zeta)=0.\end{aligned} \tag{2.18}$$

Together, these would give 10 equations and determinant of the matrix formed by the coefficients is continuous as a function of α, β and φ. We show that this determinant is non-zero, which would give that $\widehat{F}_{ij}(\zeta)=0$ for $0 \le i, j \le 3$. In order to show that the determinant is non-vanishing, it is enough to observe that as $\alpha, \beta, \varphi \to 0$ in these 10 equations, we would get the same set of equations as in (2.10) - (2.16). However, we have already shown that $\widehat{F}_{ij}(\zeta_0)=0$ for $0 \le i, j \le 3$, using these equations. By continuity of the determinant, we have that the matrix of coefficients formed by the 10 equations mentioned above has non-zero determinant when α, β, φ close to 0. Hence we have $\widehat{F}_{ij}(\zeta)=0$ for $0 \le i, j \le 3$, where $\zeta=(-\sin\varphi, \sin\alpha\cos\beta, \cos\alpha, \sin\alpha\sin\beta)$, with α, β and φ are near 0. Repeating the same argument as above, we can show that $\widehat{F}_{ij}(\lambda\zeta)=0$ for $0 \le i, j \le 3$, where ζ is as above and $\lambda > 0$. This concludes the Lemma 2.2 for the case of $n=3$.

Now we consider the general case $n \ge 4$.

As before, first let us show that $\widehat{F}_{ij}(\zeta_0)=0$ for all $0 \le i, j \le n$, where recall that $\zeta_0 = (0, 0, 1, 0, \cdots, 0)$. We fix $\theta_0 = (1, 0, 0, \cdots, 0) \in \mathbb{S}^{n-1}$. Note that $(1, \theta_0)\cdot\zeta_0 = 0$. Consider

$$\theta_k(a)=\cos ae_1+\sin ae_k \text{ for } k\ge 3, \tag{2.19}$$

$$\theta_{kl}(a)=\cos ae_1+\frac{1}{\sqrt{2}}\sin ae_k+\frac{1}{\sqrt{2}}\sin ae_l \text{ for } 3\le k<l\le n \tag{2.20}$$

where $e_j \in \mathbb{R}^n$ be vector in $\mathbb{R}^n$ whose j^{th} entry is 1 and other entries are zero. If a is near 0, then $\theta_0(a)$ is near θ_0. Also note that $(1, \theta_k(a))\cdot\zeta_0 = 0$ and $(1, \theta_{kl}(a))\cdot\zeta_0 = 0$. Now substituting this choice of ζ_0, $\theta_k(a)$ and $\theta_{kl}(a)$ into (2.3), we get,

$$\left(\widehat{F}_{00}+2\cos a\widehat{F}_{01}+2\sin a\widehat{F}_{0k}+\cos^2 a\widehat{F}_{11}+2\sin a\cos a\widehat{F}_{1k}+\sin^2 a\widehat{F}_{kk}\right)(\zeta_0)=0 \text{ for } k\ge 3. \tag{2.21}$$

$$\begin{aligned}\Big(&\widehat{F}_{00}+2\cos a\widehat{F}_{01}+\sqrt{2}\sin a\widehat{F}_{0k}+\sqrt{2}\sin a\widehat{F}_{0l}+\cos^2 a\widehat{F}_{11}+\sqrt{2}\sin a\cos a\widehat{F}_{1k}\\ &+\sqrt{2}\sin a\cos a\widehat{F}_{1l}+\frac{\sin^2 a}{2}\left(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)\Big)(\zeta_0)=0 \text{ for } 3\le k<l\le n.\end{aligned} \tag{2.22}$$

Differentiating (2.21) 4 times and letting $a \to 0$, we arrive at the following equations:

$$\left(\widehat{F}_{00}+2\widehat{F}_{01}+\widehat{F}_{11}\right)(\zeta_0)=0 \tag{2.23}$$
$$\left(\widehat{F}_{0k}+\widehat{F}_{1k}\right)(\zeta_0)=0 \tag{2.24}$$
$$\left(\widehat{F}_{01}-\widehat{F}_{11}+\widehat{F}_{kk}\right)(\zeta_0)=0 \tag{2.25}$$
$$\left(\widehat{F}_{0k}+4\widehat{F}_{1k}\right)(\zeta_0)=0 \tag{2.26}$$
$$\left(-\widehat{F}_{01}+4\widehat{F}_{11}-4\widehat{F}_{kk}\right)(\zeta_0)=0. \tag{2.27}$$

Similarly, we differentiate (2.22) with respect to a 4 times and let $a\to 0$. We get,

$$\begin{aligned}\Big(&-2\sin a\widehat{F}_{01}+\sqrt{2}\cos a\widehat{F}_{0k}+\sqrt{2}\cos a\widehat{F}_{0l}-\sin 2a\widehat{F}_{11}+\sqrt{2}\cos 2a\widehat{F}_{1k}\\&+\sqrt{2}\cos 2a\widehat{F}_{1l}+\frac{\sin 2a}{2}\left(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)\Big)(\zeta_0)=0;\ 3\le k<l\le n\end{aligned} \tag{2.28}$$

$$\begin{aligned}\Big(&-2\cos a\widehat{F}_{01}-\sqrt{2}\sin a\widehat{F}_{0k}-\sqrt{2}\sin a\widehat{F}_{0l}-2\cos 2a\widehat{F}_{11}-2\sqrt{2}\sin 2a\widehat{F}_{1k}\\&-2\sqrt{2}\sin 2a\widehat{F}_{1l}+\cos 2a\left(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)\Big)(\zeta_0)=0;\ 3\le k<l\le n.\end{aligned} \tag{2.29}$$

$$\begin{aligned}\Big(&2\sin a\widehat{F}_{01}-\sqrt{2}\cos a\widehat{F}_{0k}-\sqrt{2}\cos a\widehat{F}_{0l}+4\sin 2a\widehat{F}_{11}-4\sqrt{2}\cos 2a\widehat{F}_{1k}\\&-4\sqrt{2}\cos 2a\widehat{F}_{1l}-2\sin 2a\left(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)\Big)(\zeta_0)=0;\ 3\le k<l\le n.\end{aligned} \tag{2.30}$$

$$\begin{aligned}\Big(&2\cos a\widehat{F}_{01}+\sqrt{2}\sin a\widehat{F}_{0k}+\sqrt{2}\sin a\widehat{F}_{0l}+8\cos 2a\widehat{F}_{11}+8\sqrt{2}\sin 2a\widehat{F}_{1k}\\&+8\sqrt{2}\sin 2a\widehat{F}_{1l}-4\cos 2a\left(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)\Big)(\zeta_0)=0;\ 3\le k<l\le n.\end{aligned} \tag{2.31}$$

Letting $a\to 0$ in (2.28) - (2.31), we have,

$$\left(\widehat{F}_{0k}+\widehat{F}_{0l}+\widehat{F}_{1k}+\widehat{F}_{1l}\right)(\zeta_0)=0 \tag{2.32}$$
$$\left(-2\widehat{F}_{01}-2\widehat{F}_{11}+\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll}\right)(\zeta_0)=0;\quad k\ge 3 \tag{2.33}$$
$$\left(\widehat{F}_{0k}+\widehat{F}_{0l}+4\widehat{F}_{1k}+4\widehat{F}_{1l}\right)(\zeta_0)=0;\quad k\ge 3. \tag{2.34}$$
$$\left(2\widehat{F}_{01}+8\widehat{F}_{11}-4(\widehat{F}_{kk}+2\widehat{F}_{kl}+\widehat{F}_{ll})\right)(\zeta_0)=0;\ 3\le k<l\le n. \tag{2.35}$$

Now we consider (2.23) - (2.26) and (2.32) - (2.35) combined with the following two equations:

$$\widehat{F}_{02}(\zeta_0)=\widehat{F}_{12}(\zeta_0)=\widehat{F}_{22}(\zeta_0)=\cdots=\widehat{F}_{n2}(\zeta_0)=0; \tag{2.36}$$
$$\left(\widehat{F}_{00}+\widehat{F}_{11}+\widehat{F}_{22}+\widehat{F}_{33}+\cdots+\widehat{F}_{nn}\right)(\zeta_0)=0, \tag{2.37}$$

since $\delta(F)=\mathrm{trace}(F)=0$.

We now show that these equations imply that $\widehat{F}_{ij}(\zeta_0)=0$ for all $0\le i,j\le n$.

Adding (2.35) and (2.33), we get,

$$2\widehat{F}_{11} - (\widehat{F}_{kk} + 2\widehat{F}_{kl} + \widehat{F}_{ll})(\zeta_0) = 0 \text{ for } 3 \leq k < l \neq n. \tag{2.38}$$

Subtracting (2.34) from (2.32), we get,

$$(\widehat{F}_{1k} + \widehat{F}_{1l})(\zeta_0) = (\widehat{F}_{0k} + \widehat{F}_{0l})(\zeta_0) = 0 \text{ for } 3 \leq k < l \neq n. \tag{2.39}$$

Adding (2.25) and (2.27), we get,

$$\widehat{F}_{11}(\zeta_0) = \widehat{F}_{kk}(\zeta_0) \text{ for } k \geq 3 \text{ and } \widehat{F}_{01}(\zeta_0) = 0. \tag{2.40}$$

Now combined with the previous equation, we have from (2.23) that

$$\widehat{F}_{00}(\zeta_0) = -\widehat{F}_{11}(\zeta_0). \tag{2.41}$$

From (2.24) and (2.26), we have that

$$\widehat{F}_{1k}(\zeta_0) = \widehat{F}_{0k}(\zeta_0) = 0 \text{ for } k \geq 3. \tag{2.42}$$

Now we already know from (2.36) that $\widehat{F}_{22}(\zeta_0) = 0$. Using (2.40) and (2.41) in (2.37), we get that

$$(n-2)\widehat{F}_{11}(\zeta_0) = 0. \tag{2.43}$$

This then implies that

$$\widehat{F}_{mm}(\zeta_0) = 0 \text{ for all } 0 \leq m \leq n. \tag{2.44}$$

Now from (2.38), this then implies that

$$\widehat{F}_{kl}(\zeta_0) = 0 \text{ for all } 3 \leq k < l \leq n. \tag{2.45}$$

Now combined with (2.36), we now have that

$$\widehat{F}_{ij}(\zeta_0) = 0 \text{ for all } 0 \leq i, j \leq n. \tag{2.46}$$

Next our goal is to show that if ζ is any non-zero space-like vector in a small enough conical neighborhood (in the Euclidean sense) of ζ_0, then $\widehat{F}_{ij}(\zeta) = 0$, for $0 \leq i, j \leq n$ as well. We recall that a non-zero vector $\zeta = (\zeta^0, \zeta^1, \zeta^2, \cdots, \zeta^n)$ is space-like if $|\zeta^0| < \|(\zeta^1, \zeta^2, \cdots, \zeta^n)\|$, where the norm $\|\cdot\|$ refers to the Euclidean norm.

We start with a unit vector in $\mathbb{R}^n$, $\zeta' := (\zeta^1, \zeta^2, \cdots, \zeta^n)$, and let us choose $\zeta^0 = -\sin\varphi$. Then $(-\sin\varphi, \zeta^1, \zeta^2, \cdots, \zeta^n)$ is a space-like vector if $-\pi/2 < \varphi < \pi/2$.

Let us recall that in showing $\widehat{F}_{ij}(\zeta_0) = 0$, we considered a perturbation $\theta_0(a)$ (see (2.19)) of the vector $\theta_0 = (1, 0, \cdots, 0)$. Note that we required that $\theta_0(a)$ was close enough to θ_0 and $(1, \theta_0(a)) \cdot \zeta_0 = 0$. As in the proof for the case $n = 3$, the calculations below are motivated by these requirements for the vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \cdots, \zeta^n)$.

Since we are interested in a non-zero space-like vector in a small enough conical neighborhood of ζ_0, let us choose ζ' as

$$\zeta' = (\cos\varphi_1 \sin\varphi_2, \cos\varphi_2, \sin\varphi_1 \sin\varphi_2 \cos\varphi_3, \cdots, \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}).$$

Then clearly ζ' is close to $(0, 1, 0, \cdots, 0) \in \mathbb{R}^n$ whenever φ_i for $1 \leq i \leq n-1$ are close enough to 0, and choosing φ close to 0, we get that the space-like vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \cdots, \zeta^n)$ is close enough to $\zeta_0 = (0, 0, 1, 0, \cdots, 0) \in \mathbb{R}^{1+n}$.

Next choose $\theta_0(\varphi) := \cos\varphi e_1 + \sin\varphi e_2$ close to θ_0 when φ is close to 0 and the perturbation of $\theta_0(\varphi)$ for a close to 0 by

$$\theta_k(\varphi,a) = \cos a\cos\varphi e_1 + \sin\varphi e_2 + \sin a\cos\varphi e_k \text{ for } k\geq 3,$$
$$\theta_{kl}(\varphi,a) = \cos a\cos\varphi e_1 + \sin\varphi e_2 + \frac{1}{\sqrt{2}}\sin a\cos\varphi e_k + \frac{1}{\sqrt{2}}\sin a\cos\varphi e_l \text{ for } 3\leq k<l\leq n.$$

Let us consider the orthogonal matrix A such that $A\zeta' = e_2$. Let us denote the entries of this matrix by $A=(a_{ij})$. Define $\Theta_0(\varphi)$ and $\Theta_k(a,\varphi)$ and $\Theta_{kl}(\varphi,a)$ by

$$\Theta_k(\varphi,a) = A^T\left(\theta_k(\varphi,a)\right) = \begin{bmatrix} a_{11}\cos a\cos\varphi + a_{21}\sin\varphi + a_{k1}\sin a\cos\varphi \\ \vdots \\ a_{1n}\cos a\cos\varphi + a_{2n}\sin\varphi + a_{kn}\sin a\cos\varphi \end{bmatrix} = \begin{bmatrix} A_1(a) \\ \vdots \\ A_n(a)\end{bmatrix} \text{ where } k\geq 3$$

and

$$\Theta_{kl}(\varphi,a) = A^T\left(\theta_{kl}(\varphi,a)\right) = \begin{bmatrix} a_{11}\cos a\cos\varphi + a_{21}\sin\varphi + \frac{1}{\sqrt{2}}\left(a_{k1}+a_{l1}\right)\sin a\cos\varphi \\ \vdots \\ a_{1n}\cos a\cos\varphi + a_{2n}\sin\varphi + \frac{1}{\sqrt{2}}\left(a_{kn}+a_{ln}\right)\sin a\cos\varphi \end{bmatrix} = \begin{bmatrix} B_1(a) \\ \vdots \\ B_n(a)\end{bmatrix},$$

where $3\leq k<l\leq n$. We first note that if a,φ, and φ_i for $1\leq i\leq n-1$, are close enough to 0, then $\Theta_k(\varphi,a)$ and $\Theta_{kl}(\varphi,a)$ are close enough to θ_0. Denoting $\widetilde{\Theta}_k(\varphi,a) = (1,\Theta_k(\varphi,a)$ and $\widetilde{\Theta}_{k,l}(\varphi,a) = (1,\Theta_{k,l}(\varphi,a)$, we have that $LF(t,x,\widetilde{\Theta}_k(\varphi,a))=0$ for $k\geq 3$ and $LF(t,x,\widetilde{\Theta}_{kl}(\varphi,a))=0$ for $3\leq k<l\leq n$.

Next we show that for all φ,a and φ_i for $1\leq i\leq n-1$ close enough to $0, (1,\widetilde{\Theta}_k(\varphi,a))\cdot\zeta=0$ and $\left(1,\widetilde{\Theta}_{kl}(\varphi,a)\right)\cdot\zeta=0$.

To see this, consider

$$\begin{aligned}(-\sin\varphi,\zeta')\cdot(1,\Theta_k(\varphi,a)) = -\sin\varphi + \langle\zeta',\Theta_k(\varphi)\rangle &= -\sin\varphi + \langle\zeta', A^T\left(\theta_k(\varphi,a)\right)\rangle \\ &= -\sin\varphi + \langle A\zeta',\theta_k(\varphi,a)\rangle.\end{aligned}$$

Note that the matrix A is chosen such that $A\left(\zeta'\right) = (0,1,0,\cdots,0)$. Since $\theta_k(\varphi,a) = \cos a\cos\varphi e_1 + \sin\varphi e_2 + \sin a\cos\varphi e_k$, $k\geq 3$, we now get that $(1,\Theta_k(\varphi,a))\cdot\zeta=0$. Similarly we can check that $(1,\Theta_{kl}(\varphi,a))\cdot\zeta=0$.

Using this choice of $\widetilde{\Theta}_k(\varphi,a)$ in (2.3), we have

$$\begin{aligned}\Bigg(&\widehat{F}_{00} + 2A_1(a)\widehat{F}_{01} + 2A_2(a)\widehat{F}_{02} + 2A_3(a)\widehat{F}_{03} + \cdots + 2A_n(a)\widehat{F}_{0n} \\ &+ (A_1(a))^2\widehat{F}_{11} + 2A_1(a)A_2(a)\widehat{F}_{12} + 2A_1(a)A_3(a)\widehat{F}_{13} + \cdots + 2A_1(a)A_n(a)\widehat{F}_{1n} \\ &+ (A_2(a))^2\widehat{F}_{22} + 2A_2(a)A_3(a)\widehat{F}_{23} + 2A_2(a)A_4(a)\widehat{F}_{24} + \cdots + 2A_2(a)A_n(a)\widehat{F}_{2n} \\ &\qquad\vdots \\ &+ \left(A_{n-1}(a)\right)^2\widehat{F}_{n-1,n-1} + 2A_{n-1}(a)A_n(a)\widehat{F}_{n-1,n} + \left(A_n(a)\right)^2\widehat{F}_{nn}\Bigg)(\zeta) = 0.\end{aligned} \tag{2.47}$$

Next using the choice $\widetilde{\Theta}_{kl}(\varphi, a)$ in (2.3), we get

$$\begin{aligned}
&\Big(\widehat{F}_{00} + 2B_1(a)\widehat{F}_{01} + 2B_2(a)\widehat{F}_{02} + 2B_3(a)\widehat{F}_{03} + \cdots + 2B_n(a)\widehat{F}_{0n} \\
&+ (B_1(a))^2\widehat{F}_{11} + 2B_1(a)B_2(a)\widehat{F}_{12} + 2B_1(a)B_3(a)\widehat{F}_{13} + \cdots + 2B_1(a)B_n(a)\widehat{F}_{1n} \\
&+ (B_2(a))^2\widehat{F}_{22} + 2B_2(a)B_3(a)\widehat{F}_{23} + 2B_2(a)B_4(a)\widehat{F}_{24} + \cdots + 2B_2(a)B_n(a)\widehat{F}_{2n} \\
&\qquad\qquad\vdots \\
&+ (B_{n-1}(a))^2\,\widehat{F}_{n-1,n-1} + 2B_{n-1}(a)B_n(a)\widehat{F}_{n-1,n} + (B_n(a))^2\,\widehat{F}_{nn}\Big)(\zeta) = 0.
\end{aligned} \tag{2.48}$$

We differentiate each of Equations (2.47) and (2.48), 4 times and let $a \to 0$. Arguing similarly to the case of $n = 3$, we will arrive at the fact that $\widehat{F}_{ij}(\zeta) = 0$, and also $\widehat{F}_{ij}(\lambda\zeta) = 0$ for $\lambda > 0$. $\square$

Proof of Theorem 1.1. By Lemma 2.2, we have an open cone of space-like vectors ζ along which the Fourier transform $\widehat{F}_{ij}(\zeta) = 0$. Since F_{ij} for $1 \le i, j \le n$ are extended by zero outside Ω, therefore using Paley-Wiener theorem, we have that $F_{ij} \equiv 0$ for all $0 \le i, j \le n$. $\square$

Next we prove the decomposition result stated in Theorem 1.2.

Proof of Theorem 1.2. Assume the decomposition is true. Taking trace on both sides in (1.4), we get,

$$\mathrm{trace}(F) = \mathrm{trace}(\widetilde{F}) + \mathrm{trace}(\lambda g) + \mathrm{trace}(\mathrm{d}v).$$

Now by assumption, $\mathrm{trace}(\widetilde{F}) = 0$ and $\mathrm{trace}(\lambda g) = (n-1)\lambda$. Also $\mathrm{trace}(\mathrm{d}v) = \delta v$. Therefore

$$\mathrm{trace}(F) = (n-1)\lambda + \delta v. \tag{2.49}$$

Let us take divergence on both sides of (1.4). Using the fact that $\widetilde{F}$ is divergence free

$$\delta F = \delta\,(\lambda g) + \delta \mathrm{d}v.$$

Writing the above equation in expanded form, we have

$$\begin{bmatrix} \partial_j F_{0j} \\ \partial_j F_{1j} \\ \vdots \\ \partial_j F_{nj} \end{bmatrix} = \begin{bmatrix} -\partial_0\lambda \\ \partial_1\lambda \\ \vdots \\ \partial_n\lambda \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \Delta v_0 + \partial^2_{0j}v_j \\ \Delta v_1 + \partial^2_{1j}v_j \\ \vdots \\ \Delta v_n + \partial^2_{nj}v_j \end{bmatrix}. \tag{2.50}$$

Now using the expression for λ from (2.49) in (2.50), we get

$$\frac{1}{2}\begin{bmatrix} \Delta v_0 + \partial^2_{0j}v_j \\ \Delta v_1 + \partial^2_{1j}v_j \\ \vdots \\ \Delta v_n + \partial^2_{nj}v_j \end{bmatrix} + \frac{1}{n-1}\begin{bmatrix} -\partial_0\mathrm{trace}(F) \\ \partial_1\mathrm{trace}(F) \\ \vdots \\ \partial_n\mathrm{trace}(F) \end{bmatrix} - \frac{1}{n-1}\begin{bmatrix} -\partial^2_{0j}v_j \\ \partial^2_{1j}v_j \\ \vdots \\ \partial^2_{nj}v_j \end{bmatrix} = \begin{bmatrix} \partial_j F_{0j} \\ \partial_j F_{1j} \\ \vdots \\ \partial_j F_{nj} \end{bmatrix}.$$

Thus the equation for v is

$$\begin{bmatrix} \Delta v_0 + \left(1+\frac{2}{n-1}\right)\partial^2_{0j}v_j \\ \Delta v_1 + \left(1-\frac{2}{n-1}\right)\partial^2_{1j}v_j \\ \vdots \\ \Delta v_n + \left(1-\frac{2}{n-1}\right)\partial^2_{nj}v_j \end{bmatrix} = 2\begin{bmatrix} \partial_j F_{0j} \\ \partial_j F_{1j} \\ \vdots \\ \partial_j F_{nj} \end{bmatrix} - \frac{2}{n-1}\begin{bmatrix} -\partial_0 \mathrm{trace}(F) \\ \partial_1 \mathrm{trace}(F) \\ \vdots \\ \partial_n \mathrm{trace}(F) \end{bmatrix} := \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}. \tag{2.51}$$

This set of equations can be written as

$$\begin{aligned} &\Delta v_0 + \left(1+\frac{2}{n-1}\right)\partial^2_{0k}v_k = u_0 \\ &\Delta v_j + \left(1-\frac{2}{n-1}\right)\partial^2_{jk}v_k = u_j, \text{ for } 1 \le j \le n. \end{aligned} \tag{2.52}$$

We first note that for $n = 3$, the above system of equations becomes

$$\begin{cases} 3\partial_0^2 v_0 + \partial_1^2 v_0 + \partial_2^2 v_0 + \partial_3^2 v_0 + 2\left(\partial^2_{01}v_1 + \partial^2_{02}v_2 + \partial^2_{03}v_3\right) = u_0, \text{ in } \Omega \\ \Delta v_1 = u_1, \text{in } \Omega \\ \Delta v_2 = u_2, \text{ in } \Omega \\ \Delta v_3 = u_3, \text{ in } \Omega \\ v_0 = v_1 = v_2 = v_3 = 0, \text{ on } \partial\Omega. \end{cases} \tag{2.53}$$

Equation (2.53) is a decoupled system of equations for v with zero Dirichlet boundary data and hence it is uniquely solvable. Then we use (2.49) to solve for λ. This completes the proof of Theorem 1.2 for $n = 3$.

Now in what follows, we assume that $n \ge 4$.

For simplicity, we denote $\alpha = 1 + \frac{2}{n-1}$, $\beta = 1 - \frac{2}{n-1}$ and $A(t, x; \nabla)$ the following operator (here and below $\nabla = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$):

$$A(t,x;\nabla) = \begin{bmatrix} \Delta + \alpha\partial_0^2 & \alpha\partial^2_{01} & \alpha\partial^2_{02} & \cdots & \alpha\partial^2_{0n} \\ \beta\partial^2_{10} & \Delta + \beta\partial_1^2 & \beta\partial^2_{12} & \cdots & \beta\partial^2_{1n} \\ \beta\partial^2_{20} & \beta\partial^2_{21} & \Delta + \beta\partial_2^2 & \cdots & \beta\partial^2_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta\partial^2_{n0} & \beta\partial^2_{n1} & \beta\partial^2_{n2} & \cdots & \Delta + \beta\partial_n^2 \end{bmatrix}. \tag{2.54}$$

Then we have (2.52) with the homogeneous boundary condition can be written as

$$\begin{cases} A(t,x;\nabla)v(t,x) = u(t,x) \quad (t,x) \in \Omega, \\ v(t,x) = 0 \quad (t,x) \in \partial\Omega \end{cases} \tag{2.55}$$

where $v(t,x) = (v_0(t,x), v_1(t,x), \cdots, v_n(t,x))^T$ and $u(t,x) = (u_0(t,x), u_1(t,x), \cdots, u_n(t,x))^T$ are two column vectors. Our goal is to show that the boundary value problem (2.55) is uniquely solvable. To this end, we show (see [15, 21]) that $A(t, x; \nabla)$ is strongly elliptic with zero kernel and zero co-kernel.

We first prove strong ellipticity. The symbol $A(t,x;\xi)$ of operator $A(t,x;\nabla)$ is (up to a sign) given by

$$A(t,x;\xi) = \begin{bmatrix} |\xi|^2+\alpha\xi_0^2 & \alpha\xi_0\xi_1 & \alpha\xi_0\xi_2 & \cdots & \alpha\xi_0\xi_n \\ \beta\xi_1\xi_0 & |\xi|^2+\beta\xi_1^2 & \beta\xi_1\xi_2 & \cdots & \beta\xi_1\xi_n \\ \beta\xi_2\xi_0 & \beta\xi_2\xi_1 & |\xi|^2+\beta\xi_2^2 & \cdots & \beta\xi_2\xi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta\xi_n\xi_0 & \beta\xi_n\xi_1 & \beta\xi_n\xi_2 & \cdots & |\xi|^2+\beta\xi_n^2, \end{bmatrix} \tag{2.56}$$

where $\xi = (\xi_0, \xi_1, \cdots, \xi_n)$. To prove strong ellipticity for $A(t,x;\nabla)$ it is enough to show that

$$P(t,x;\xi) = \frac{A(t,x;\xi)+A^T(t,x;\xi)}{2}$$

is positive definite. Now

$$P(t,x;\xi) = \begin{bmatrix} |\xi|^2+\alpha\xi_0^2 & \xi_0\xi_1 & \xi_0\xi_2 & \cdots & \xi_0\xi_n \\ \xi_1\xi_0 & |\xi|^2+\beta\xi_1^2 & \beta\xi_1\xi_2 & \cdots & \beta\xi_1\xi_n \\ \xi_2\xi_0 & \beta\xi_2\xi_1 & |\xi|^2+\beta\xi_2^2 & \cdots & \beta\xi_2\xi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n\xi_0 & \beta\xi_n\xi_1 & \beta\xi_n\xi_2 & \cdots & |\xi|^2+\beta\xi_n^2 \end{bmatrix}.$$

Let $\eta \in \mathbb{R}^{1+n} \setminus \{0\}$. Then $\eta^T P(t,x;\xi)\eta$ is given by

$$\begin{aligned} \eta^T P(t,x;\xi)\eta &= |\xi|^2|\eta|^2 + (\alpha-1)\,\xi_0^2\eta_0^2 + \xi_0\eta_0\,(\xi\cdot\eta) + (1-\beta)\xi_0\eta_0\,(\xi\cdot\eta - \xi_0\eta_0) + \beta\xi\cdot\eta\,(\xi\cdot\eta-\xi_0\eta_0) \\ &= |\xi|^2|\eta|^2 + (\alpha+\beta-2)\,\xi_0^2\eta_0^2 + \beta\,(\xi\cdot\eta)^2 + 2\,(1-\beta)\,(\xi\cdot\eta)\,\xi_0\eta_0. \end{aligned}$$

Now using the value of α and β, we have

$$\begin{aligned} \eta^T P(t,x;\xi)\eta &= |\xi|^2|\eta|^2 + \frac{n-3}{n-1}\,(\xi\cdot\eta)^2 + \frac{4}{n-1}\,(\xi_0\eta_0)\,(\xi\cdot\eta) \\ &= \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+(n-3)\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)^2 + 4\left(\frac{\xi_0\eta_0}{|\xi||\eta|}\right)\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)\right). \end{aligned}$$

Let us write the vectors ξ and η as $\xi = (\xi_0, \xi')$ and $\eta = (\eta_0, \eta')$. Now, for simplicity, we define $A = \frac{\xi_0\eta_0}{|\xi||\eta|}$ and $B = \frac{\xi'\cdot\eta'}{|\xi||\eta|}$, then clearly $|A| \le 1$, $|B| \le 1$ and $|A+B| \le 1$. Using these in the above equation, we have

$$\begin{aligned} \eta^T P(t,x;\xi)\eta &= \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+(n-3)\,(A+B)^2 + 4A\,(A+B)\right) \\ &= \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+(n+1)\,A^2 + 2(n-1)AB + (n-3)B^2\right) \\ &\ge \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+(n+1)A^2 - (n-1)A^2 - (n-1)B^2 + (n-3)B^2\right) \\ &\ge \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+2A^2-2B^2\right) \ge \frac{n-3}{n-1}|\xi|^2|\eta|^2. \end{aligned}$$

This proves that $P(t,x,\xi)$ is positive definite and hence $A(t,x;\nabla)$ is strongly elliptic for $n \ge 4$.

Next we show that (2.55) with $u = 0$ on the right hand side has only the zero solution. Multiplying the first equation in (2.52) by v_0 and second equation in (2.52) by v_j and integrating over Ω, we get the following set of equations

$$\int\limits_{\Omega} |\nabla v_0(t,x)|^2 \,\mathrm{d}t\mathrm{d}x + \left(1 + \frac{2}{n-1}\right) \int\limits_{\Omega} \nabla \cdot v(t,x) \partial_0 v_0(t,x) \,\mathrm{d}t\mathrm{d}x = 0 \tag{2.57}$$

and for $1 \le j \le n$

$$\int\limits_{\Omega} |\nabla v_j(t,x)|^2 \,\mathrm{d}t\mathrm{d}x + \left(1 - \frac{2}{n-1}\right) \int\limits_{\Omega} \nabla \cdot v(t,x) \partial_j v_j(t,x) \,\mathrm{d}t\mathrm{d}x = 0. \tag{2.58}$$

Adding the set of equations in (2.57) and (2.58), we get

$$\int\limits_{\Omega} \sum_{j=0}^{n} |\nabla v_j(t,x)|^2 \,\mathrm{d}t\mathrm{d}x + \left(1 - \frac{2}{n-1}\right) \int\limits_{\Omega} |\nabla \cdot v(t,x)|^2 \,\mathrm{d}t\mathrm{d}x + \frac{4}{n-1} \int\limits_{\Omega} \nabla \cdot v(t,x) \partial_0 v_0(t,x) \,\mathrm{d}t\mathrm{d}x = 0. \tag{2.59}$$

For simplicity, let us denote $a = \partial_0 v_0$, $b = \sum_{j=1}^{n} \partial_j v_j$ and $c = \sum_{j=0}^{n} |\nabla v_j|^2 - |\partial_0 v_0|^2$. Using these in (2.59), we have

$$\int\limits_{\Omega} \left(c + a^2 + \frac{n-3}{n-1}(a+b)^2 + \frac{4}{n-1}\left(a^2 + ab\right)\right) \mathrm{d}t\mathrm{d}x = 0.$$

Rewriting this, we get,

$$\int\limits_{\Omega} \left(2na^2 + 2\,(n-1)\,ab + (n-3)\,b^2 + (n-1)\,c\right) \mathrm{d}t\mathrm{d}x = 0.$$

Now let us view the integrand in the above equation as a quadratic equation in a and its discriminant $D(t,x)$ is given by

$$\begin{aligned} D(t,x) &= 4(n-1)^2 b^2 - 8n\left((n-3)b^2 + (n-1)c\right) \\ &= 4\Big(\left(n^2 - 2n + 1\right) b^2 - 2n(n-3)b^2 - 2n(n-1)c\Big) \\ &= 4\Big(\left(-n^2 + 4n + 1\right) b^2 - 2n(n-1)c\Big). \end{aligned}$$

Now

$$c = \sum_{j=0}^{n} |\nabla v_j|^2 - |\partial_0 v_0|^2 = \sum_{i,j=0}^{n} |\partial_i v_j|^2 - |\partial_0 v_0|^2 \ge \sum_{j=1}^{n} |\partial_j v_j|^2.$$

Also

$$b^2 = \Big|\sum_{j=1}^{n} \partial_j v_j\Big|^2 = \sum_{j=1}^{n} |\partial_j v_j|^2 + 2 \sum_{1 \le j < k \le n} \mathrm{Re}\left(\partial_j v_j \overline{\partial_k v_k}\right) \le n \sum_{j=1}^{n} |\partial_j v_j|^2 \le nc.$$

Thus we have that $nc \ge b^2$ and using this we get

$$D(t,x) \le 4\left(-n^2 + 4n + 1 - 2n + 2\right) b^2 = 4\left(-n^2 + 2n + 3\right) b^2 < 0 \text{ if } b^2 \ne 0 \text{ and } n \ge 4.$$

However if $D(t,x) < 0$, we have the integrand in (2.59) is strictly positive which is not possible since the integral in (2.59) is zero. Hence we have $b = 0$ and using this in (2.59), we have $\sum_{j=0}^{n} |\nabla v_j|^2 = 0$ in Ω. This implies $v_j(t,x) = c_j$ for $0 \le j \le n$ where c_j is some constant. Now using the boundary condition we have that $v_j(t,x)=0$ in Ω. Hence $\mathrm{Ker}(A(t,x;\nabla))=\{0\}$.

Finally, we show that the co-kernel of $A(t,x;\nabla)$ is 0 as well. We proceed as follows. Let $w\in(\mathrm{Image}(A(t,x;\nabla)))^{\perp}$. That is, consider w such that

$$\langle w, A(t,x;\nabla)v\rangle = 0 \text{ for all } v\in C^{\infty}(\Omega) \text{ with } v=0 \text{ on } \partial\Omega. \tag{2.60}$$

This, in particular, gives

$$\langle A^{*}(t,x;\nabla)w, v\rangle = 0 \text{ for all } v\in C_c^{\infty}(\Omega), \tag{2.61}$$

where

$$A^{*}(t,x;\partial)=\begin{bmatrix} \Delta+\alpha\partial_0^2 & \beta\partial_{10}^2 & \beta\partial_{20}^2 & \cdots & \beta\partial_{n0}^2\\ \alpha\partial_{01}^2 & \Delta+\beta\partial_1^2 & \beta\partial_{21}^2 & \cdots & \beta\partial_{n1}^2\\ \alpha\partial_{02}^2 & \beta\partial_{12}^2 & \Delta+\beta\partial_2^2 & \cdots & \beta\partial_{n2}^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \alpha\partial_{0n}^2 & \beta\partial_{1n}^2 & \beta\partial_{2n}^2 & \cdots & \Delta+\beta\partial_n^2 \end{bmatrix}. \tag{2.62}$$

By integration by parts in (2.60), combined with (2.54) and the fact that $v|_{\partial\Omega}=0$, we have

$$0=\langle w, A(t,x,\nabla)v\rangle_{L^2(\Omega)}=\langle A^{*}(t,x,\nabla)w, v\rangle_{L^2(\Omega)}+\langle w, B(t,x,\partial_\nu)v\rangle_{L^2(\partial\Omega)} \tag{2.63}$$

where $B(t,x,\partial_\nu)$ is the boundary operator we arrive at after integration by parts. The first term on the right hand side of (2.63) is 0 by (2.61). Next we show that $w=0$ on $\partial\Omega$. Let u be an arbitrary vector field on $\partial\Omega$ with $C^{\infty}(\partial\Omega)$ coefficients. We show that there exists a vector field v in Ω with C^{∞} coefficients such that

$$B(t,x,\partial_\nu)v=u, \text{ on } \partial\Omega, \text{ and } v|_{\partial\Omega}=0. \tag{2.64}$$

The boundary operator B has a smooth extension to a small enough neighbourhood of the boundary. With this extension, we can consider (2.64) as an initial value problem for a system of first order ODEs with smooth coefficients, the solution of which exists in a small enough neighborhood of the boundary. This solution can now be extended smoothly to all of Ω which we denote by v. Using this in (2.63), we get that $w|_{\partial\Omega}=0$. Thus, finally to show that the co-kernel of $A(t,x;\partial)$ is 0, we have to show that the following BVP

$$\begin{cases} A^{*}(t,x;\partial)w=0 \text{ for } (t,x)\in\Omega\\ w(t,x)=0 \text{ for } (t,x)\in\partial\Omega \end{cases} \tag{2.65}$$

has only the zero solution where $A^{*}(t,x;\partial)$ is the adjoint for operator $A(t,x;\partial)$. Using the expression for $A^{*}(t,x;\partial)$ from (2.62) in (2.65), we have the following set of equations for w_j for $0\le j\le n$ with zero Dirichlet boundary condition.

$$\begin{aligned} &\Delta w_0+(\alpha-\beta)\partial_0^2 w_0+\beta\partial_{k0}^2 w_k=0\\ &\Delta w_j+(\alpha-\beta)\partial_{0j} w_0+\beta\partial_{kj}^2 w_k=0,\ 1\le j\le n. \end{aligned} \tag{2.66}$$

Now multiplying the first equation in (2.66) by w_0 and second equation by w_j and integrating over Ω, we have

$$\int_{\Omega} |\nabla w_0(t,x)|^2 \mathrm{d}t\mathrm{d}x + (\alpha-\beta)\int \partial_0 w_0(t,x)\partial_0 w_0(t,x)\mathrm{d}t\mathrm{d}x + \beta\sum_{k=0}^{n}\int_{\Omega} \partial_0 w_0(t,x)\partial_k w_k(t,x)\mathrm{d}t\mathrm{d}x = 0;$$

$$\int_{\Omega} |\nabla w_j(t,x)|^2 \mathrm{d}t\mathrm{d}x + (\alpha-\beta)\int_{\Omega} \partial_0 w_0(t,x)\partial_j w_j(t,x)\mathrm{d}t\mathrm{d}x + \beta\sum_{k=0}^{n} \partial_j w_j(t,x)\partial_k w_k(t,x)\mathrm{d}t\mathrm{d}x;\ 1\le j\le n.$$

Adding the above set of equations and substituting the expressions for α and β, we get

$$\sum_{j=0}^{n}\int_{\Omega} |\nabla w_j(t,x)|^2\mathrm{d}t\mathrm{d}x + \frac{4}{n-1}\int_{\Omega} \nabla\cdot w(t,x)\partial_0 w_0(t,x)\mathrm{d}t\mathrm{d}x + \left(1-\frac{2}{n-1}\right)\int_{\Omega} |\nabla\cdot w(t,x)|^2\mathrm{d}t\mathrm{d}x = 0.$$

This equation is exactly the same as that of (2.59). Hence repeating the same arguments as before, we conclude that $w(t,x) = 0$. Thus we have co-kernel$(A) = \{0\}$ for $n \geq 4$. This completes the proof of the decomposition theorem for $n \geq 4$. □

Proof of Theorem 1.3. Now combining the results of Theorems 1.1 and 1.2, we conclude Theorem 1.3. For, given $F \in C^\infty(\overline{\Omega}, S^2\mathbb{R}^{1+n})$, by Theorem 1.3, we can decompose $F = \widetilde{F} + \lambda g + \mathrm{d}v$, with $\widetilde{F}, \lambda, v \in C^\infty(\overline{\Omega})$ satisfying $\delta(\widetilde{F}) = \mathrm{trace}(\widetilde{F}) = 0$ and $v|_{\partial\Omega} = 0$, and g is the Minkowski metric. It is straightforward to see that λg and $\mathrm{d}v$ above are in the kernel of the light ray transform; see [10] as well. The fact that $\mathrm{d}v$ with $v|_{\partial\Omega} = 0$ lies in the kernel of the light ray transform follows by fundamental theorem of calculus and λg lies in the kernel because g has signature $(-1, 1, \cdots, 1)$, and light ray transform integrates F along lines in the direction $\widetilde{\theta} = (1, \theta)$ with $|\theta| = 1$. Therefore, we conclude that $LF(t, x, \widetilde{\theta}) = L\widetilde{F}(t, x, \widetilde{\theta}) = 0$. Finally, to conclude, we apply Theorem 1.1 for $\widetilde{F}$, after extending $\widetilde{F} = 0$ outside $\overline{\Omega}$.

□

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† TIFR Centre for Applicable Mathematics, Bangalore 560065, India.
E-mail: vkrishnan@tifrbng.res.in

‡ TIFR Centre for Applicable Mathematics, Bangalore 560065, India.
E-mail: soumen@tifrbng.res.in

* Beijing Computational Science Research Center, Beijing 100193, China.
E-mail: mvashisth@csrc.ac.cn